On $I$-Statistical Convergence

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Abstract. In this paper we investigate the notion of $I$-statistical convergence and introduce $I$-st limit points and $I$-st cluster points of real number sequence and also studied some of its basic properties.

Keywords: $I$-limit point, $I$-cluster point, $I$-statistically Convergent.


1. Introduction

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of $I$-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal $I$ of subsets of the set $N$ to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of $I$-convergence was further extended to $I$-statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done

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by Savas and Das [17], Debnath and Debnath [3], Mursaleen et al [15], Et et al. [5] and many others [9, 10, 22, 23]. In [16], Savas and Das introduced the I-statistical convergence and I-λ-statistical convergence and the relation between them. Also they studied these concept in the notion of [V, λ]- summability method.

In the present paper we return to the view of I-statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a I-statistical analogue of the set of limit points and cluster points of a real number sequence.

2. Definitions and Preliminaries

Definition 2.1. [8] If \( K \) is a subset of the positive integers \( N \), then \( K_n \) denotes the set \( \{ k \in K : k \leq n \} \). The natural density of \( K \) is given by \( D(K) = \lim_{n \to \infty} \frac{|K_n|}{n} \).

Definition 2.2. [8] A sequence \( (x_n) \) is said to be statistically convergent to \( x_0 \) if for each \( \varepsilon > 0 \), the set \( A(\varepsilon) = \{ k \in N : d(x_k, x_0) \geq \varepsilon \} \) has natural density zero. \( x_0 \) is called the statistical limit of the sequence \( (x_n) \) and we write \( \text{st-lim}_{n \to \infty} x_n = x_0 \).

Definition 2.3. [7] If \( (x_{k(j)}) \) be a subsequence of a sequence \( x = (x_n) \) and density of \( K = \{ k(j) : j \in N \} \) is zero then \( (x_{k(j)}) \) is called a thin subsequence. Otherwise \( (x_{k(j)}) \) is called a non-thin subsequence of \( x \).

\( x_0 \) is said to be a statistical limit point of a sequence \( (x_n) \), if there exist a non-thin subsequence of \( (x_n) \) which converges to \( x_0 \).

Let \( A_x \) denotes the set of all statistical limit points of the sequence \( (x_n) \).

Definition 2.4. [7] \( x_0 \) is said to be a statistical cluster point of a sequence \( x = (x_n) \), provided that for each \( \varepsilon > 0 \) the density of the set \( \{ k \in N : d(x_k, x_0) < \varepsilon \} \) is not equal to 0.

Let \( \Gamma_x \) denotes the set of all statistical cluster points of the sequence \( (x_n) \).

Definition 2.5. [12] Let \( X \) is a non-empty set. A family of subsets \( I \subset P(X) \) is called an ideal on \( X \) if and only if

(i) \( \emptyset \in I \);  
(ii) for each \( A, B \in I \) implies \( A \cup B \in I \);  
(iii) for each \( A \in I \) and \( B \supset A \) implies \( B \in I \).

Definition 2.6. [12] Let \( X \) is a non-empty set. A family of subsets \( F \subset P(X) \) is called a filter on \( X \) if and only if

(i) \( \emptyset \notin F \);  
(ii) for each \( A, B \in F \) implies \( A \cap B \in F \);  
(iii) for each \( A \in F \) and \( B \supset A \) implies \( B \in F \).
On $I$-statistical convergence

An ideal $I$ is called non-trivial if $I \neq \emptyset$ and $X \notin I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal $I$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in $X$ if and only if $I \supset \{\{x\} : x \in X\}$

Definition 2.7. [12] Let $I \subset P(N)$ be a non-trivial ideal on $N$. A sequence $(x_n)$ is said to be $I$-convergent to $x_0$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ belongs to $I$. $x_0$ is called the $I$-limit of the sequence $(x_n)$ and we write $I\text{-lim}_{n \to \infty} x_n = x_0$.

Definition 2.8. [12] $x_0$ is said to be $I$-limit point of a sequence $x = (x_n)$ provided that there is a subset $K = \{k_1 < k_2 < ...\} \subset N$ such that $K \notin I$ and $\text{lim}_{k \to \infty} x_{k_i} = x_0$.

Let $I(A_x)$ denotes the set of all $I$-limit points of the sequence $x$.

Definition 2.9. [12] $x_0$ is said to be $I$-cluster point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ the set $\{k \in N : d(x_k, x_0) < \varepsilon\} \notin I$.

Let $I(I_x)$ denotes the set of all $I$-cluster points of the sequence $x$.

Definition 2.10. [16] A sequence $x = (x_n)$ is said to be $I$-statistically convergent to $x_0$ if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\{n \in N : \frac{1}{n} \sum_{k \leq n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\} \in I.$$  

$x_0$ is called $I$-statistical limit of the sequence $(x_n)$ and we write, $I\text{-stlim} x_n = x_0$.

Throughout the paper we consider $I$ as an admissible ideal.

3. Main Results

Theorem 3.1. If $(x_n)$ be a sequence such that $I\text{-stlim} x_n = x_0$, then $x_0$ determined uniquely.

Proof. If possible let the sequence $(x_n)$ be $I$-statistically convergent to two different numbers $x_0$ and $y_0$

i.e, for any $\varepsilon > 0$, $\delta > 0$ we have,

$$A_1 = \{n \in N : \frac{1}{n} \sum_{k \leq n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$$

and $A_2 = \{n \in N : \frac{1}{n} \sum_{k \leq n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$

Therefore, $A_1 \cap A_2 \neq \emptyset$, since $A_1 \cap A_2 \in \mathcal{F}(I)$.

Let $m \in A_1 \cap A_2$ and take $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$ so, $\frac{1}{m} \sum_{k \leq m} |\{k \leq m : d(x_k, x_0) \geq \varepsilon\}| < \delta$

and $\frac{1}{m} \sum_{k \leq m} |\{k \leq m : d(x_k, y_0) \geq \varepsilon\}| < \delta$

i.e, for maximum $k \leq m$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus, we must have

$$\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset$$

a contradiction, as the neighbourhood of $x_0$ and $y_0$ are disjoint.

Hence the theorem is proved. □
Theorem 3.2. For any sequence \((x_n)\), \(st-lim x_n = x_0\) implies \(I-stlim x_n = x_0\).

Proof. Let \(st-lim x_n = x_0\).

Then for each \(\varepsilon > 0\), the set \(A(\varepsilon) = \{k \leq n : d(x_k, x_0) \geq \varepsilon\}\) has natural density zero.

i.e., \(\lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} \delta_{d(x_k, x_0) \geq \varepsilon} = 0\)

So for every \(\varepsilon > 0\) and \(\delta > 0\),

\[\{n \in N : \frac{1}{n} \sum_{k \leq n} \delta_{d(x_k, x_0) \geq \varepsilon} \geq \delta\}\]

is a finite set and therefore belongs to \(I\), as \(I\) is an admissible ideal.

Hence \(I-stlim x_n = x_0\). □

But the converse is not true.

Example 3.3. Let \(I = \zeta\) be the class of \(A \subseteq N\) that intersect a finite number of \(\Delta_j\)'s where \(N = \bigcup_{j=1}^{\infty} \Delta_j\) and \(\Delta_i \cap \Delta_j = \emptyset\) for \(i \neq j\).

Let \(x_n = \frac{1}{n}\) and so \(\lim_{n \to \infty} d(x_n, 0) = 0\). Put \(\varepsilon_n = d(x_n, 0)\) for \(n \in N\).

Now define a sequence \((y_n)\) by \(y_n = x_j\) if \(n \in \Delta_j\)

Let \(\eta > 0\). Choose \(\nu \in N\) such that \(\varepsilon_\nu < \eta\). Then

\[A(\eta) = \{n \in N : d(y_n, 0) \geq \eta\} \subseteq \Delta_1 \cup \cdots \cup \Delta_\nu \in \zeta\]

Now, \(\{k \leq n : d(y_k, 0) \geq \eta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\}\)

i.e., \(\frac{1}{n} \sum_{k \leq n} \delta_{d(y_k, 0) \geq \eta} \leq \{n \in N : d(y_n, 0) \geq \eta\}\)

so for any \(\delta > 0\),

\[\{n \in N : \frac{1}{n} \sum_{k \leq n} \delta_{d(y_k, 0) \geq \eta} \geq \delta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\} \in \zeta\]

Therefore \((y_n)\) is \(\zeta\)-statistically convergent to \(0\).

But \((y_n)\) is not a statistically convergent.

Theorem 3.4. For any sequence \((x_n)\), \(I-lim x_n = x_0\) implies \(I-stlim x_n = x_0\).

Proof. The proof is obvious. But the converse is not true. □

Example 3.5. If we take \(I = I_f\) the sequence \((x_n)\),

where \(x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & otherwise \end{cases}\)

is \(I\)-statistically convergent to \(1\). But \((x_n)\) is not \(I\)-convergent.

Theorem 3.6. If each subsequence of \((x_n)\) is \(I\)-statistically convergent to \(\xi\) then \((x_n)\) is also \(I\)-statistically convergent to \(\xi\).

Proof. Suppose \((x_n)\) is not \(I\)-statistically convergent to \(\xi\), then there exists \(\varepsilon > 0\) and \(\delta > 0\) such that

\[A = \{n \in N : \frac{1}{n} \sum_{k \leq n} \delta_{d(x_k, \xi) \geq \varepsilon} \geq \delta\} \notin I\].

Since \(I\) is admissible ideal so \(A\) must be an infinite set.

Let \(A = \{n_1 < n_2 < \ldots < n_m < \ldots\} \subseteq N\) and \(y_m = x_{n_m}\) for \(m \in N\). Then \((y_m)\), as a subsequence of \((x_n)\) which is not \(I\)-statistically convergent to \(\xi\),

But the converse is not true. We can easily show this from example 3.5.
Theorem 3.7. Let \((x_n)\) and \((y_n)\) be two sequences then

(i) \(I^\text{-st} \lim x_n = x_0\) and \(c \in \mathbb{R}\) implies \(I^\text{-st} \lim cx_n = cx_0\).

(ii) \(I^\text{-st} \lim x_n = x_0\) and \(I^\text{-st} \lim y_n = y_0\) implies \(I^\text{-st} \lim (x_n + y_n) = x_0 + y_0\).

Proof. (i) If \(c = 0\), we have nothing to prove.

So we assume that \(c \neq 0\).

Now, \(\frac{1}{n}|\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| = \frac{1}{n}|\{k \leq n : |c|d(x_k, x_0) \geq \varepsilon\}| \leq \frac{1}{n}|\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|}\}| < \delta\).

Therefore, \(\{n \in N : \frac{1}{n}|\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)\).

i.e, \(I^\text{-st} \lim cx_n = cx_0\).

(ii) We have \(A_1 = \{n \in N : \frac{1}{n}|\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)\)

and \(A_2 = \{n \in N : \frac{1}{n}|\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)\).

Since \(A_1 \cap A_2 \neq \emptyset\), therefore for all \(n \in A_1 \cap A_2\) we have,

\(\frac{1}{n}|\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| \leq \frac{1}{n}|\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| + \frac{1}{n}|\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \delta\).

i.e, \(\{n \in N : \frac{1}{n}|\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)\).

Hence \(I^\text{-st} \lim (x_n + y_n) = (x_0 + y_0)\). \(\square\)

Definition 3.8. A sequence \(x = (x_n)_{n \in N}\) of elements of \(X\) is said to be \(I^\text{-statistical} \lim\) convergent to \(\xi \in X\) if and only if there exists a set \(M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(I)\), such that \(st\lim d(x_{m_k}, \xi) = 0\).

Theorem 3.9. If \(I^*\text{-st} \lim_{n \to \infty} x_n = \xi\) then \(I^\text{-st} \lim_{n \to \infty} x_n = \xi\).

Proof. Let \(I^*\text{-st} \lim_{n \to \infty} x_n = \xi\). By assumption there exist a set \(H \in I\) such that for \(M = N \setminus H = \{m_1 < m_2 < \ldots < m_k < \ldots\}\) we have \(st\text{-lim} x_{m_k} = \xi\).

i.e, \(\lim_{n \to \infty} \frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| = 0\).

so for any \(\delta > 0\), \(\{n \in N : \frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\} \in I\) since \(I\) is an admissible ideal.

Now, \(A(\varepsilon, \delta) = \{n \in N : \frac{1}{n}|\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\}\)

\(\subset H \cup \{n \in N : \frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\} \in I\)

i.e, \(I^\text{-st} \lim_{n \to \infty} x_n = \xi\). \(\square\)

But the converse may not be true.

From example 3.3, we have \(\xi^*\text{-st} \lim_{n \to \infty} y_n = 0\).

Suppose that \(\zeta^*\text{-st} \lim_{n \to \infty} y_n = 0\). Then there exist a set \(H \in \zeta\) such that for \(M = N \setminus H = \{m_1 < m_2 < \ldots < m_k < \ldots\}\) we have \(st\text{-lim} y_{m_k} = 0\). By definition of \(\zeta\) there exist a \(p \in N\) such that \(H \subset \Delta_1 \cup \ldots \cup \Delta_p\). But then \(\Delta_{p+1} \subset M\), so for infinitely many \(m_k \in \Delta_{p+1}\),

\(D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} = 2^{-(p+1)} > 0\) for \(0 < \eta < \frac{1}{p+1}\)

i.e, \(D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} \neq 0\), which is a contradicts \(st\text{-lim} y_{m_k} = 0\).

Hence \(\zeta^*\text{-st} \lim_{n \to \infty} y_n \neq 0\).
Definition 3.10. An element $x_0$ is said to be an $I$-statistical limit point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < \ldots \} \subset N$ such that $M \notin I$ and $\stlim x_{m_k} = x_0$.

$I-S(A_x)$ denotes the set of all $I$-statistical limit points of the sequence $(x_n)$.

Theorem 3.11. If $(x_n)$ be a sequence such that $\stlim x_n = x_0$ then $I-S(A_x) = \{x_0\}$.

Proof. Since $(x_n)$ is $I$-statistically convergent to $x_0$, so for each $\varepsilon > 0$ and $\delta > 0$ the set,

$$A = \{n \in N: \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \geq \delta\} \in I,$$

where $I$ is an admisible ideal.

Suppose $I-S(A_x)$ contains $y_0$ different from $x_0$, i.e., $y_0 \notin I-S(A_x)$.

So there exist a $M \subset N$ such that $M \notin I$ and $\stlim x_{m_k} = y_0$.

Let $B = \{n \in M: \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, y_0) \geq \varepsilon\} \geq \delta\}$. So $B$ is a finite set and therefore $B \in I$ and so $B^c = \{n \in M: \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, y_0) < \varepsilon\} < \delta\} \in F(I)$.

Again let $A_1 = \{n \in M: \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \geq \delta\}$. So $A_1 \subseteq A \in I$.

i.e., $A_1 \in F(I)$. Therefore $A_1^c \cap B^c \neq \emptyset$, since $A_1^c \cap B^c \in F(I)$.

Let $\frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, y_0) < \varepsilon\} < \delta$

and $\frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, x_0) \leq \varepsilon\} \geq \delta$

i.e, for maximum $k \leq n$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus we must have,

$$\{k \leq p : d(x_k, x_0) < \varepsilon\} \cap \{k \leq p : d(x_k, y_0) < \varepsilon\} \neq \emptyset$$

a contradiction, as the neighbourhood of $x_0$ and $y_0$ are disjoint.

Hence $I-S(A_x) = \{x_0\}$. $\square$

Definition 3.12. [15] An element $x_0$ is said to be an $I$-statistical cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$

$$\{n \in N: \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \geq \delta\} \notin I.$$

$I-S(I_x)$ denotes the set of all $I$-statistical cluster points of the sequence $(x_n)$.

Theorem 3.13. For any sequence $x = (x_n)$, $I-S(I_x)$ is closed.

Proof. Let $y_0$ be a limit point of the set $I-S(I_x)$ then for any $\varepsilon > 0$, $I-S(I_x) \cap B(y_0, \varepsilon) \neq \emptyset$, where $B(y_0, \varepsilon) = \{z \in X : d(z, y_0) < \varepsilon\}$.

Let $z_0 \in I-S(I_x) \cap B(y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B(z_0, \varepsilon_1) \subseteq B(y_0, \varepsilon)$.

Then we have $\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d(x_k, y_0) \geq \varepsilon_1\}$

$\Rightarrow \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \geq \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, y_0) \geq \varepsilon_1\}$

Now for any $\delta > 0$,

$$\{n \in N: \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} < \delta\}$$
For any sequence \( x = (x_n), I-S(A_x) \subseteq I-S(\Gamma_x) \).

**Proof.** Let \( x_0 \in I-S(A_x) \). Then there exist a set \( M = \{m_1 < m_2 < \cdots \} \notin I \) such that, \( st\text{-}lim x_{m_k} = x_0 \Rightarrow \lim_{k \to \infty} \frac{1}{k} \{m_i \leq k : d(x_{m_i}, x_0) \geq \varepsilon\} = 0 \).

Take \( \delta > 0 \), so there exist \( k_0 \in N \) such that for \( n > k_0 \) we have, \( \frac{1}{n} \{m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon\} < \delta \).

Let \( A = \{n \in N : \frac{1}{n} \{m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon\} < \delta\} \).

Also, \( A \supseteq M/\{m_1 < m_2 < \cdots < m_{k_0}\} \). Since \( I \) is an admissible ideal and \( M \notin I \), therefore \( A \notin I \). So by definition of \( I \)-statistical cluster point \( x_0 \in I-S(\Gamma_x) \).

Hence the theorem is proved. \( \square \)

**Theorem 3.15.** If \( x = (x_n) \) and \( y = (y_n) \) be two sequences such that \( \{n \in N : x_n \neq y_n\} \notin I \), then

(i) \( I-S(A_x) = I-S(A_y) \) and (ii) \( I-S(\Gamma_x) = I-S(\Gamma_y) \).

**Proof.** (i) Let \( x_0 \in I-S(A_x) \). So by definition there exist a set \( K = \{k_1 < k_2 < k_3 < \cdots \} \) of \( N \) such that \( K \notin I \) and \( st\text{-}lim x_{k_n} = x_0 \).

Since \( \{n \in K : x_n \neq y_n\} \subseteq \{n \in N : x_n \neq y_n\} \notin I \),

therefore \( K' = \{n \in K : x_n = y_n\} \notin I \) and \( K' \subseteq K \).

So we have \( st\text{-}lim y_{k_n} = x_0 \).

This shows that \( x_0 \in I-S(A_y) \) and therefore \( I-S(A_x) \subseteq I-S(A_y) \).

By symmetry \( I-S(A_y) \subseteq I-S(A_x) \).

Hence \( I-S(A_y) = I-S(A_x) \).

(ii) Let \( x_0 \in I-S(\Gamma_x) \). So by definition for each \( \varepsilon > 0 \) the set, \( A = \{n \in N : \frac{1}{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} < \delta\} \notin I \).

Let \( B = \{n \in N : \frac{1}{n} \{k \leq n : d(y_k, x_0) \geq \varepsilon\} < \delta\} \). We have to prove that \( B \notin I \).

Suppose \( B \in I \). So, \( B^c = \{n \in N : \frac{1}{n} \{k \leq n : d(y_k, x_0) \geq \varepsilon\} \geq \delta\} \in \mathcal{F}(I) \).

By hypothesis the set \( C = \{n \in N : x_n = y_n\} \in \mathcal{F}(I) \).

Therefore \( B^c \cap C \in \mathcal{F}(I) \). Also it is clear that \( B^c \cap C \subseteq A^c \in \mathcal{F}(I) \),

i.e., \( A \in I \), which is a contradiction.

Hence \( B \notin I \) and thus the result is proved. \( \square \)

**Theorem 3.16.** If \( g \) is a continuous function on \( X \) then it preserves \( I \)-statistical convergence in \( X \).

**Proof.** Let \( I\text{-}st\lim_{n \to \infty} x_n = \xi \).

Since \( g \) is continuous, then for each \( \varepsilon_1 > 0 \), there exist \( \varepsilon_2 > 0 \) such that if \( x \in B(\xi, \varepsilon_1) \) then \( g(x) \in B(g(\xi), \varepsilon_2) \).
Also we have,
\[ C(\varepsilon_1, \delta) = \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} < \delta \} \in \mathcal{F}(I) \]

Now, \( \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} \supseteq \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} \)
so, \( \frac{1}{n} \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} \geq \frac{1}{n} \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} \)

for \( \delta > 0, \) \( \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} \| < \delta \} \)
\( \subseteq \{ n \in N : \frac{1}{n} \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} \| < \delta \} \in \mathcal{F}(I) \)

since \( C(\varepsilon_1, \delta) \in \mathcal{F}(I) \).
Hence the theorem is proved. \( \square \)

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