

A study of Nilpotent groups through right transversals

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ABSTRACT. In group theory nilpotency of a group has a great importance. In this paper we have studied some concept of nilpotency through right transversals. We have also studied prime power groups and frattini subgroups through right transversals.

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1. INTRODUCTION:

An extension of a group H is a group G , which contains H as a normal subgroup. We shall call a group G a general extension of a subgroup H if G contains H as a subgroup (not necessarily as a normal subgroup). A pair (G, H) , where H is a subgroup of a group G , we will be called a **general extension**. We have a category E , whose objects are general extensions and a morphism from (G_1, H_1) to (G_2, H_2) is a group homomorphism ϕ from G_1 to G_2 such that $\phi(H_1) \subseteq H_2$. The category E will be called the **category of general extensions**

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Definition 1.1: A groupoid (S, \circ) (a nonempty set S with a binary operation \circ) is called a **right quasigroup** if $\forall x, y \in S$, the equation $X \circ x = y$ where X is unknown in the equation, has a unique solution. If $\exists e \in S$ such that $e \circ x = x \circ e = x \forall x \in S$, then (S, \circ) is called a right quasigroup with identity.

Throughout the Paper, a right quasigroup will always be assumed to be with identity.

Let G be a group and H a subgroup of G . A **right transversal** (right coset representative system) to H in G is a subset S of G obtained by selecting one and only one member from each right coset of $G \text{ mod } H$.

Throughout the Paper, a right transversal will be assumed to contain the identity of the group.

Let S be a right transversal to H in G . Let $x, y \in S$. Define $\{xoy\} = S \cap Hxy$. Then z , where $\{z\} = S \cap Hyx^{-1}$ is the unique solution of the equation $X \circ x = y$, where X is unknown in the equation. Therefore (S, \circ) is a right quasigroup with identity. This right quasigroup structure on S is said to be the right quasigroup structure induced by the group G on the right transversal S .

2. C-GROUPOID AND RIGHT QUASI GROUP

Definition 2.1: A quadruple (S, H, σ, f) , where S is a groupoid with identity e , H a group which acts on S from right through a given action θ , σ a map from S to H^H (the set of all maps from H to H) and f a map from $S \times S$ to H , is called a **c-groupoid** if it satisfies the following conditions:

$$C_1) xoy = y \Rightarrow x = e$$

$$C_2) \forall x \in S \exists x' \in S \text{ such that } x' \circ x = e$$

$$C_3) \forall x \in S, \text{ let } \sigma_x \text{ denote the image } \sigma(x) \text{ of } x \text{ under the map } \sigma. \text{ Then } \sigma_e = I_H, \text{ the identity map on } H.$$

$$C_4) f(x, e) = f(e, x) = 1, \text{ the identity of } H$$

$$C_5) \sigma_x(h_1 h_2) = \sigma_x(h_1) \sigma_{x\theta h_1}(h_2)$$

$$C_6) (xoy)oz = x\theta f(y, z) \circ (yoz)$$

$$C_7) (xoy)\theta h = x\theta \sigma_y(h) \circ y\theta h \quad C_7) (xoy)\theta h = x\theta \sigma_y(h) \circ y\theta h$$

$$C_8) f(x, y)f(xoy, z) = \sigma_x(f(y, z))f(x\theta f(y, z), yoz)$$

$$C_9) f(x, y)\sigma_{xoy}(h) = \sigma_x(\sigma_y(y))f(x\theta \sigma_y(h), y\theta h)$$

where $x, y \in S$ and $h_1, h_2 \in H$

Remark 2.2 From the above definition it follows that if (S, H, σ, f) is a c-groupoid, then

$$(i) \sigma_x(1) = 1 \text{ for all } x \in S.$$

$$(ii) e\theta h = e \text{ for all } h \in H.$$

Example 2.3: Let S be a right transversal to a subgroup H of a group G . Let $x, y \in S$ and $h \in H$. Then $x \circ y = f(x, y) x \circ y$ for some $f(x, y) \in H$ and $x \circ y \in S$. Also $x \circ h = \sigma_x(h) x \circ h$ for some $\sigma_x(h) \in H$ and $x \circ h \in S$. This gives us a map $f: S \times S \rightarrow H$ and a map $\sigma: S \rightarrow H^H$ defined by $f((x, y)) = f(x, y)$

and $\sigma(x)(h) = \sigma_x(h)$. Then (S, H, σ, f) is a c-groupoid. Conversely we have the following result.

Theorem 2.4 Given a c-groupoid (S, H, σ, f) there is a group G which contains H as a subgroup and S as a right transversal of H in G such that the corresponding c-groupoid is (S, H, σ, f) . [Ref. 8]

The pair (G, H) where G is the group described in the above Theorem, will be termed as the general extension associated to the c-groupoid (S, H, σ, f) .

3. RIGHT QUASIGROUPS AND GENERAL EXTENSIONS

Let S be a set and $\text{Sym}(S)$, the symmetric group on S (for the product in $\text{Sym}(S)$ we adopt the convention $(r \cdot s)(x) = s(r(x))$; $r, s \in \text{Sym}(S)$ and $x \in S$).

Now, let (S, \circ) be a right quasigroup with identity e . Let $y, z \in S$. Define a map $f^s(y, z)$ from S to S as follows: for $x \in S$, $f^s(y, z)(x)$ is the unique solution of the equation $X \circ (y \circ z) = (x \circ y) \circ z$. It can be checked that $f^s(y, z) \in \text{Sym}(S)$.

Definition 3.1: The subgroup G_s of $\text{Sym}(S)$ generated by the subset $\{f^s(y, z) : y, z \in S\}$ of $\text{Sym}(S)$ will be called the **group torsion** (or **associator**) of S .

Remark 3.2 Since $(e \circ y) \circ z = e \circ (y \circ z)$, for all $y, z \in S$, G_s is a subgroup of $\text{Sym}(S \setminus \{e\})$ also. Clearly, we have

Proposition 3.3: The group torsion G_s of a right quasigroup S is trivial if and only if S is a group.

Proposition 3.4 A subgroup H of a group G is normal if and only if group torsion of every transversal to H in G is trivial.

Proposition 3.5 Let S be a right quasigroup with identity e . Then, there exists a group G^S containing S as a right transversal such that if there is any group G containing S as a right transversal, then there is a unique homomorphism from G to G^S which is identity on S . **Proof:** By Theorems 2.5 and 2.10 it is sufficient to construct a c-groupoid (S, H^S, σ^S, f^S) , where S is the given right quasigroup such that given any c-groupoid (S, H, σ, f) , where S is again the given right quasigroup, there exists a unique c-homomorphism (p, q, g) from (S, H, σ, f) to (S, H^S, σ^S, f^S) such that $p = I_s$, the identity map on S . Take $H^S = \text{Sym}(S - \{e\})$, the symmetric group on $S - \{e\}$. We extend the natural action of H^S on $S - \{e\}$ to the action θ^S of H^S on S defining $e \theta^S h = e, \forall h \in H^S$. Already, we have a map $f^S : S \times S \rightarrow H^S$ defined by $f^S((x, y)) = f^S(x, y)$. Next we define a map $\sigma_y^S : S \rightarrow S$ by the equation

$$\sigma_y^S(h)(x) \circ (y \theta^S h) = (x \circ y) \theta^S h$$

It can be seen that $\sigma_y^S(h) \in \text{Sym}(S - \{e\}) = H^S$. Finally, it can be checked that (S, H^S, σ^S, f^S) is a c-groupoid, which satisfies the required property. #

The group G^S described in the theorem 3.5 will be termed the **general extension** of $\text{Sym}(S - \{e\})$ by the right quasigroup S .

4. NILPOTENT GROUP

A group G is said to be **nilpotent** if $\text{Ln}(G) = \{e\}$ or equivalently $\text{Zn}(G) = G$ for some n . A group G is said to be **nilpotent of class n** if $\text{Ln}(G) = \{e\}$ but $\text{Ln}-1(G) \neq \{e\}$ or equivalently $\text{Zn}(G) = G$ but $\text{Zn}-1(G) \neq G$.

Proposition 4.1: Let G be a nilpotent group and H a subgroup of G then for every transversal S to H in G there exists $x \in S - \{e\}$ such that $x\theta h = x \forall h \in H$. In particular $x\theta f(y, z) = x \forall y, z \in S$ and $G_S \subseteq \text{Sym}(S - \{e, x\})$ and $G_S \subseteq N_{G_S S}(G_S)$.

Proof: Let G be a nilpotent group, take H be any subgroup of G then $H \subset N_G(H)$

$\exists g \in N_G(H) - H$ such that $gHg^{-1} = H$ i.e.; $\exists g \in G - H$ such that $ghg^{-1} \in H$ i.e.

$ghg^{-1} = h'$ for all $h \in H$ and h' is an element of H .

Since $g \in G = HS \Rightarrow g = h_1x$ where $h_1 \in H, x \in S - \{e\}$

Since $g \notin H$ so that $(h_1x)h(h_1x)^{-1} = h'$

$$h_1xhx^{-1}h_1^{-1} = h' \text{ or } xhx^{-1} = h'' \text{ for some } h'' \in H$$

$$\Rightarrow xh = h''x$$

$$\Rightarrow x\theta h = x \forall h \in H$$

i.e. $\exists x \in S - \{e\}$ such that $x\theta h = x$

In particular $x\theta f(y, z) = x \forall y, z \in S$

Since $f(S \times S) \subseteq H$

If q be the permutation representation of $f(S \times S)$ on S defined by $q(f(y, z)) = f^S(y, z)$ then \exists a homomorphism $\psi : f(S \times S) \rightarrow G_S S$ defined by

$\psi(hx) = q(h)x$. Then we see that

$$x\theta^S f^S(y, z) = x \forall y, z \in S$$

$$\text{i.e. } f^S(y, z)(x) = x \forall y, z \in S$$

so that $f^S(y, z) \in \text{Sym}(S - \{e, x\}) \forall y, z \in S$

$$\Rightarrow G_S \subseteq \text{Sym}(S - \{e, x\})$$

Now

$$\begin{aligned} & x f^S(y, z) x^{-1} \\ &= \sigma_x^S(f^S(y, z)) x \theta^S f^S(y, z) x^{-1} \\ &= \sigma_x^S(f^S(y, z)) x (f^S(x', x))^{-1} x' \\ &= \sigma_x^S(f^S(y, z)) \sigma_x^S(f^S(x', x))^{-1} f^S(x \theta^S (f^S(x', x^{-1}))^{-1}, x') x \theta^S (f^S(x', x))^{-1} o x' \\ &= \sigma_x^S(f^S(y, z)) \sigma_x^S((f^S(x', x))^{-1} f^S(x, x') x o x') \\ &= \sigma_x^S(f^S(y, z)) \sigma_x^S((f^S(x', x))^{-1}) f^S(x', x) \in G_S \end{aligned}$$

(R.Lal, Journal of algebra-1996, Transversals in group-3.6)

$$\therefore G_S \subset N_{G_S S} (G_S).$$

Proposition 4.2: Let G be a group of order p^{m+1} , where p is a prime.

(i) If G has a nilpotent class > 1 , then all transversals of $Z_{c-1}(G)$ in G are isomorphic with trivial group torsion and their order is at least p^2 .

(ii) For every divisor of $|G|$, G has a proper transversal of that order.

Proof: (i) To prove this it is sufficient to prove that $G/Z_{c-1}(G)$ is not cyclic (i.e. not prime cyclic). Suppose $|G/Z_{c-1}(G)| = p$

$$\text{Then } Z(G/Z_{c-1}(G)) = G/Z_{c-1}(G) \quad \text{--- (1)}$$

$$\text{Consider } G/Z_{c-2}(G) / (Z_{c-1}(G)/Z_{c-2}(G)) \approx G/Z_{c-1}(G) \quad \text{--- (2)}$$

$$\text{But } Z_{c-1}(G)/Z_{c-2}(G) = Z(G/Z_{c-2}(G)) \quad \text{--- (3)}$$

$\Rightarrow G/Z_{c-2}(G) / Z(G/Z_{c-2}(G))$ is cyclic $\Rightarrow G/Z_{c-2}(G)$ is abelian such that $Z_{c-1}(G)/Z_{c-2}(G) = Z(G/Z_{c-2}(G)) = G/Z_{c-2}(G) \Rightarrow Z_{c-1}(G) = G$, a contradiction

$$\therefore |G/Z_{c-1}(G)| \geq p^2.$$

(ii) since $Z_{c-1} \trianglelefteq G$. Hence the result. (ii) Since for every divisor of $|G|$, G has a subgroup of that order. Hence if $|G| = dd_1$ Since d_1 divides $|G|$, G has a subgroup H of order d_1 so that $[G : H] = d \Rightarrow G$ has a proper transversal of order d .

Definition 4.3: Let G be a group. If G has no maximal subgroups, then define the **Frattini subgroup** of G denoted by $\Phi(G)$ to be G itself. If G has maximal subgroups then the Frattini subgroup $\Phi(G)$ of G is defined to be the intersection of all maximal subgroups of G .

(Wielandt)4.4: Let G be a finite group. Then G is nilpotent $\Leftrightarrow G' \subseteq \text{frat}(G)$.

Proposition 4.5: Let G be a finite group. Then $G' \subseteq \text{frat}(G)$ if and only if each right transversal of $\text{frat}(G)$ is contained in some right transversal of G' , where G' is the commutator subgroup of G .

Proof: $G' \subseteq \text{frat}(G)$, thus

$$\begin{aligned} G'g_1 &= G'g_2 \\ \Rightarrow g_1g_2^{-1} &\in G' \\ \Rightarrow g_1g_2^{-1} &\in \text{frat}(G) \\ \Rightarrow \text{frat}(G)g_1 &= \text{frat}(G)g_2 \end{aligned}$$

This shows that $\text{frat}(G)g_1 \neq \text{frat}(G)g_2 \Rightarrow G'g_1 \neq G'g_2$. Let S be any right transversal of $\text{frat}(G)$. Let $1 \neq g \in S$, then $\Rightarrow \text{frat}(G)g \neq \text{frat}(G) \Rightarrow G'g \neq G' \Rightarrow g \in S_1$ for some right transversal of G' . Since $g_1 \neq g_2 \in S \Rightarrow \text{frat}(G)g_1 \neq \text{frat}(G)g_2$

$$\Rightarrow G'g_1 \neq G'g_2$$

\Rightarrow we have a right transversal S_1 of G' containing g_1 and g_2 . This shows that every right transversal of $\text{frat}(G)$ is contained in some right transversal of G' . Conversely, let every right transversal of $\text{frat}(G)$ is contained in some right transversal of G' . From this it immediately follows that

$$\text{frat}(G)g_1 \neq \text{frat}(G)g_2 \Rightarrow G'g_1 \neq G'g_2$$

i.e. $G'g_1 = G'g_2 \Rightarrow \text{frat}(G)g_1 = \text{frat}(G)g_2$ Let $g \in G' \Rightarrow G'g = G' \Rightarrow \text{frat}(G)g = \text{frat}(G)$

$$\Rightarrow g \in \text{frat}(G)$$

$$\Rightarrow G' \subseteq \text{frat}(G).$$

From this result it immediately follows that:

Proposition 4.6: Let G be a finite group. G is nilpotent \Leftrightarrow every right transversal of $\text{frat}(G)$ is contained in some right transversal of G' .

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