Iranian Journal of Mathematical Sciences and Informatics
Vol. 13, No. 2 (2018), pp 111-132
DOI: 10.7508/ijmsi.2018.13.010

A Numerical Scheme for Solving Nonlinear Fractional Volterra Integro-Differential Equations

Parisa Rahimkhani[a], Yadollah Ordokhani,[*,a] Esmail Babolian[b]

[a]Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran.
[b]Department of Computer Science, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran.

E-mail: P.rahimkhani@alzahra.ac.ir
E-mail: ordokhani@alzahra.ac.ir
E-mail: babolian@khu.ac.ir

Abstract. In this paper, a Bernoulli pseudo-spectral method for solving nonlinear fractional Volterra integro-differential equations is considered. First existence of a unique solution for the problem under study is proved. Then the Caputo fractional derivative and Riemman-Liouville fractional integral properties are employed to derive a new approximate formula for unknown function of the problem. The suggested technique transforms this type of equations to the solution of a system of algebraic equations. In the next step, the error analysis of the proposed method is investigated. Finally, the technique is applied to some problems to show its validity and applicability.

Keywords: Fractional Volterra integro-differential equations, Bernoulli pseudo-spectral method, Caputo derivative.

2010 Mathematics subject classification: 34K28, 26A33, 34A12.

[*]Corresponding Author

Received 25 March 2016; Accepted 25 May 2017

1. Introduction

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to an arbitrary order. A history of the development of fractional differential operators can be found in [26].

In the last decades, many authors have demonstrated applications of fractional calculus in the nonlinear oscillation of earthquakes [11], viscoelastic materials [2], colored noise [22], solid mechanics [30], fluid-dynamic traffic [12], continuum and statistical mechanics [21], economics [3], anomalous transport [25], bioengineering [20] and dynamics of interfaces between nanoparticles and substrates [5]. Owing to the increasing applications, a considerable attention has been given to finding exact and numerical solutions of fractional differential equations. In general, most of the fractional differential equations do not have exact solution. Therefore various methods for the approximate solutions of these equations are extended. These methods include variational iteration method [35], finite difference method [24], Adomian decomposition method [27], homotopy analysis method [10], Legendre collocation method [31], second kind Chebyshev wavelet method [37], CAS wavelet method [32], Bernoulli wavelet method [28] and Fractional-order Bernoulli wavelets [29].

In this article, we consider the following equation

$$D^{\nu}y(x)-\lambda\int_{0}^{x}k(x,t)F(y(t))dt=f(x), \qquad 0\leqslant x\leqslant 1, n-1<\nu\leqslant n, \tag{1.1}$$

$$y^{(i)}(0)=\delta_i, \qquad i=0,1,\ldots,n-1, n\in\mathbb{N}, \tag{1.2}$$

where $y^{(i)}(x)$ stands for the ith-order derivative of $y(x)$; $\delta_i, i=0,1,...,n-1$, are real constants; $f\in C([0,1],\mathbb{R}), k\in C([0,1]^2,\mathbb{R})$ are given functions; $y(x)$ is the unknown function; $D^{\nu}(n-1<\nu\leq n)$ is the fractional derivative in the Caputo sense and $F(y(x))$ is a polynomial of $y(x)$ with constant coefficients.

The Bernoulli polynomials play an important role in various branches of mathematical analysis, namely the theory of modular forms [19], the theory of distributions in p-adic analysis [17], the study of polynomial expansions of analytic functions [4], etc. Recently, new applications of the Bernoulli polynomials have also been found in mathematical physics, in connection with theory of Korteweg-de Vries equation [8], Lame equation [9] and in the study of vertex algebras [7].

In the pseudo-spectral methods ([1],[33]), there are basically two steps to obtain a numerical approximation to a solution of differential equations. First, an appropriate finite or discrete representation of the solution must be chosen. The second step is to obtain a system of algebraic equations.

In the present paper, we use the Bernoulli pseudo-spectral method for solving the fractional integro-differential equation (1.1). For this purpose, we must introduce an appropriate representation of the solution based on Bernoulli polynomials. Then we can reduce fractional integro-differential equation to a system

of algebraic equations which can be solved easily.
The structure of this paper is as follows. In section 2, we describe the basic definitions of fractional calculated and Bernoulli polynomials. In section 3, we investigate the existence and uniqueness of solution for the nonlinear fractional integro-differential equation (1.1). Section 4, is devoted to the proposed method for solving equation (1.1). In section 5, we introduce the error analysis of the proposed method. In section 6, we report our numerical findings and demonstrate the accuracy of the proposed method by considering six numerical examples. The conclusion is given in section 7.

2. Basic Definitions

In this section, first we give necessary definitions of the fractional calculus. Then, we state some properties of Bernoulli polynomials which are used further in this paper. Finally, function approximation via these conceptions is introduced.

2.1. **The fractional integral and derivative.**

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\nu \geq 0$ is defined as [16]

$$I^{\nu} f(x) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^x \frac{f(s)}{(x-s)^{1-\nu}} ds, & \nu > 0, \ x > 0, \\ f(x), & \nu = 0. \end{cases} \tag{2.1}$$

For the Riemann-Liouville fractional integral we have [16]

$$I^{\nu} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)} x^{\nu+\beta}, \qquad \beta > -1.$$

The Riemann-Liouville fractional integral is a linear operator, namely

$$I^{\nu}(\lambda f(x) + \mu g(x)) = \lambda I^{\nu} f(x) + \mu I^{\nu} g(x),$$

where λ and μ are real constants.

Definition 2.2. Caputo's fractional derivative of order ν is defined as [16]

$$D^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_0^x \frac{f^{(n)}(s)}{(x-s)^{\nu+1-n}} ds, \qquad n-1 < \nu \leqslant n, \ n \in \mathbb{N}, \ x > 0. \tag{2.2}$$

For the Caputo derivative we have the following two basic properties [16]

$$D^{\nu} I^{\nu} f(x) = f(x),$$
$$I^{\nu} D^{\nu} f(x) = f(x) - \sum_{i=0}^{n-1} f^{(i)}(0) \frac{x^i}{i!}.$$

2.2. **Bernoulli polynomials.**

Definition 2.3. Bernoulli polynomials of order m can be defined with the following formula [23]

$$B_m(x) = \sum_{i=0}^{m} \left(\begin{array}{c} m \\ i \end{array} \right) B_{m-i} x^i, \qquad x \in [0,1], \tag{2.3}$$

where $B_i := B_i(0)$, $i = 0, 1, ..., m$, are Bernoulli numbers. The first few Bernoulli polynomials are

$B_0(x) = 1,$

$B_1(x) = x - \frac{1}{2},$

$B_2(x) = x^2 - x + \frac{1}{6},$

$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$

These polynomials satisfy the following formula [23]

$$\int_0^1 B_n(x) B_m(x) dx = (-1)^{n-1} \frac{m!n!}{(m+n)!} B_{m+n}, \qquad m, n \geqslant 1. \tag{2.4}$$

According to [18], Bernoulli polynomials are form a complete basis on the interval [0, 1].

2.3. **Function approximation.** Let $H = L^2[0,1]$ and $\{B_0, B_1, ..., B_{m-1}\}$ be the set of Bernoulli polynomials and

$$Y = span\{B_0, B_1, ..., B_{m-1}\}.$$

Let g be an arbitrary element in H. Since Y is a finite dimensional and closed subspace, g has a unique best approximation out of Y such as $g_0 \in Y$, that is

$$\forall y \in Y, \ \ \|g - g_0\| \leq \|g - y\|,$$

this implies that

$$\forall y \in Y, \ \ < g - g_0, y >= 0, \tag{2.5}$$

where $<, >$ denotes inner product. Since $g_0 \in Y$, there exist unique coefficients $c_0, c_1, ..., c_{m-1}$, such that

$$g(x) \simeq g_0(x) = \sum_{i=0}^{m-1} c_i B_i(x) = C^T B(x), \tag{2.6}$$

where C and $B(x)$ are m vectors given by

$$C = [c_0, c_1, ..., c_{m-1}]^T, \qquad B(x) = [B_0(x), B_1(x), ..., B_{m-1}(x)]^T. \tag{2.7}$$

Using Eq. (2.5) we get

$$\langle g - C^T B, B_i \rangle = 0, \quad i = 0, 1, ..., m-1.$$

For simplicity, we write

$$C^T \langle B, B \rangle = \langle g, B \rangle, \tag{2.8}$$

where $< B, B >$ is an $m \times m$ matrix. Let

$$D = \langle B, B \rangle = \int_0^1 B(x) B^T(x) dx. \tag{2.9}$$

The matrix D in Eq. (2.9) can be calculated by using Eq. (2.4). Therefore, from relations (2.8) and (2.9), we obtain

$$C^T = \langle g, B \rangle D^{-1}. \tag{2.10}$$

The main approximate formula for $y(x)$ is given in the following theorem.

Theorem 2.4. *Let $D^\nu y(x)$ be approximated by the Bernoulli polynomials ($D^\nu y(x) \simeq \sum_{k=0}^{m-1} c_k B_k(x)$), and also suppose $n-1 < \nu \leqslant n$. Then*

$$y(x) \simeq \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k b_{k,r}^{(\nu)} x^{r+\nu} + \sum_{i=0}^{n-1} \delta_i \frac{x^i}{i!}, \tag{2.11}$$

where $b_{k,r}^{(\nu)} = \begin{pmatrix} k \\ r \end{pmatrix} \frac{\Gamma(r+1)}{\Gamma(r+1+\nu)} B_{k-r}$.

Proof. Applying operator I^ν, on both sides of $D^\nu y(x) \simeq \sum_{k=0}^{m-1} c_k B_k(x)$, we have

$$\begin{aligned} y(x) - \sum_{i=0}^{n-1} y^{(i)}(0) \frac{x^i}{i!} &\simeq I^\nu (\sum_{k=0}^{m-1} c_k B_k(x)) = I^\nu (\sum_{k=0}^{m-1} c_k \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} B_{k-r} x^r) \\ &= \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k \begin{pmatrix} k \\ r \end{pmatrix} B_{k-r} I^\nu (x^r) \\ &= \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k \begin{pmatrix} k \\ r \end{pmatrix} B_{k-r} \frac{\Gamma(r+1)}{\Gamma(r+1+\nu)} x^{r+\nu} \\ &= \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k b_{k,r}^{(\nu)} x^{r+\nu}. \end{aligned}$$

Then, we obtain

$$y(x) \simeq \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k b_{k,r}^{(\nu)} x^{r+\nu} + \sum_{i=0}^{n-1} \delta_i \frac{x^i}{i!}, \tag{2.12}$$

where $b_{k,r}^{(\nu)} = \begin{pmatrix} k \\ r \end{pmatrix} \frac{\Gamma(r+1)}{\Gamma(r+1+\nu)} B_{k-r}$. □

3. Existence and Uniqueness of Solution

In this section, we will investigate the existence and uniqueness of solution for the nonlinear fractional integro-differential equation (1.1). Since any two norms in $\mathbb{R}$ are equivalent, to be more concert, we will use the sup norm $\|.\|_\infty$, which for any $Y = [y_1, y_2, \dots, y_n]^T$ is given as

$$\|Y\|_\infty = max_{1\leq j\leq n}|y_j|.$$

Definition 3.1. Let $f \in C([0,1],\mathbb{R})$ and $H \in C([0,1]^2\times\mathbb{R},\mathbb{R})$. These functions satisfy the Lipschitz conditions if there exist real constants $\xi, \eta > 0$, such that [14]

$$|f(x) - f(t)| \leq \xi|x-t|, \qquad \|H(x,t,y) - H(x,t,z)\| \leq \eta\|y-z\|, \tag{3.1}$$

where $x, t \in [0,1]$ and $y, z \in \mathbb{R}$.

Let $y \in C([0,1],\mathbb{R})$, be the solution of nonlinear fractional integro-differential equation (1.1). We can write Eqs. (1.1) and (1.2) in the following form

$$\begin{aligned} &D^\nu y(x) = f(x) + \int_0^x H(x,t,y(t))dt, && 0 \leqslant x \leqslant 1, n-1 < \nu \leqslant n, \\ &y^{(i)}(0) = \delta_i, && i = 0,1,...,n-1, n \in \mathbb{N}. \end{aligned} \tag{3.2}$$

Applying operator I^ν on both sides of (3.2) and by using relation (9) in [13] for $\beta = 0$, we obtain an operator $\mathcal{A} : C([0,1]\times\mathbb{R}) \to C([0,1]\times\mathbb{R})$ such that

$$\mathcal{A}y(x) = \sum_{i=0}^{n-1}\delta_i\frac{x^i}{i!} + \frac{1}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}f(t)dt + \frac{1}{\Gamma(\nu+1)}\int_0^x(x-t)^\nu H(x,t,y(t))dt.$$

The fact that the problem (1.1), (1.2) has a unique solution, is equivalent to finding a fixed point y of the operator $\mathcal{A}$, i.e. $\mathcal{A}y = y$.
In the similar manner, for $\Delta = [0,1]$ and any $g \in C(\Delta,\mathbb{R})$ we consider the norm

$$\|g\| = max_{\tau\in\Delta}\|g(\tau)\|.$$

It is an obvious fact that the space $C(\Delta,\mathbb{R})$ with this norm is a Banach space. It is also clear that for $g \in C([0,1],\mathbb{R})$ and $x \in [0,1]$ we have

$$\|\int_0^x g(t)dt\| \leq \|\int_0^x |g(t)|dt\|.$$

Theorem 3.2. *Let $f \in C([0,1],\mathbb{R})$ and $H \in C([0,1]^2\times\mathbb{R},\mathbb{R})$, satisfy the Lipschitz conditions (3.1). If $\frac{\eta}{\Gamma(\nu+2)} \leq \frac{1}{2}$, then the problem (1.1) has a unique solution.*

Proof. Choose $\mu \geq 2(\sum_{i=0}^{n-1}\frac{|\delta_i|}{i!} + \frac{\xi+M_1}{\Gamma(\nu+1)} + \frac{M_2}{\Gamma(\nu+2)})$ and let $|f(0)| = M_1$ and $\|H(x,t,0)\| = M_2$. Then we can show that $\mathcal{A}Z_\mu \subset Z_\mu$ where $Z_\mu = \{y \in$

$C([0,1],\mathbb{R}) : |y| \leq \mu\}$. So let $y \in Z_\mu$. Then we get

$$\begin{aligned}
\|\mathcal{A}y(x)\| &\leq \sum_{i=0}^{n-1} \frac{|\delta_i|}{i!} + \frac{1}{\Gamma(\nu)} \| \int_0^x (x-t)^{\nu-1} |f(t)| dt \| \\
&+ \frac{1}{\Gamma(\nu+1)} \| \int_0^x (x-t)^{\nu} |H(x,t,y(t))| dt \| \\
&\leq \sum_{i=0}^{n-1} \frac{|\delta_i|}{i!} + \frac{1}{\Gamma(\nu)} \| \int_0^x (x-t)^{\nu-1} (|f(t)-f(0)| + |f(0)|) dt \| \\
&+ \frac{1}{\Gamma(\nu+1)} \| \int_0^x (x-t)^{\nu} (|H(x,t,y(t)) - H(x,t,0)| + |H(x,t,0)|) dt \| \\
&\leq \sum_{i=0}^{n-1} \frac{|\delta_i|}{i!} + \frac{1}{\Gamma(\nu)} \| (\xi + M_1) \int_0^x (x-t)^{\nu-1} dt \| \\
&+ \frac{1}{\Gamma(\nu+1)} \| (\mu\eta + M_2) \int_0^x (x-t)^{\nu} dt \| \\
&\leq \sum_{i=0}^{n-1} \frac{|\delta_i|}{i!} + \frac{\xi + M_1}{\Gamma(\nu+1)} + \frac{\mu\eta + M_2}{\Gamma(\nu+2)} \\
&\leq \sum_{i=0}^{n-1} \frac{|\delta_i|}{i!} + \frac{\xi + M_1}{\Gamma(\nu+1)} + \frac{M_2}{\Gamma(\nu+2)} + \mu \frac{\eta}{\Gamma(\nu+2)} \\
&\leq \frac{\mu}{2} + \frac{\mu}{2} = \mu.
\end{aligned}$$

Now, let $y_1, y_2 \in C([0,1],\mathbb{R})$ and $x \in [0,1]$. Then we obtain

$$\begin{aligned}
\|\mathcal{A}y_1(x) - \mathcal{A}y_2(x)\| &\leq \frac{1}{\Gamma(\nu+1)} \| \int_0^x (x-t)^{\nu} |H(x,t,y_1(t)) - H(x,t,y_2(t))| dt \| \\
&\leq \| \frac{x^{\nu+1}}{\Gamma(\nu+2)} \eta \|y_1 - y_2\| \| \leq \frac{\eta}{\Gamma(\nu+2)} \|y_1 - y_2\| \\
&\leq \frac{1}{2} \|y_1 - y_2\| = \Omega_{\eta,\nu} \|y_1 - y_2\|.
\end{aligned}$$

Therefore, since $\Omega_{\eta,\nu} < 1$, the result follows by the contraction mapping theorem. □

4. The Proposed Method

Consider the nonlinear fractional-order Volterra integro-differential equation (1.1) and (1.2). We approximate $D^\nu y(x)$ as

$$D^\nu y(x) \simeq \sum_{k=0}^{m-1} c_k B_k(x). \tag{4.1}$$

From Eqs. (1.1), (1.2), (4.1) and Theorem 2.4, we have

$$\sum_{k=0}^{m-1} c_k B_k(x) - \lambda \int_0^x k(x,t) F(\sum_{k=0}^{m-1}\sum_{r=0}^{k} c_k b_{k,r}^{(\nu)} t^{r+\nu} + \sum_{i=0}^{n-1} \delta_i \frac{t^i}{i!}) dt = f(x). \quad (4.2)$$

Now, we collocate (4.2) at the zeros $x_p, p = 0, 1, ..., m-1$, of shifted Legendre polynomial $L_m(x)$

$$\sum_{k=0}^{m-1} c_k B_k(x_p) - \lambda \int_0^{x_p} k(x_p,t) F(\sum_{k=0}^{m-1}\sum_{r=0}^{k} c_k b_{k,r}^{(\nu)} t^{r+\nu} + \sum_{i=0}^{n-1} \delta_i \frac{t^i}{i!}) dt = f(x_p). \quad (4.3)$$

Then, we transfer the t−interval $[0, x_p]$ into τ −interval $[-1, 1]$ by change of variable $\tau = \frac{2}{x_p} t - 1$,

$$\sum_{k=0}^{m-1} c_k B_k(x_p) - \lambda \frac{x_p}{2} \int_{-1}^{1} k(x_p, \frac{x_p}{2}(\tau+1)) \quad (4.4)$$

$$F(\sum_{k=0}^{m-1}\sum_{r=0}^{k} c_k b_{k,r}^{(\nu)} (\frac{x_p}{2}(\tau+1))^{r+\nu} + \sum_{i=0}^{n-1} \delta_i \frac{(\frac{x_p}{2}(\tau+1))^i}{i!}) d\tau = f(x_p).$$

By using the Gauss −Legendre integration formula [34], for $p = 0, 1, ..., m-1$, we have:

$$\sum_{k=0}^{m-1} c_k B_k(x_p) - \lambda \frac{x_p}{2} \sum_{q=1}^{m} \omega_q k(x_p, \frac{x_p}{2}(\tau_q+1))$$

$$F(\sum_{k=0}^{m-1}\sum_{r=0}^{k} c_k b_{k,r}^{(\nu)} (\frac{x_p}{2}(\tau_q+1))^{r+\nu} + \sum_{i=0}^{n-1} \delta_i \frac{(\frac{x_p}{2}(\tau_q+1))^i}{i!}) = f(x_p), \quad (4.5)$$

where $\tau_q,\ \ q = 1, 2, ..., m$, are zeros of Legendre polynomial $P_m(x)$ and $\omega_q = \frac{-2}{(m+1)P'_m(\tau_q)P_{m+1}(\tau_q)},\ \ q = 1, 2, ..., m$. Eq. (4.5), gives a system of m nonlinear algebraic equations which can be solved, for the unknowns $c_k,\ \ k = 0, 1, ..., m-1$, using Newton's iterative method. Finally, $y(x)$ can be approximated by (2.11).

5. Error Analysis

In this section, to check the accuracy of the proposed method, some error analysis of the method will be presented for the fractional Volterra integro-differential equations.

Theorem 5.1. *Assume that $y \in L^2[0, 1]$ be an arbitrary function approximated by the truncated Bernoulli serie $\sum_{k=0}^{m-1} c_k B_k(x)$, then the coefficients $c_k, k = 0, 1, ..., m-1$, can be calculated from the following relation* [36]

$$c_k = \frac{1}{k!} \int_0^1 y^{(k)}(x) dx.$$

Also, we have

$$|c_k| \le \tfrac{Y_k}{k!},$$

where Y_k is the maximum of $|y^{(k)}|$ in the interval [0, 1].

Theorem 5.1, implies that Bernoulli coefficients decay rapidly with increasing of k.
Since, in real problems, we have to solve equations with unknown exact solutions, therefore, when we obtain an approximate solution we can not say that this solution is good or bad unless we are able to calculate the error function $E_m(x) = y_m(x) - y(x)$. We introduce a method similar to [13], let $y_m(x)$ be an approximate solution of (1.1).
We can write Eqs. (1.1) and (1.2) in the following form

$$\begin{aligned} &D^{\nu} y(x) = f(x) + \textstyle\int_0^x H(x,t,y(t))dt, \qquad 0 \leqslant x \leqslant 1, n-1 < \nu \leqslant n,\\ &y^{(i)}(0) = \delta_i, \qquad i = 0,1,...,n-1, n \in \mathbb{N}. \end{aligned} \tag{5.1}$$

Applying operator I^{ν} on both sides of (5.1), yields
$y(x) = \sum_{i=0}^{n-1} \delta_i \frac{x^i}{i!} + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt + \frac{1}{\Gamma(\nu+1)} \int_0^x (x-t)^{\nu} H(x,t,y(t))dt.$
Thus $y_m(x)$ satisfies the following equation

$$\begin{aligned} y_m(x) &= \sum_{i=0}^{n-1} \delta_i \frac{x^i}{i!} + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt \\ &+ \frac{1}{\Gamma(\nu+1)} \int_0^x (x-t)^{\nu} H(x,t,y_m(t))dt \\ &+ R_m(x). \end{aligned}$$

Where $R_m(x)$ is the residual function that depends only on $y_m(x)$ and can be obtained from the following relation

$$\begin{aligned} R_m(x) &= y_m(x) - \sum_{i=0}^{n-1} \delta_i \frac{x^i}{i!} - \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt \\ &- \frac{1}{\Gamma(\nu+1)} \int_0^x (x-t)^{\nu} H(x,t,y_m(t))dt. \end{aligned} \tag{5.2}$$

Now, we define the error function

$$E_m(x) = y_m(x) - y(x),$$

where

$$E_m(x) = \frac{1}{\Gamma(\nu+1)} \int_0^x (x-t)^{\nu} (H(x,t,y_m(t)) - H(x,t,y(t)))dt + R_m(x). \tag{5.3}$$

By using Taylor's Theorem [13], we can write

$$\begin{aligned} H(x,t,y_m(t)) - H(x,t,y(t)) &= H(x,t,y_m(t)) - H(x,t,y_m(t)-E_m(t)) \\ &\simeq \frac{\partial}{\partial y_m} H(x,t,y_m(t))E_m(t) \\ &- \frac{1}{2}\frac{\partial^2}{\partial y_m^2} H(x,t,y_m(t))E_m^2(t). \end{aligned}$$

Therefore, from the above equation and (5.3), we get

$$\begin{aligned} E_m(x) &= \frac{1}{\Gamma(\nu+1)}\int_0^x (x-t)^\nu (\frac{\partial}{\partial y_m} H(x,t,y_m(t))E_m(t) \\ &- \frac{1}{2}\frac{\partial^2}{\partial y_m^2} H(x,t,y_m(t))E_m^2(t))dt + R_m(x). \end{aligned}$$

Thus, we obtain a nonlinear Volterra integro-differential equation in which the error function $E_m(x)$ is unknown. Obviously, we can apply the proposed method for this equation to find an approximation of the error function $E_m(x)$.

6. Numerical Results

In this section, we present six numerical examples to illustrative our method and to demonstrate its efficiency. We have performed all numerical computations using a computer program written in Mathematica.

Example 6.1. Consider the following equation [32]

$$D^\nu y(x) - \int_0^x e^{-t}[y(t)]^2 dt = 1, \qquad 0 \leqslant x \leqslant 1, \;\; 3 < \nu \leqslant 4, \tag{6.1}$$

subject to the initial conditions $y(0) = y'(0) = y''(0) = y'''(0) = 1$. The exact solution of this problem, when $\nu = 4$, is $y(x) = e^x$.
First, we investigate the conditions of existence and uniqueness of solution for this problem. We have

$$f(x) = 1, \qquad H(x,t,y(t)) = e^{-t}y^2(t).$$

It's clear that

$$f \in C([0,1], R), \qquad H \in C([0,1]^2 \times R, R).$$

We have

$$
\begin{aligned}
\|H(x,t,y_1(t)) - H(x,t,y_2(t))\| &= \|e^{-t}y_1^2(t) - e^{-t}y_2^2(t)\| \\
&\leq (\|y_1(t)\| + \|y_2(t)\|)\|y_1(t) - y_2(t)\| \\
&\leq 5.80\|y_1(t) - y_2(t)\|,
\end{aligned}
$$

using $\|y_1(t)\| \leq 2.90$ and $\|y_2(t)\| \leq 2.90$. Therefore, $f(x)$ (constant function) and $H(x,t,y(t))$ satisfy the Lipschitz conditions (3.1).
Also, we have

$$\tfrac{\eta}{\Gamma(\nu+2)} \leq \tfrac{5.80}{\Gamma(3+2)} = \tfrac{5.80}{24} < \tfrac{1}{2}.$$

This means that Theorem 3.2, can be applied to this example with $\eta = 5.80$. Therefore, this problem has a unique solution. We have solved this equation with proposed method of section 4. Fig. 1 displays the absolute error between the exact and approximate solutions for $m = 4, 6, 8$. From these results we can conclude that our numerical results are in perfect agreement with the exact solution for $\nu = 4$. From the Comparison between the CAS wavelet method [32] and the presented method, we find that our method has higher degree of accuracy. Also, the numerical results for $\nu = 3.25, 3.50, 3.75, 4$ and the exact solution are given in Fig. 2. From this figure we see as ν approaches 4, the corresponding solutions of the fractional order integro-differential equation approaches to the exact solution of integer order integro-differential equation.

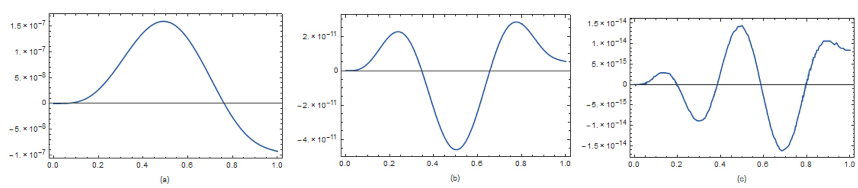
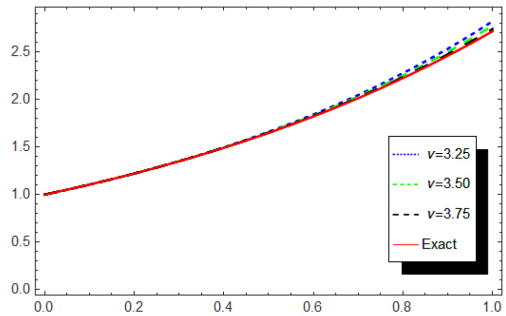


FIGURE 1. The absolute error between the exact and approximate solutions for $\nu = 4$: (a) $m = 4$, (b) $m = 6$, (c) $m = 8$ for Example 6.1.

EXAMPLE 6.2. Consider the following equation [37]

$$D^{\nu}y(x) - \int_0^x [y(t)]^2 dt = -1, \qquad 0 \leqslant x \leqslant 1, \ \ 0 < \nu \leqslant 1, \tag{6.2}$$

subject to the initial condition $y(0) = 0$.
Table 1 shows the numerical solutions for $\nu = 0.9, 1$ and $m = 8$ by using the present method and the second kind Chebyshev wavelet method [37] for



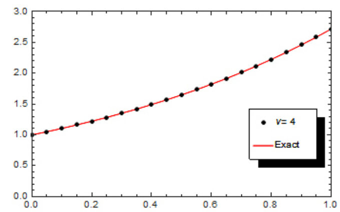


FIGURE 2. Comparison of $y(x)$ for $m = 3$, with $\nu = 3.25, 3.50, 3.75, 4$ and the exact solution for Example 6.1.

$k = 6, M = 2$. From Table 1, we can see that the numerical results obtained by our method are in high agreement with the exact solution for $\nu = 1$. Therefore, we state that the solutions for $\nu = 0.9$ are also credible. The numerical results for $\nu = 0.7, 0.8, 0.9, 1$ are displayed in Fig. 3. This figure shows that as $\nu \to 1$, the corresponding solutions of fractional order differential equation approach to the solutions of integer order differential equation.
We know that the exact solutions for the values $\nu \neq 1$ are not known. Therefore, to show efficiency of our method, we use estimated error of section 5. Let

$$H_m(x,t,y_m(t)) = y_m^2(t), \qquad \frac{\partial}{\partial y_m}H_m(x,t,y_m(t)) = 2y_m(t),$$
$$\frac{\partial^2}{\partial y_m^2}H_m(x,t,y_m(t)) = 2.$$

From Eqs. (5.2) and (5.3) we obtain

$$R_m(x) = y_m(x) + \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}dt - \frac{1}{\Gamma(\nu+1)}\int_0^x (x-t)^{\nu} y_m^2(t)dt, \quad (6.3)$$

and

$$E_m(x) = \frac{1}{\Gamma(\nu+1)}\int_0^x (x-t)^{\nu}((2y_m(t)E_m(t)) - (E_m^2(t)))dt + R_m(x). \quad (6.4)$$

Also, we approximate $E_m(x)$ as

$$E_m(x) \simeq \sum_{k=0}^{m-1} a_k B_k(x), \quad (6.5)$$

where the coefficients $a_k, k = 0, 1, \ldots, m-1$ are unknown.
By substituting Eq. (6.5) in Eq. (6.4), we get

$$\begin{aligned}\sum_{k=0}^{m-1} a_k B_k(x) &- \frac{1}{\Gamma(\nu+1)} \int_0^x (x-t)^\nu ((2y_m(t) \sum_{k=0}^{m-1} a_k B_k(t)) \\ &- (\sum_{k=0}^{m-1} a_k B_k(t))^2) dt - R_m(x) = 0. \end{aligned} \tag{6.6}$$

Now, we collocate (6.6) at the zeros $x_p, p = 0, 1, ..., m-1$ of shifted Legendre polynomial $L_m(x)$

$$\begin{aligned}\sum_{k=0}^{m-1} a_k B_k(x_p) &- \frac{1}{\Gamma(\nu+1)} \int_0^{x_p} (x_p-t)^\nu ((2y_m(t) \sum_{k=0}^{m-1} a_k B_k(t)) \\ &- (\sum_{k=0}^{m-1} a_k B_k(t))^2) dt - R_m(x_p) = 0.\end{aligned}$$

Then, we transfer the t$-$interval $[0, x_p]$ into τ $-$interval $[-1, 1]$ by change of variable $\tau = \frac{2}{x_p} t - 1$

$$\begin{aligned}&\sum_{k=0}^{m-1} a_k B_k(x_p) - \frac{x_p}{2\Gamma(\nu+1)} \int_{-1}^1 ((x_p - \frac{x_p}{2}(\tau+1))^\nu ((2y_m(\frac{x_p}{2}(\tau+1)) \\ &\sum_{k=0}^{m-1} a_k B_k(\frac{x_p}{2}(\tau+1))) - (\sum_{k=0}^{m-1} a_k B_k(\frac{x_p}{2}(\tau+1)))^2)) - R_m(x_p) = 0.\end{aligned}$$

By using the Gauss $-$Legendre integration formula [34], for $p = 0, 1, ..., m-1$, we have:

$$\begin{aligned}&\sum_{k=0}^{m-1} a_k B_k(x_p) - \frac{x_p}{2\Gamma(\nu+1)} \sum_{q=1}^{m} \omega_q ((x_p - \frac{x_p}{2}(\tau_q+1))^\nu ((2y_m(\frac{x_p}{2}(\tau_q+1)) \\ &\quad \sum_{k=0}^{m-1} a_k B_k(\frac{x_p}{2}(\tau_q+1))) - (\sum_{k=0}^{m-1} a_k B_k(\frac{x_p}{2}(\tau_q+1)))^2)) - R_m(x_p) = 0,\end{aligned}$$

where $\tau_q,\ \ q = 1, 2, ..., m$ are zeros of Legendre polynomial $P_m(x)$ and $\omega_q = \frac{-2}{(m+1)P'_m(\tau_q)P_{m+1}(\tau_q)},\ \ q = 1, 2, ..., m$. This relation gives a system of m nonlinear algebraic equations which can be solved for the unknowns $a_k,\ \ k = 0, 1, ..., m-1$ using Newton's iterative method. Finally, $E_m(x)$ given in (6.5) can be calculated. Table 2 displays the estimated error for various values of ν with $m = 10$.

EXAMPLE 6.3. Consider the following equation [37, 32]

Table 1. Comparison of the numerical solutions with the Ref. [37] for $m = 8$ and various values of ν for Example 6.2.

x	Exact solution	Present method		Ref. [37]	
	$\nu = 1$	$\nu = 1$	$\nu = 0.9$	$\nu = 1$	$\nu = 0.9$
0	0	0	0	0	− 0.00017
0.0625	−0.06250	−0.06250	− 0.08576	−0.06250	− 0.08588
0.1250	−0.12498	−0.12498	− 0.15997	−0.12498	− 0.16003
0.1875	−0.18740	−0.18740	− 0.23025	−0.18740	− 0.23029
0.2500	−0.24968	−0.24968	− 0.29791	−0.24968	− 0.29794
0.3125	−0.31171	−0.31171	− 0.36344	−0.31171	− 0.36345
0.3750	−0.37336	−0.37336	− 0.42702	−0.37336	− 0.42702
0.4375	−0.43446	−0.43446	− 0.48866	−0.43447	− 0.48865
0.5000	−0.49482	−0.49482	− 0.54829	−0.49483	− 0.54828
0.5625	−0.55423	−0.55423	− 0.60576	−0.55424	− 0.60575
0.6250	−0.61243	−0.61243	− 0.66089	−0.61245	− 0.66090
0.6875	−0.66917	−0.66917	− 0.71347	−0.66918	− 0.71349
0.7500	−0.72415	−0.72415	− 0.76327	−0.72418	− 0.76332
0.8125	−0.77710	−0.77709	− 0.81007	−0.77712	− 0.81012
0.8750	−0.82767	−0.82767	− 0.85360	−0.82770	− 0.85365
0.9375	−0.87557	−0.87557	− 0.89363	−0.87561	− 0.89366

$$D^{\nu}y(x)+\int_0^x [y(t)]^2dt = \sinh(x)+\frac{1}{2}\cosh(x)\sinh(x)-\frac{x}{2}, \quad 0 \leqslant x \leqslant 1, \quad 1 < \nu \leqslant 2, \tag{6.7}$$

subject to the initial conditions $y(0) = 0, y'(0) = 1$. The exact solution of this problem, when $\nu = 2$ is $y(x) = \sinh(x)$.

Fig. 4 shows the absolute error between the exact and approximate solutions for various values of m. From the comparison between our results and results in Refs. [37] and [32], we find that Bernoulli pseudo-spectral method can reach higher degree of accuracy when solving this problem. Also, the numerical results for $\nu = 1.25, 1.50, 1.75, 2$ and the exact solution with $m = 4$ are presented

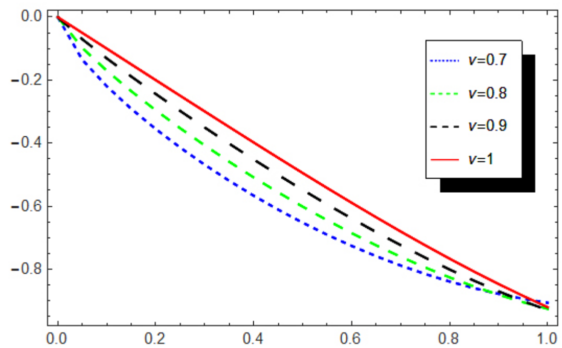


FIGURE 3. Comparison of $y(x)$ for $m = 6$, with $\nu = 0.7, 0.8, 0.9, 1$ for Example 6.2.

TABLE 2. The estimated errors for $m = 10$ and various values of ν for Example 6.2.

x	$\nu = 0.6$	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$
0	1.14×10^{-6}	7.14×10^{-7}	2.76×10^{-7}	6.61×10^{-8}
0.1	1.15×10^{-8}	2.85×10^{-9}	5.29×10^{-9}	2.36×10^{-9}
0.2	7.75×10^{-8}	5.25×10^{-8}	2.08×10^{-8}	4.94×10^{-9}
0.3	1.26×10^{-7}	7.85×10^{-8}	3.16×10^{-8}	8.06×10^{-9}
0.4	1.10×10^{-8}	1.74×10^{-8}	1.43×10^{-8}	4.98×10^{-9}
0.5	8.31×10^{-8}	2.29×10^{-8}	7.06×10^{-10}	1.28×10^{-9}
0.6	2.88×10^{-7}	1.51×10^{-7}	5.12×10^{-8}	1.12×10^{-8}
0.7	2.96×10^{-7}	1.36×10^{-7}	3.68×10^{-8}	5.70×10^{-9}
0.8	4.32×10^{-7}	2.13×10^{-7}	6.36×10^{-8}	1.15×10^{-8}
0.9	6.89×10^{-7}	3.77×10^{-7}	1.29×10^{-7}	2.74×10^{-8}
1	1.11×10^{-6}	6.52×10^{-7}	2.39×10^{-7}	5.50×10^{-8}

in Fig. 5. This figure shows that as $\nu \to 2$, the approximate solutions tend to the exact solution.

EXAMPLE 6.4. Consider the following equation [32]

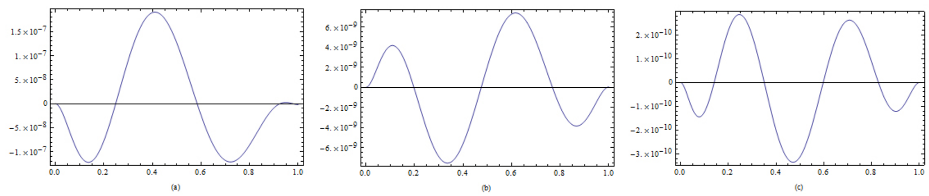


FIGURE 4. The absolute errors between the exact and approximate solutions for $\nu = 2:$ (a) $m = 5$, (b) $m = 6$, (c) $m = 7$ for Example 6.3.

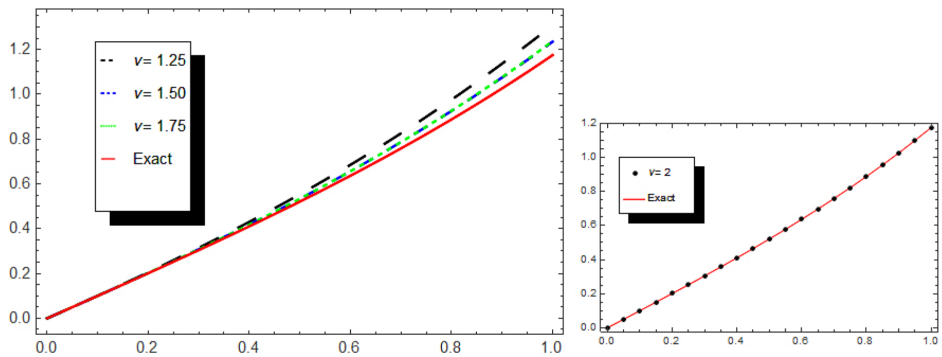


FIGURE 5. Comparison of $y(x)$ for $m = 4$, with $\nu = 1.25, 1.50, 1.75, 2$ and the exact solution for Example 6.3.

$$D^{\frac{6}{5}}y(x) - \int_0^x (x-t)^2[y(t)]^3 dt = \frac{5}{2\Gamma(\frac{4}{5})}x^{\frac{4}{5}} - \frac{1}{252}x^9, \qquad 0 \leqslant x \leqslant 1, \quad (6.8)$$

subject to the initial conditions $y(0) = y'(0) = 0$. The exact solution of this problem is $y(x) = x^2$.
The absolute errors for some values of m are shown in Fig. 6. Also, the comparisons between the exact and approximate solutions for various choices of m are given in Fig. 7. From Figs. 6 and 7, we can see that the obtained results using our method are in good agreement with the exact solution and with the approximate solutions for $\nu = \frac{6}{5}$ in [32].

EXAMPLE 6.5. Consider the following equation [37]

$$D^{\nu}y(x) - \int_0^x [y(t)]^3 dt = e^x - \frac{1}{3}e^{3x} + \frac{1}{3}, \qquad 0 \leqslant x \leqslant 1, \quad 0 < \nu \leqslant 1, \quad (6.9)$$

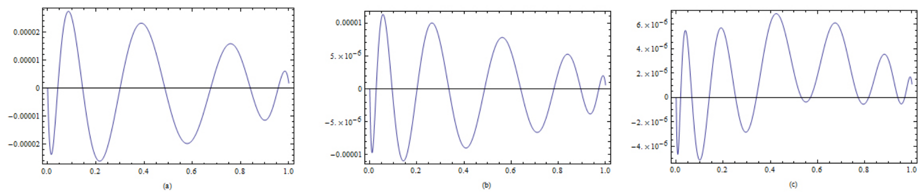


FIGURE 6. The absolute errors between the exact and approximate solutions for $\nu = \frac{6}{5}$ with : (a) $m = 8$, (b) $m = 10$, (c) $m = 12$ for Example 6.4.

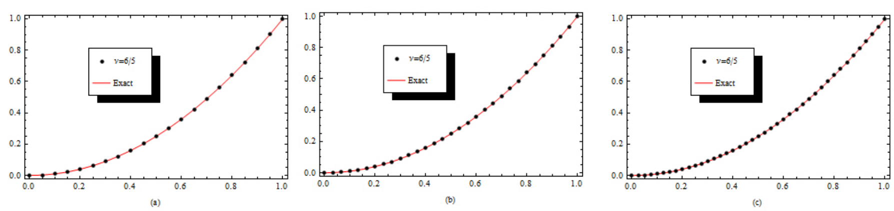


FIGURE 7. The comparison between the exact and approximate solutions for : (a) $m = 4$, (b) $m = 6$, (c) $m = 8$ for Example 6.4.

subject to the initial condition $y(0) = 1$. The exact solution of this problem, when $\nu = 1$, is $y(x) = e^x$.
Table 3 denotes the approximate solutions obtained for different values of x by using the present method for $m = 4, 6, 8$ and the second kind Chebyshev wavelet method [37] for $k = 5, M = 2$ with $\nu = 1$, together with the exact solution. From Table 3, we can conclude that our numerical solutions are in a good agreement with the exact solution when $\nu = 1$. Also, the numerical results for $y(x)$ with $m = 12$ and $\nu = 0.7, 0.8, 0.9, 1$ and the exact solution are plotted in Fig. 8. From the graphical results, it is clear that the approximate solutions converge to the exact solution.

EXAMPLE 6.6. Consider the following equation [15]

$$D^\nu y(x)+\int_0^x x^3 \cos t e^{y(t)} dt = x^3(-1+e^{\sin x})-\sin x, \qquad 0 \leqslant x \leqslant 1, \quad 1 < \nu \leq 2, \tag{6.10}$$

subject to the initial conditions $y(0) = 0, y'(0) = 1$. The exact solution of this problem, when $\nu = 2$ is $y(x) = \sin x$.
Table 4 shows the numerical solutions for $\nu = 1.75, 2$ by using the present

TABLE 3. Comparison of the numerical solutions with the Ref. [37] for $\nu = 1$ for Example 6.5.

x	Exact solution	Present method			Ref. [37]
	$\nu = 1$	$m = 4$	$m = 6$	$m = 8$	$k = 5, M = 2$
0	1	1	1	1	1.000122
0.1	1.105171	1.104782	1.105174	1.105171	1.105345
0.2	1.221403	1.220677	1.221408	1.221403	1.221645
0.3	1.349859	1.348907	1.349864	1.349859	1.350196
0.4	1.491825	1.490795	1.491828	1.491825	1.492295
0.5	1.648721	1.647767	1.648722	1.648721	1.649382
0.6	1.822119	1.821352	1.822117	1.822119	1.823061
0.7	2.013753	2.013182	2.013751	2.013753	2.015118
0.8	2.225541	2.224992	2.225543	2.225541	2.227565
0.9	2.459603	2.458620	2.459611	2.459603	2.462682

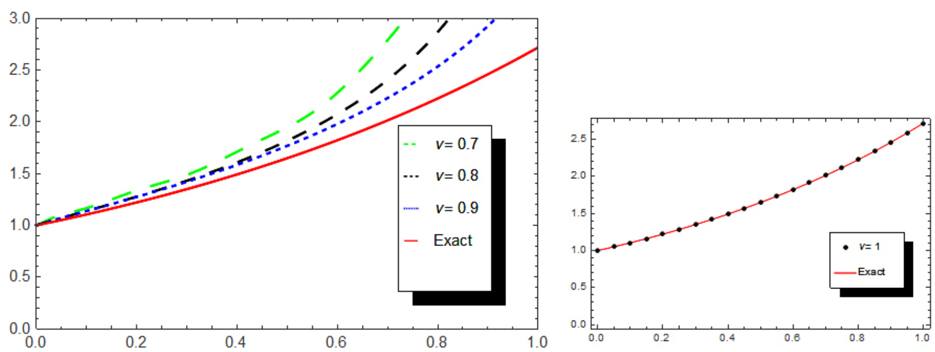


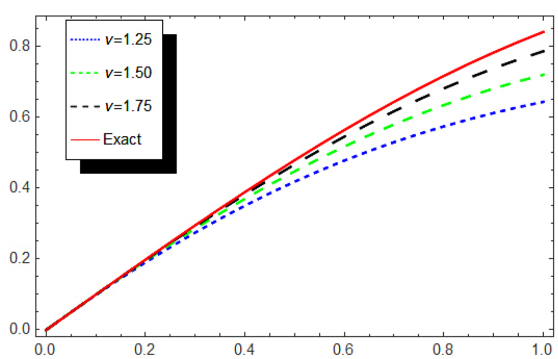
FIGURE 8. Comparison of $y(x)$ for $m = 12$, with $\nu = 0.7, 0.8, 0.9, 1$ and the exact solution for Example 6.5.

method, when $m = 7$, the Tau method [15] and the exact solution. From Table 4, we can see that the numerical results obtained by our method are in high agreement with the exact solution for $\nu = 2$. Therefore, we state that the solutions for $\nu = 1.75$ are also credible. Also, the numerical results for $m = 3$ with $\nu = 1.25, 1.50, 1.75, 2$ and the exact solution are displayed in Fig.

9. This figure shows that as $\nu \to 2$, the approximate solutions tend to the exact solution.

TABLE 4. Comparison of our numerical solutions with the Ref. [15] for various values of ν for Example 6.6.

x	Exact solution	Present method		Ref. [15]	
	$\nu = 2$	$\nu = 1.75$	$\nu = 2$	$\nu = 1.75$	$\nu = 2$
0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.099833	0.099598	0.099833	0.099419	0.099833
0.2	0.198669	0.197301	0.198669	0.196812	0.198669
0.3	0.295520	0.291793	0.295520	0.290975	0.295520
0.4	0.389418	0.381968	0.389418	0.380815	0.347526
0.5	0.479425	.466862	0.479425	0.465333	0.479427
0.6	0.564642	0.545634	0.564642	0.543605	0.564648
0.7	0.644218	0.617561	0.644218	0.614770	0.644233
0.8	0.717356	0.682046	0.717356	0.678009	0.717397
0.9	0.783327	0.738628	0.783327	0.732528	0.783420
1	0.841471	0.787000	0.841471	0.777544	0.841666



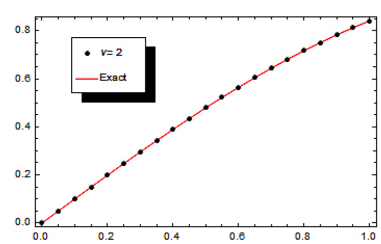


FIGURE 9. Comparison of $y(x)$ for $m = 3$, with $\nu = 1.25, 1.50, 1.75, 2$ and the exact solution for Example 6.6.

7. Conclusion

In this paper, the Bernoulli spectral method is implemented for solving nonlinear fractional Volterra integro-differential equations. The properties of Bernoulli polynomials together with the Gaussian integration method are utilized to reduce the proposed problem to the solution a system of algebraic equations which is solved by using Newton's iteration method. Special attention is given to the study of existence and uniqueness of solution for problem (1.1) and the error analysis for presented method. From the obtained numerical results we can see that the obtained solutions using the suggested scheme are in excellent agreement with the exact solution when ν is integer and with more accuracy compared with CAS wavelet method, second kind Chebyshev wavelet method and Tau method. Moreover, only a small number of Bernoulli polynomials are needed to obtain a satisfactory result.

Acknowledgments

Authors are very grateful to one of the reviewers for carefully reading the paper and for his(her) comments and suggestions which have improved the paper.

References

1. A. Azizia, J. Saeidianb, S. Abdi, Applying Legendre Wavelet Method with Regularization for a Class of Singular Boundary Value Problems, *Iranian Journal of Mathematical Sciences and Informatics*, **11**(2), (2016), 57−69.
2. R. L. Bagley, P. J. Torvik, Fractional Calculus in the Transient Analysis of Viscoelastically Damped Structures, *AIAA. J.*, **23**(6), (1985), 918−925.
3. R. T. Baillie, Long Memory Processes and Fractional Integration in Econometrics, *J. Econometrics*, **73**(1), (1996), 5−59.
4. R. P. Boas, R. C. Buck, *Polynomial Expansions of Analytic Functions*, New York, Springer-Verlag, 1964.
5. T. S. Chow, Fractional Dynamics of Interfaces Between Soft-Nanoparticles and Rough Substrates, *Phys. Lett. A.*, **342**(1−2), (2005), 148−155.
6. E. H. Doha, A. H. Bhrawy, S. S. Ezz-Eldien, Efficient Chebyshev Spectral Methods for Solving Multi-Term Fractional Orders Differential Equations, *Appl. Math. Model.*, **35**(12), (2011), 5662−5672.
7. B. Doyon, J. Lepowsky J, A. Milas, Twisted Vertex Operators and Bernoulli Polynomials, *Commun. Contemp. Math.*, **8**(2), (2006), 247−307.
8. D. B. Fairlie, A. P. Veselov, Faulhaber and Bernoulli Polynomials and Solitons, *Physica D: Nonlinear Phenomena*, **152−153**, (2001), 47−50.
9. M. P. Grosset, A. P. Veselov, Elliptic Faulhaber Polynomials and Lame Densities of States, *Int. Math. Res. Not.*, **2016**(5), (2006), 1−21.
10. I. Hashim, O. Abdulaziz, S. Momani, Homotopy Analysis Method for Fractional IVPs, *Commun. Nonl. Sci. Numer. Simul.*, **14**(3), (2009) 674−684.
11. J. H. He, Nonlinear Oscillation with Fractional Derivative and its Applications, *International Conference on Vibrating Engineering* , (1998), 288−291.

12. J. H. He, Some Applications of Nonlinear Fractional Differential Equations and Their Approximations, *Bull. Sci. Technol.*, **2**, (1999), 86−90.
13. M. H. Heydari, M. R. Hooshmandasl, F. Mohammadi, C. Cattani, Wavelets Method for Solving Systems of Nonlinear Singular Fractional Volterra Integro-Diffrential Equations, *Communications in Nonlinear Science and Numerical Simulation*, **19**(1), (2014), 37−48.
14. J. K. Hunter, B. Nachtergaele, *Applied Analysis*, University of california, 2000.
15. S. Karimi Vanani, A. Amintaei, Operational Tau Approximation for a General Class of Fractional Integro-Differential Equations, *Comput. Appl. Math.*, **30**(3), (2011), 655−674.
16. E. Keshavarz, Y. Ordokhani, M. Razzaghi, Bernoulli Wavelet Operational Matrix of Fractional Order Integration and its Applications in Solving the Fractional Order Differential Equations, *Appl. Math. Model.*, **38**(24), (2014), 6038−6051.
17. N. Koblitz, *p-Adic Numbers, p-adic Analysis and Zeta-Functions*, Second ed, New York, Springer-Verlag, 1984.
18. E. Kreyszig, *Introductory Functional Analysis with Applications*, New York, John Wiley and Sons Press, 1978.
19. S. Lang, *Introduction to Modular Forms*, Springer-Verlag, 1976.
20. R. L. Magin, *Fractional Calculus in Bioengineering*, Crit. Rev. Biomed. Eng., 2004.
21. F. Mainardi, *Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics. Fractals and Fractional Calculus in Continuum Mechanics*, New York, Springer Verlag, 1997.
22. B. Mandelbrot, Some Noises with $1/f$ Spectrum, a Bridge between direct Current and White Noise, *IEEE Trans Inform Theory*, **2**, (1967), 289−98.
23. S. Mashayekhi, Y. Ordokhani, M. Razzaghi, Hybrid Functions Approach for Optimal Control described by Integro−Differential Equations, *Appl. Math. Model.*, **37**(5), (2013), 3355− 3368.
24. M. Meerschaert, C. Tadjeran, Finite Difference Approximations for Two-Sided Space-Fractional Partial Differential Equations, *Appl. Numer. Math.*, **56**(1), (2006), 80−90.
25. R. Metzler, J. Klafter, The Restaurant at the End of the Random Walk: Recent Developments in the Description of Anomalous Transport by Fractional Dynamics, *Journal of Physics A: Mathematical and General*, **37**, (2004), 161−208.
26. K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, New York, Wiley, 1993.
27. S. Momani, K. Al-Khaled, Numerical Solutions for Systems of Fractional Differential Equations by the Decomposition Method, *Appl. Math. Comput.*, **162**(3), (2005), 1351−1365.
28. P. Rahimkhani, Y. Ordokhani, E. Babolian, A New Operational Matrix based on Bernoulli Wavelets for Solving Fractional Delay Differential Equations, *Numer. Algor.*, **74**, (2017), 223−245.
29. P. Rahimkhani, Y. Ordokhani, E. Babolian, Fractional-order Bernoulli Wavelets and Their Applications, *Appl. Math. Model.*, **40**, (2016), 8087−8107.
30. Y. A. Rossikhin, M. V. Shitikova, Applications of Fractional Calculus to Dynamic Problems of Linear and Nonlinear Hereditary Mechanics of Solids, *Appl. Mech. Rev.*, **50**(1), (1997), 15−67.
31. A. Saadatmandi, M. Dehghan, A Legendre Collocation Method for Fractional Integro-Differential Equations, *Vibration and Control*, **17**(13), (2011), 2050−2058.
32. H. Saeedi, M. Mohseni Moghadam, Numerical Solution of Nonlinear Volterra Integro-Differential Equations of Arbitrary Order by CAS Wavelets, *Commun. Nonl. Sci. Numer. Simul.*, **16**(3), (2011), 1216−1226.

33. S. Sedaghat, Y. Ordokhani, Stability and Numerical Solution of Time Variant Linear Systems with Delay in Both the State and Control, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(1), (2012), 43−57.
34. J. Stoer, R. Bulirsch, *Introduction to Numerical Analysis*, Second Edition, Springer-Verlag, 2002.
35. N. H. Sweilam, M. M. Khader, R. F. Al-Bar, Numerical Studies for a Multi-Order Fractional Differential Equation, *Phys. Lett. A.*, **371**(1−2), (2007), 26−33.
36. E. Tohidi, A. H. Bhrawy, K. Erfani, A Collocation Method Based on Bernoulli Operational Matrix for Numerical Solution of Generalized Pantograph Equation, *Appl. Math. Model.*, **37**(6), (2013), 4283−4294.
37. L. Zhu, Q. Fan, Numerical Solution of Nonlinear Fractional-Order Volterra Integro-Differential Equations by SCW, *Commun. Nonl. Sci. Numer. Simul.*, **18**(5), (2013), 1203−1213.