On $I$-Statistical Convergence

Shyamal Debnath$^a$, Debjani Rakshit$^b$

$^a$Department of Mathematics, Tripura University, Agartala-799022, India.
$^b$Department of FST, ICFAI University, Tripura, Kamalghat, West Tripura-799210, India.

E-mail: shyamalnitamath@gmail.com
E-mail: debjanirakshit88@gmail.com

Abstract. In this paper we investigate the notion of $I$-statistical convergence and introduce $I$-st limit points and $I$-st cluster points of real number sequence and also studied some of its basic properties.

Keywords: $I$-limit point, $I$-cluster point, $I$-statistically Convergent.


1. INTRODUCTION

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of $I$-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal $I$ of subsets of the set $N$ to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of $I$-convergence was further extended to $I$-statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done.

*Corresponding Author

Received 04 February 2016; Accepted 27 September 2017
©2018 Academic Center for Education, Culture and Research TMU
by Savas and Das [17], Debnath and Debnath [3], Mursaleen et.al [15], Et et al. [5] and many others [9, 10, 22, 23]. In [16], Savas and Das introduced the \(I\)-statistical convergence and \(I\-\lambda\)-statistical convergence and the relation between them. Also they studied these concept in the notion of \([V, \lambda]\)- summability method.

In the present paper we return to the view of \(I\)-statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a \(I\)-statistical analogue of the set of limit points and cluster points of a real number sequence.

2. Definitions and Preliminaries

**Definition 2.1.** [8] If \(K\) is a subset of the positive integers \(N\), then \(K_n\) denotes the set \(\{k \in K : k \leq n\}\). The natural density of \(K\) is given by \(D(K) = \lim_{n \to \infty} \frac{|K_n|}{n}\).

**Definition 2.2.** [8] A sequence \((x_n)\) is said to be statistically convergent to \(x_0\) if for each \(\varepsilon > 0\), the set \(A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}\) has natural density zero. \(x_0\) is called the statistical limit of the sequence \((x_n)\) and we write \(st\-\lim_{n \to \infty} x_n = x_0\).

**Definition 2.3.** [7] If \((x_{k(j)})\) be a subsequence of a sequence \(x = (x_n)\) and density of \(K = \{k(j) : j \in N\}\) is zero then \((x_{k(j)})\) is called a thin subsequence. Otherwise \((x_{k(j)})\) is called a non-thin subsequence of \(x\).

\(x_0\) is said to be a statistical limit point of a sequence \((x_n)\), if there exist a non-thin subsequence of \((x_n)\) which converges to \(x_0\).

Let \(\Lambda_x\) denotes the set of all statistical limit points of the sequence \((x_n)\).

**Definition 2.4.** [7] \(x_0\) is said to be a statistical cluster point of a sequence \(x = (x_n)\), provided that for each \(\varepsilon > 0\) the density of the set \(\{k \in N : d(x_k, x_0) < \varepsilon\}\) is not equal to 0.

Let \(\Gamma_x\) denotes the set of all statistical cluster points of the sequence \((x_n)\).

**Definition 2.5.** [12] Let \(X\) is a non-empty set. A family of subsets \(I \subset P(X)\) is called an ideal on \(X\) if and only if
  
  (i) \(\emptyset \in I\);
  
  (ii) for each \(A, B \in I\) implies \(A \cup B \in I\);
  
  (iii) for each \(A \in I\) and \(B \subset A\) implies \(B \in I\).

**Definition 2.6.** [12] Let \(X\) is a non-empty set. A family of subsets \(\mathcal{F} \subset P(X)\) is called a filter on \(X\) if and only if
  
  (i) \(\emptyset \notin \mathcal{F}\);
  
  (ii) for each \(A, B \in \mathcal{F}\) implies \(A \cap B \in \mathcal{F}\);
  
  (iii) for each \(A \in \mathcal{F}\) and \(B \supset A\) implies \(B \in \mathcal{F}\).
An ideal $I$ is called non-trivial if $I \neq \emptyset$ and $X \not\in I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal $I$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in $X$ if and only if $I \supset \{\{x\} : x \in X\}$

**Definition 2.7.** [12] Let $I \subset P(N)$ be a non-trivial ideal on $N$. A sequence $(x_n)$ is said to be $I$-convergent to $x_0$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ belongs to $I$. $x_0$ is called the $I$-limit of the sequence $(x_n)$ and we write $I\text{-lim}_{n \to \infty} x_n = x_0$.

**Definition 2.8.** [12] $x_0$ is said to be $I$-limit point of a sequence $x = (x_n)$ provided that there is a subset $K = \{k_1 < k_2 < \ldots\} \subset N$ such that $K \not\in I$ and $\lim x_{k_i} = x_0$.

Let $I(A_x)$ denotes the set of all $I$-limit points of the sequence $x$.

**Definition 2.9.** [12] $x_0$ is said to be $I$-cluster point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ the set $\{k \in N : d(x_k, x_0) < \varepsilon\} \notin I$.

Let $I(I_x)$ denotes the set of all $I$-cluster points of the sequence $x$.

**Definition 2.10.** [16] A sequence $x = (x_n)$ is said to be $I$-statistically convergent to $x_0$ if for every $\varepsilon > 0$ and every $\delta > 0$, 
\[\{n \in N : \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \geq \delta\} \in I.\]

$x_0$ is called $I$-statistical limit of the sequence $(x_n)$ and we write, $I\text{-stlim} x_n = x_0$.

Throughout the paper we consider $I$ as an admissible ideal.

### 3. Main Results

**Theorem 3.1.** If $(x_n)$ be a sequence such that $I\text{-stlim} x_n = x_0$, then $x_0$ determined uniquely.

**Proof.** If possible let the sequence $(x_n)$ be $I$-statistically convergent to two different numbers $x_0$ and $y_0$

- i.e, for any $\varepsilon > 0$, $\delta > 0$ we have, $A_1 = \{n \in N : \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} < \delta\} \in \mathcal{F}(I)$
- and $A_2 = \{n \in N : \frac{1}{n} \sum_{k=1}^{n} \{k \leq n : d(x_k, y_0) \geq \varepsilon\} < \delta\} \in \mathcal{F}(I)$
- Therefore, $A_1 \cap A_2 \neq \emptyset$, since $A_1 \cap A_2 \in \mathcal{F}(I)$.
- Let $m \in A_1 \cap A_2$ and take $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$
- so, $\frac{1}{m} \sum_{k=1}^{m} \{k \leq m : d(x_k, x_0) \geq \varepsilon\} < \delta$
- and $\frac{1}{m} \sum_{k=1}^{m} \{k \leq m : d(x_k, y_0) \geq \varepsilon\} < \delta$

- i.e, for maximum $k \leq m$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.
- Thus, we must have $\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of $x_0$ and $y_0$ are disjoint.
- Hence the theorem is proved. \(\square\)
Theorem 3.2. For any sequence \((x_n)\), st-lim\(x_n = x_0\) implies I-stlim\(x_n = x_0\).

Proof. Let st-lim\(x_n = x_0\).

Then for each \(\varepsilon > 0\), the set \(A(\varepsilon) = \{k \leq n : d(x_k, x_0) \geq \varepsilon\}\) has natural density zero.

i.e, \(\lim_{n \to \infty} \frac{1}{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} = 0\)

So for every \(\varepsilon > 0\) and \(\delta > 0\),

\(\{n \in N : \frac{1}{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \geq \delta\}\) is a finite set and therefore belongs to \(I\), as \(I\) is an admissible ideal.

Hence I-stlim\(x_n = x_0\). □

But the converse is not true.

Example 3.3. Let \(I = \zeta\) be the class of \(A \subset N\) that intersect a finite number of \(\Delta_j\)′s where \(N = \bigcup_{j=1}^\infty \Delta_j\) and \(\Delta_i \cap \Delta_j = \emptyset\) for \(i \neq j\).

Let \(x_n = \frac{1}{n}\) and so \(\lim_{n \to \infty} d(x_n, 0) = 0\). Put \(\epsilon_n = d(x_n, 0)\) for \(n \in N\).

Now define a sequence \((y_n)\) by \(y_n = x_j\) if \(n \in \Delta_j\)

Let \(\eta > 0\). Choose \(\nu \in N\) such that \(\epsilon_\nu < \eta\). Then

\(A(\eta) = \{n : d(y_n, 0) \geq \eta\} \subseteq \Delta_1 \cup \ldots \cup \Delta_\nu \subset \zeta\).

Now, \(\{k \leq n : d(y_k, 0) \geq \eta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\}\)

i.e, \(\frac{1}{n} \{k \leq n : d(y_k, 0) \geq \eta\} \leq \{n \in N : d(y_n, 0) \geq \eta\}\)

so for any \(\delta > 0\),

\(\{n \in N : \frac{1}{n} \{k \leq n : d(y_k, 0) \geq \eta\} \geq \delta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\} \in \zeta\).

Therefore \((y_n)\) is \(\zeta\)-statistically convergent to 0.

But \((y_n)\) is not a statistically convergent.

Theorem 3.4. For any sequence \((x_n)\), I-lim\(x_n = x_0\) implies I-stlim\(x_n = x_0\).

Proof. The proof is obvious. But the converse is not true. □

Example 3.5. If we take \(I = I_f\) the sequence \((x_n)\),

where \(x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & \text{otherwise} \end{cases}\)

is I-statistically convergent to 1. But \((x_n)\) is not I-convergent.

Theorem 3.6. If each subsequence of \((x_n)\) is I-statistically convergent to \(\xi\) then \((x_n)\) is also I-statistically convergent to \(\xi\).

Proof. Suppose \((x_n)\) is not I-statistically convergent to \(\xi\), then there exists \(\varepsilon > 0\) and \(\delta > 0\) such that

\(A = \{n \in N : \frac{1}{n} \{k \leq n : d(x_k, \xi) \geq \varepsilon\} \geq \delta\} \notin I\). Since \(I\) is admissible ideal so \(A\) must be an infinite set.

Let \(A = \{n_1 < n_2 < \ldots < n_m < \ldots\}\). Let \(y_m = x_{n_m}\) for \(m \in N\). Then \((y_m)_{m \in N}\) is a subsequence of \((x_n)\) which is not I-statistically convergent to \(\xi\), a contradiction. Hence the theorem is proved. □

But the converse is not true. We can easily show this from example 3.5.
Theorem 3.7. Let \((x_n)\) and \((y_n)\) be two sequences then
(i) \(I\)-statlim\(x_n = x_0\) and \(c \in \mathbb{R}\) implies \(I\)-statlim\(cx_n = cx_0\).
(ii) \(I\)-statlim\(x_n = x_0\) and \(I\)-statlim\(y_n = y_0\) implies \(I\)-statlim\(x_n + y_n = x_0 + y_0\).

Proof. (i) If \(c = 0\), we have nothing to prove.
So we assume that \(c \neq 0\).
Now, \(\frac{1}{n}|\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| = \frac{1}{n}|\{k \leq n : c|d(x_k, x_0) \geq \varepsilon\}| \leq \frac{1}{n}|\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|}\}| < \delta\)
Therefore, \(\{n \in N : \frac{1}{n}|\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|}\}| < \delta\}\) \(\in \mathcal{F}(I)\) 
\(\therefore I\)-statlim\(cx_n = cx_0\).

(ii) We have \(A_1 = \{n \in N : \frac{1}{n}|\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)\) and \(A_2 = \{n \in N : \frac{1}{n}|\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)\).
Since \(A_1 \cap A_2 \neq \emptyset\), therefore for all \(n \in A_1 \cap A_2\) we have,
\(\frac{1}{n}|\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| \leq \frac{1}{n}|\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| + \frac{1}{n}|\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \delta\).
i.e., \(\{n \in N : \frac{1}{n}|\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| < \delta\}\) \(\in \mathcal{F}(I)\).
Hence \(I\)-statlim\(x_n + y_n = (x_0 + y_0)\).

Definition 3.8. A sequence \(x = (x_n)_{n \in N}\) of elements of \(X\) is said to be \(I\)-statistical convergent to \(\xi \in X\) if and only if there exists a set \(M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(I)\), such that \(st\)-lim \(d(x_{m_k}, \xi) = 0\).

Theorem 3.9. If \(I^*\)-statlim\(n \to \infty x_n = \xi\) then \(I\)-statlim\(n \to \infty x_n = \xi\).

Proof. Let \(I^*\)-statlim\(n \to \infty x_n = \xi\). By assumption there exist a set \(H \in I\) such that for \(M = N \setminus H = \{m_1 < m_2 < \ldots < m_k < \ldots\}\) we have \(st\)-lim \(x_{m_k} = \xi\).
i.e., \(st\)-lim\(x_{m_k} = \xi\) so for any \(\delta > 0\), \(\{n \in N : \frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\}\) \(\in I\) since \(I\) is an admissible ideal.
Now, \(A(\varepsilon, \delta) = \{n \in N : \frac{1}{n}|\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\}\)
\(\subset H \cup \{n \in N : \frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\}\) \(\in I\)
i.e., \(I\)-statlim\(n \to \infty x_n = \xi\).

But the converse may not be true.
From example 3.3. we have \(\zeta\)-statlim\(n \to \infty y_n = 0\).
Suppose that \(\zeta^*\)-statlim\(n \to \infty y_n = 0\). Then there exist a set \(H \in \zeta\) such that for \(M = N \setminus H = \{m_1 < m_2 < \ldots < m_k < \ldots\}\) we have \(st\)-lim \(y_{m_k} = 0\). By definition of \(\zeta\) there exist a \(p \in N\) such that \(H \subset \Delta_1 \cup \ldots \cup \Delta_p\). But then \(\Delta_{p+1} \subset M\), so for infinitely many \(m_k \in \Delta_{p+1}\),
\(D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} = 2^{-p+1} > 0\) for \(0 < \eta < \frac{1}{p+1}\)
i.e., \(D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} \neq 0\), which is a contradicts \(st\)-lim \(y_{m_k} = 0\).
Hence \(\zeta^*\)-statlim\(n \to \infty y_n \neq 0\).
Definition 3.10. An element $x_0$ is said to be an $I$-statistical limit point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < \ldots\} \subset N$ such that $M \notin I$ and $\text{st-lim} x_{m_k} = x_0$.

$I-S(I_x)$ denotes the set of all $I$-statistical limit points of the sequence $(x_n)$.

Theorem 3.11. If $(x_n)$ be a sequence such that $I$-st-lim $x_n = x_0$ then $I-S(I_x) = \{x_0\}$.

Proof. Since $(x_n)$ is $I$-statistically convergent to $x_0$, so for each $\varepsilon > 0$ and $\delta > 0$ the set,

$A = \{n \in N : \frac{1}{n} \{ k \leq n : d(x_k, x_0) \geq \varepsilon \} \geq \delta \} \in I$, where $I$ is an admissible ideal.

Suppose $I-S(I_x)$ contains $y_0$ different from $x_0$, i.e., $y_0 \in I-S(I_x)$.

So there exist a $M \subset N$ such that $M \notin I$ and $\text{st-lim} X_{m_k} = y_0$.

Let $B = \{n \in M : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} \geq \delta \}$. So $B$ is a finite set and therefore $B \in I$ and so $B^c = \{n \in M : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} < \delta \} \in F(I)$.

Again let $A_1 = \{n \in M : \frac{1}{n} \{ k \leq n : d(x_k, x_0) \geq \varepsilon \} \geq \delta \}$. So $A_1 \subset A \in I$.

Therefore $A_1 \cap B^c \neq \emptyset$, since $A_1 \cap B^c \in F(I)$

Let $p \in A_1 \cap B^c$ and take $\varepsilon = \frac{d(x_k, y_0)}{2} > 0$

so $\frac{1}{n} \{ k \leq p : d(x_k, x_0) \geq \varepsilon \} \geq \delta$

and $\frac{1}{n} \{ k \leq p : d(x_k, y_0) \geq \varepsilon \} < \delta$

i.e., for maximum $k \leq p$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus we must have,

$\{k \leq p : d(x_k, x_0) < \varepsilon\} \cap \{k \leq p : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of $x_0$ and $y_0$ are disjoint.

Hence $I-S(I_x) = \{x_0\}$.

Definition 3.12. [15] An element $x_0$ is said to be an $I$-statistical cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$

$\{n \in N : \frac{1}{n} \{ k \leq n : d(x_k, x_0) \geq \varepsilon \} \geq \delta \} \notin I$.

$I-S(I_x)$ denotes the set of all $I$-statistical cluster points of the sequence $(x_n)$.

Theorem 3.13. For any sequence $x = (x_n)$, $I-S(I_x)$ is closed.

Proof. Let $y_0$ be a limit point of the set $I-S(I_x)$ then for any $\varepsilon > 0$, $I-S(I_x) \cap B(y_0, \varepsilon) \neq 0$, where $B(y_0, \varepsilon) = \{z \in R : d(z, y_0) < \varepsilon\}$

Let $z_0 \in I-S(I_x) \cap B(y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B(z_0, \varepsilon_1) \subseteq B(y_0, \varepsilon)$.

Then we have $\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d(x_k, y_0) \geq \varepsilon_1\}$

$\Rightarrow \frac{1}{n} \{ k \leq n : d(x_k, z_0) \geq \varepsilon_1 \} \geq \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon_1 \}$

Now for any $\delta > 0$,

$\{n \in N : \frac{1}{n} \{ k \leq n : d(x_k, z_0) \geq \varepsilon_1 \} \geq \delta\}$
On I- statistical convergence

\[ \subseteq \{ n \in \mathbb{N} : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} \geq \delta \} \]

Since \( z_0 \in I-S(\Gamma_x) \) therefore, \( \{ n \in \mathbb{N} : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} \geq \delta \} \notin I. \)

i.e, \( y_0 \in I-S(\Gamma_x) \). Hence the theorem is proved. \( \square \)

**Theorem 3.14.** For any sequence \( x = (x_n) \), \( I-S(A_x) \subseteq I-S(\Gamma_x) \).

*Proof.* Let \( x_0 \in I-S(A_x) \). Then there exist a set \( M = \{ m_1 < m_2 < \ldots \} \notin I \) such that, \( st\text{-}lim x_{m_k} = x_0 \Rightarrow \lim_{k \to \infty} \frac{1}{k} \{ m_i \leq k : d(x_{m_i}, x_0) \geq \varepsilon \} = 0. \)

Take \( \delta > 0 \), so there exist \( k_0 \in \mathbb{N} \) such that for \( n > k_0 \) we have,

\[ \frac{1}{n} \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} < \delta. \]

Let \( A = \{ n \in \mathbb{N} : \frac{1}{n} \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} \geq \delta \} \).

Also, \( A \supseteq M/ \{ m_1 < m_2 < \ldots < m_k \} \). Since \( I \) is an admissible ideal and \( M \notin I \), therefore \( A \notin I \). So by definition of \( I \)-statistical cluster point \( x_0 \in I-S(\Gamma_x) \).

Hence the theorem is proved. \( \square \)

**Theorem 3.15.** If \( x = (x_n) \) and \( y = (y_n) \) be two sequences such that

\( \{ n \in \mathbb{N} : x_n \neq y_n \} \notin I \), then

(i) \( I-S(A_x) = I-S(A_y) \) and (ii) \( I-S(\Gamma_x) = I-S(\Gamma_y) \).

*Proof.* (i) Let \( x_0 \in I-S(A_x) \). So by definition there exist a set \( K = \{ k_1 < k_2 < k_3 < \cdots \} \) of \( \mathbb{N} \) such that \( K \notin I \) and \( st\text{-}lim x_{k_n} = x_0. \)

Since \( \{ n \in K : x_n \neq y_n \} \subset \{ n \in \mathbb{N} : x_n \neq y_n \} \notin I \),

therefore \( K' = \{ n \in K : x_n = y_n \} \notin I \) and \( K' \subseteq K. \)

So we have \( st\text{-}lim y_{k_n} = x_0. \)

This shows that \( x_0 \in I-S(A_y) \) and therefore \( I-S(A_x) \subseteq I-S(A_y). \)

By symmetry \( I-S(A_y) \subseteq I-S(A_x). \)

Hence \( I-S(A_y) = I-S(A_x) \).

(ii) Let \( x_0 \in I-S(\Gamma_x) \). So by definition for each \( \varepsilon > 0 \) the set,

\( A = \{ n \in \mathbb{N} : \frac{1}{n} \{ k \leq n : d(x_k, x_0) \geq \varepsilon \} \geq \delta \} \notin I. \)

Let \( B = \{ n \in \mathbb{N} : \frac{1}{n} \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} \geq \delta \} \). We have to prove that \( B \notin I. \)

Suppose \( B \in I \). So, \( B^c = \{ n \in \mathbb{N} : \frac{1}{n} \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} \geq \delta \} \in \mathcal{F}(I). \)

By hypothesis the set \( C = \{ n \in \mathbb{N} : x_n = y_n \} \in \mathcal{F}(I). \)

Therefore \( B^c \cap C \in \mathcal{F}(I) \). Also it is clear that \( B^c \cap C \subseteq A^c \in \mathcal{F}(I), \)

i.e, \( A \in I \), which is a contradiction.

Hence \( B \notin I \) and thus the result is proved. \( \square \)

**Theorem 3.16.** If \( g \) is a continuous function on \( X \) then it preserves \( I \)-statistical convergence in \( X \).

*Proof.* Let \( I\text{-}st\text{-}lim_{n \to \infty} x_n = \xi. \)

Since \( g \) is continuous, then for each \( \varepsilon_1 > 0 \), there exist \( \varepsilon_2 > 0 \) such that if \( x \in B(\xi, \varepsilon_1) \) then \( g(x) \in B(g(\xi), \varepsilon_2) \).
Also we have,
\[ C(\varepsilon_1, \delta) = \{ n \in N : \frac{1}{n} | \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} | < \delta \} \in \mathcal{F}(I) \]

Now, \( \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} \supseteq \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} \)

so,
\[ \frac{1}{n} \{ k \leq n : \frac{1}{n} \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} \} \]

for \( \delta > 0 \), \( \{ n \in N : \frac{1}{n} | \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} | < \delta \} \)

since \( C(\varepsilon_1, \delta) \in \mathcal{F}(I) \).

Hence the theorem is proved. \( \square \)

**ACKNOWLEDGMENTS**

The authors are grateful to CSIR for their financial support to carry out this research work through the Project No. 25(0236)/14/EMR-II.

**REFERENCES**

5. M. Et, A. Alotaibi, S. A. Mohiuddine, On \( (\Delta^m, I) \)-statistical convergence of order \( \alpha \), The scientific world journal, vol. 2014.