On $I$-Statistical Convergence

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Abstract. In this paper we investigate the notion of $I$-statistical convergence and introduce $I$-st limit points and $I$-st cluster points of real number sequence and also studied some of its basic properties.

Keywords: $I$-limit point, $I$-cluster point, $I$-statistically Convergent.


1. Introduction

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of $I$-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal $I$ of subsets of the set $N$ to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of $I$-convergence was further extended to $I$-statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done.

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by Savas and Das [17], Debnath and Debnath [3], Mursaleen et.al [15], Et et al. [5] and many others [9, 10, 22, 23]. In [16], Savas and Das introduced the $I$-statistical convergence and $I$-$\lambda$-statistical convergence and the relation between them. Also they studied these concept in the notion of $[V,\lambda]$- summability method.

In the present paper we return to the view of $I$-statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a $I$-statistical analogue of the set of limit points and cluster points of a real number sequence.

2. Definitions and Preliminaries

Definition 2.1. [8] If $K$ is a subset of the positive integers $N$, then $K_n$ denotes the set \{ $k \in K : k \leq n$ \}. The natural density of $K$ is given by $D(K) = \lim_{n \to \infty} \frac{|K_n|}{n}$.

Definition 2.2. [8] A sequence $(x_n)$ is said to be statistically convergent to $x_0$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{ k \in N : d(x_k, x_0) \geq \varepsilon \}$ has natural density zero. $x_0$ is called the statistical limit of the sequence $(x_n)$ and we write $\text{st-lim}_{n \to \infty} x_n = x_0$.

Definition 2.3. [7] If $(x_{k(j)})$ be a subsequence of a sequence $x = (x_n)$ and density of $K = \{ k(j) : j \in N \}$ is zero then $(x_{k(j)})$ is called a thin subsequence. Otherwise $(x_{k(j)})$ is called a non-thin subsequence of $x$.

$x_0$ is said to be a statistical limit point of a sequence $(x_n)$, if there exist a non-thin subsequence of $(x_n)$ which converges to $x_0$.

Let $\Lambda_x$ denotes the set of all statistical limit points of the sequence $(x_n)$.

Definition 2.4. [7] $x_0$ is said to be a statistical cluster point of a sequence $x = (x_n)$, provided that for each $\varepsilon > 0$ the density of the set $\{ k \in N : d(x_k, x_0) < \varepsilon \}$ is not equal to 0.

Let $\Gamma_x$ denotes the set of all statistical cluster points of the sequence $(x_n)$.

Definition 2.5. [12] Let $X$ is a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal on $X$ if and only if

(i) $\emptyset \in I$;
(ii) for each $A, B \in I$ implies $A \cup B \in I$;
(iii) for each $A \in I$ and $B \supset A$ implies $B \in I$.

Definition 2.6. [12] Let $X$ is a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on $X$ if and only if

(i) $\emptyset \notin \mathcal{F}$;
(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$. 
An ideal \( I \) is called non-trivial if \( I \neq \emptyset \) and \( X \notin I \). The filter \( F = \mathcal{F}(I) = \{X - A : A \in I\} \) is called the filter associated with the ideal \( I \). A non-trivial ideal \( I \subset P(X) \) is called an admissible ideal in \( X \) if and only if \( I \supset \{\{x\} : x \in X\} \)

**Definition 2.7.** [12] Let \( I \subset P(N) \) be a non-trivial ideal on \( N \). A sequence \( (x_n) \) is said to be \( I \)-convergent to \( x_0 \) if for each \( \varepsilon > 0 \), the set \( A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\} \) belongs to \( I \). \( x_0 \) is called the \( I \)-limit of the sequence \( (x_n) \) and we write \( \text{lim}_{n \to \infty} x_n = x_0 \).

**Definition 2.8.** [12] \( x_0 \) is said to be \( I \)-limit point of a sequence \( x = (x_n) \) provided that there is a subset \( K = \{k_1 < k_2 < \ldots\} \subset N \) such that \( K \notin I \) and \( \lim x_{k_i} = x_0 \).

Let \( I(A) \) denotes the set of all \( I \)-limit points of the sequence \( x \).

**Definition 2.9.** [12] \( x_0 \) is said to be \( I \)-cluster point of a sequence \( x = (x_n) \) provided that for each \( \varepsilon > 0 \) the set \( \{k \in N : d(x_k, x_0) < \varepsilon\} \notin I \).

Let \( I(\Gamma_x) \) denotes the set of all \( I \)-cluster points of the sequence \( x \).

**Definition 2.10.** [16] A sequence \( x = (x_n) \) is said to be \( I \)-statistically convergent to \( x_0 \) if for every \( \varepsilon > 0 \) and every \( \delta > 0 \),
\[
\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\} \in I.
\]
\( x_0 \) is called \( I \)-statistical limit of the sequence \( (x_n) \) and we write, \( \text{I-stlim} x_n = x_0 \).

Throughout the paper we consider \( I \) as an admissible ideal.

### 3. Main Results

**Theorem 3.1.** If \( (x_n) \) be a sequence such that \( I \text{-stlim} x_n = x_0 \), then \( x_0 \) determined uniquely.

**Proof.** If possible let the sequence \( (x_n) \) be \( I \)-statistically convergent to two different numbers \( x_0 \) and \( y_0 \)

i.e, for any \( \varepsilon > 0 \), \( \delta > 0 \) we have,
\[
A_1 = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\} \in F(I)
\]
and
\[
A_2 = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta\} \in F(I)
\]
Therefore, \( A_1 \cap A_2 \neq \emptyset \), since \( A_1 \cap A_2 \in F(I) \).

Let \( m \in A_1 \cap A_2 \) and take \( \varepsilon = \frac{d(x_0, y_0)}{3} > 0 \)
so,
\[
\frac{1}{m} |\{k \leq m : d(x_k, x_0) \geq \varepsilon\}| < \delta
\]
and
\[
\frac{1}{m} |\{k \leq m : d(x_k, y_0) \geq \varepsilon\}| < \delta
\]
i.e, for maximum \( k \leq m \) will satisfy \( d(x_k, x_0) < \varepsilon \) and \( d(x_k, y_0) < \varepsilon \) for a very small \( \delta > 0 \).

Thus, we must have
\[
\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset
\]
a contradiction, as the neighbourhood of \( x_0 \) and \( y_0 \) are disjoint.

Hence the theorem is proved. \qed
Theorem 3.2. For any sequence \((x_n)\), \(\text{st-lim} x_n = x_0\) implies \(I\text{-st lim} x_n = x_0\).

Proof. Let \(\text{st-lim} x_n = x_0\).

Then for each \(\varepsilon > 0\), the set \(A(\varepsilon) = \{k \leq n : d(x_k, x_0) \geq \varepsilon\}\) has natural density zero.

i.e, \(\lim_{n \to \infty} \frac{1}{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} = 0\)

So for every \(\varepsilon > 0\) and \(\delta > 0\),

\(\{n \in N : \frac{1}{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \geq \delta\}\) is a finite set and therefore belongs to \(I\), as \(I\) is an admissible ideal.

Hence \(I\text{-st lim} x_n = x_0\). \(\square\)

But the converse is not true. We can easily show this from example 3.5.

Example 3.3. Let \(I = \zeta\) be the class of \(A \subset N\) that intersect a finite number of \(\Delta_j\)'s where \(N = \bigcup_{j=1}^{\infty} \Delta_j\) and \(\Delta_i \cap \Delta_j = \emptyset\) for \(i \neq j\).

Let \(x_n = \frac{1}{n}\) and so \(\lim_{n \to \infty} d(x_n, 0) = 0\). Put \(\epsilon_n = d(x_n, 0)\) for \(n \in N\).

Now define a sequence \((y_n)\) by \(y_n = x_j\) if \(n \in \Delta_j\)

Let \(\eta > 0\). Choose \(\nu \in N\) such that \(\epsilon_\nu < \eta\). Then

\(A(\eta) = \{n : d(y_n, 0) \geq \eta\} \subseteq \Delta_1 \cup \ldots \cup \Delta_\nu \in \zeta\).

Now, \(\{k \leq n : d(y_k, 0) \geq \eta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\}\)

i.e., \(\frac{1}{n} \{k \leq n : d(y_k, 0) \geq \eta\} \leq \{n \in N : d(y_n, 0) \geq \eta\}\)

so for any \(\delta > 0\),

\(\{n \in N : \frac{1}{n} \{k \leq n : d(y_k, 0) \geq \eta\} \geq \delta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\} \in \zeta\).

Therefore \((y_n)\) is \(\zeta\)-statistically convergent to \(0\).

But \((y_n)\) is not a statistically convergent.

Theorem 3.4. For any sequence \((x_n)\), \(I\text{-lim} x_n = x_0\) implies \(I\text{-st lim} x_n = x_0\).

Proof. The proof is obvious. But the converse is not true. \(\square\)

Example 3.5. If we take \(I = I_f\) the sequence \((x_n)\),

where \(x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & \text{otherwise} \end{cases}\)

is \(I\)-statistically convergent to \(1\). But \((x_n)\) is not \(I\)-convergent.

Theorem 3.6. If each subsequence of \((x_n)\) is \(I\)-statistically convergent to \(\xi\) then \((x_n)\) is also \(I\)-statistically convergent to \(\xi\).

Proof. Suppose \((x_n)\) is not \(I\)-statistically convergent to \(\xi\), then there exists \(\varepsilon > 0\) and \(\delta > 0\) such that

\(A = \{n \in N : \frac{1}{n} \{k \leq n : d(x_k, \xi) \geq \varepsilon\} \geq \delta\} \notin I\). Since \(I\) is admissible ideal so \(A\) must be an infinite set.

Let \(A = \{n_1 < n_2 < \ldots < n_m < \ldots\}\). Let \(y_m = x_{n_m}\) for \(m \in N\). Then \((y_m)_{m \in N}\) is a subsequence of \((x_n)\) which is not \(I\)-statistically convergent to \(\xi\), a contradiction. Hence the theorem is proved. \(\square\)

But the converse is not true. We can easily show this from example 3.5.
Theorem 3.7. Let \((x_n)\) and \((y_n)\) be two sequences then

(i) \(I\)-stlim \(x_n = x_0\) and \(c \in \mathbb{R}\) implies \(I\)-stlim \(cx_n = cx_0\).

(ii) \(I\)-stlim \(x_n = x_0\) and \(I\)-stlim \(y_n = y_0\) implies \(I\)-stlim \((x_n + y_n) = x_0 + y_0\).

Proof. (i) If \(c = 0\), we have nothing to prove.

So we assume that \(c \neq 0\).

Now, \(\frac{1}{n} \left\{ k \leq n : d(cx_k, cx_0) \geq \varepsilon \right\} = \frac{1}{n} \left\{ k \leq n : d(x_k, x_0) \geq \varepsilon \right\} \leq \frac{1}{n} \left\{ k \leq n : d(x_k, x_0) \right\} < \delta\)

Therefore, \(\{ n \in N : \frac{1}{n} \left\{ k \leq n : d(x_k, x_0) \geq \varepsilon \right\} < \delta \} \in \mathcal{F}(I)\).

i.e, \(I\)-stlim \(cx_n = cx_0\).

(ii) We have \(A_1 = \{ n \in N : \frac{1}{n} \left\{ k \leq n : d(x_k, x_0) \geq \varepsilon \right\} < \frac{\delta}{2} \} \in \mathcal{F}(I)\) and \(A_2 = \{ n \in N : \frac{1}{n} \left\{ k \leq n : d(y_k, y_0) \geq \varepsilon \right\} < \frac{\delta}{2} \} \in \mathcal{F}(I)\).

Since \(A_1 \cap A_2 \neq \emptyset\), therefore for all \(n \in A_1 \cap A_2\) we have,

\(\frac{1}{n} \left\{ k \leq n : d(x_k, x_0) \geq \varepsilon \right\} \leq \frac{1}{n} \left\{ k \leq n : d(x_k, x_0) \geq \varepsilon \right\} < \delta\).

i.e, \(\{ n \in N : \frac{1}{n} \left\{ k \leq n : d(x_k, x_0) \geq \varepsilon \right\} < \delta \} \in \mathcal{F}(I)\).

Hence \(I\)-stlim \((x_n + y_n) = (x_0 + y_0)\).

\(\square\)

Definition 3.8. A sequence \(x = (x_n)_{n \in N}\) of elements of \(X\) is said to be \(I\)-statistical convergent to \(\xi \in X\) if and only if there exists a set \(M = \{ m_1 < m_2 < ... < m_k < ... \} \in \mathcal{F}(I)\), such that \(st\text{-lim} d(x_{m_k}, \xi) = 0\).

Theorem 3.9. If \(I^*\)-stlim \(n \to \infty x_n = \xi\) then \(I\)-stlim \(n \to \infty x_n = \xi\).

Proof. Let \(I^*\)-stlim \(n \to \infty x_n = \xi\). By assumption there exist a set \(H \in I\) such that for \(M = N \setminus H = \{ m_1 < m_2 < ... < m_k < ... \}\) we have \(st\text{-lim} x_{m_k} = \xi\)

i.e, \(\lim_{n \to \infty} \frac{1}{n} \left\{ m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon \right\} = 0\)

so for any \(\delta > 0\), \(\{ n \in N : \frac{1}{n} \left\{ m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon \right\} \geq \delta \} \in I\) since \(I\) is an admissible ideal.

Now, \(A(\varepsilon, \delta) = \{ n \in N : \frac{1}{n} \left\{ k \leq n : d(x_k, \xi) \geq \varepsilon \right\} \geq \delta \}\)

\(\subset H \cup \{ n \in N : \frac{1}{n} \left\{ m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon \right\} \geq \delta \} \in I\)

i.e, \(I\)-stlim \(n \to \infty x_n = \xi\).

But the converse may not be true.

From example 3.3, we have \(\zeta\text{-st lim} n \to \infty y_n = 0\).

Suppose that \(\zeta^*\text{-st lim} n \to \infty y_n = 0\). Then there exist a set \(H \in \zeta\) such that for \(M = N \setminus H = \{ m_1 < m_2 < ... < m_k < ... \}\) we have \(st\text{-lim} y_{m_k} = 0\). By definition of \(\zeta\) there exist a \(p \in N\) such that \(H \subset \Delta_1 \cup ... \cup \Delta_p\). But then \(\Delta_{p+1} \subset M\), so for infinitely many \(m_k \in \Delta_{p+1}\),

\(D \{ m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta \} = 2^{-\eta(p+1)} > 0\) for \(0 < \eta < \frac{1}{p+1}\).

i.e, \(D \{ m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta \} \neq 0\), which is a contradicts \(st\text{-lim} y_{m_k} = 0\).

Hence \(\zeta^*\text{-st lim} n \to \infty y_n \neq 0\).
Definition 3.10. An element $x_0$ is said to be an $I$-statistical limit point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < \ldots\} \subset N$ such that $M \notin I$ and $\text{st-lim} x_{m_k} = x_0$.

$I-S(\Lambda_x)$ denotes the set of all $I$-statistical limit points of the sequence $(x_n)$.

Theorem 3.11. If $(x_n)$ be a sequence such that $I\text{-stlim} x_n = x_0$ then $I-S(\Lambda_x) = \{x_0\}$.

Proof. Since $(x_n)$ is $I$-statistically convergent to $x_0$, so for each $\varepsilon > 0$ and $\delta > 0$ the set,

$$A = \{n \in N : \frac{1}{n} \mid \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \mid \geq \delta\} \in I,$$

where $I$ is an admissible ideal.

Suppose $I-S(\Lambda_x)$ contains $y_0$ different from $x_0$, i.e., $y_0 \in I-S(\Lambda_x)$.

So there exist a $M \subset N$ such that $M \notin I$ and $\text{st-lim} X_{m_k} = y_0$.

Let $B = \{n \in M : \frac{1}{n} \mid \{k \leq n : d(x_k, y_0) \geq \varepsilon\} \mid \geq \delta\}$. So $B$ is a finite set and therefore $B \in I$ and so $B^c = \{n \in M : \frac{1}{n} \mid \{k \leq n : d(x_k, y_0) \geq \varepsilon\} \mid < \delta\} \in F(I)$.

Again let $A_1 = \{n \in M : \frac{1}{n} \mid \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \mid \geq \delta\}$. So $A_1 \subset A \in I$.

i.e., $A_1 \in F(I)$. Therefore $A_1 \cap B^c \neq \emptyset$, since $A_1 \cap B^c \in F(I)$

Let $p \in A_1 \cap B^c$ and take $\varepsilon = \frac{d(x_k, y_0)}{1} > 0$

so $\frac{1}{n} \mid \{k \leq p : d(x_k, x_0) \geq \varepsilon\} \mid < \delta$

and $\frac{1}{n} \mid \{k \leq p : d(x_k, y_0) \geq \varepsilon\} \mid < \delta$

i.e, for maximum $k \leq p$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus we must have,

$$\{k \leq p : d(x_k, x_0) < \varepsilon\} \cap \{k \leq p : d(x_k, y_0) < \varepsilon\} \neq \emptyset$$

a contradiction, as the neighbourhood of $x_0$ and $y_0$ are disjoint.

Hence $I-S(\Lambda_x) = \{x_0\}$. □

Definition 3.12. [15] An element $x_0$ is said to be an $I$-statistical cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$

$$\{n \in N : \frac{1}{n} \mid \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \mid < \delta\} \notin I.$$ 

$I-S(\Gamma_x)$ denotes the set of all $I$-statistical cluster points of the sequence $(x_n)$.

Theorem 3.13. For any sequence $x = (x_n)$, $I-S(\Gamma_x)$ is closed.

Proof. Let $y_0$ be a limit point of the set $I-S(\Gamma_x)$ then for any $\varepsilon > 0$, $I-S(\Gamma_x) \cap B(y_0, \varepsilon) \neq \emptyset$, where $B(y_0, \varepsilon) = \{z \in R : d(z, y_0) < \varepsilon\}$

Let $z_0 \in I-S(\Gamma_x) \cap B(y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B(z_0, \varepsilon_1) \subseteq B(y_0, \varepsilon)$.

Then we have $\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d(x_k, y_0) \geq \varepsilon\}$

$\Rightarrow \frac{1}{n} \mid \{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \mid \geq \frac{1}{n} \mid \{k \leq n : d(x_k, y_0) \geq \varepsilon\} \mid$

Now for any $\delta > 0$,

$$\{n \in N : \frac{1}{n} \mid \{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \mid < \delta\}$$
On $I$-statistical convergence

\[ \subseteq \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} < \delta \} \]

Since $z_0 \in I$-S($\Gamma_x$) therefore, $\{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} < \delta \} \notin I$. Hence the theorem is proved.

**Theorem 3.14.** For any sequence $x = (x_n)$, $I$-S($A_x$) $\subseteq$ I - S($\Gamma_x$).

**Proof.** Let $x_0 \in I$-S($A_x$). Then there exist a set $M = \{ m_1 < m_2 < ... \} \notin I$ such that, $st\lim x_{m_k} = x_0 \Rightarrow \lim_{k \to \infty} \frac{1}{k} \{ m_i \leq k : d(x_{m_i}, x_0) \geq \varepsilon \} = 0$.

Take $\delta > 0$, so there exist $k_0 \in N$ such that for $n > k_0$ we have, $\frac{1}{n} \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} < \delta$.

Let $A = \{ n \in N : \frac{1}{n} \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} < \delta \}$.

Also, $A \supseteq M$ $\{ m_1 < m_2 < ... < m_k \}$. Since $I$ is an admissible ideal and $M \notin I$, therefore $A \notin I$. So by definition of $I$-statistical cluster point $x_0 \in I$-S($\Gamma_x$).

Hence the theorem is proved.

**Theorem 3.15.** If $x = (x_n)$ and $y = (y_n)$ be two sequences such that

\[ \{ n \in N : x_n \neq y_n \} \notin I \], then

(i) I - S($A_x$) $= I$ - S($A_y$) and (ii) I - S($\Gamma_x$) $= I$ - S($\Gamma_y$).

**Proof.** (i) Let $x_0 \in I$-S($A_x$). So by definition there exist a set $K = \{ k_1 < k_2 < k_3 < \cdots \}$ of $N$ such that $K \notin I$ and $st\lim x_{k_n} = x_0$.

Since $\{ n \in K : x_n \neq y_n \} \subset \{ n \in N : x_n \neq y_n \} \notin I$,

therefore $K' = \{ n \in K : x_n = y_n \} \notin I$ and $K' \subseteq K$.

So we have $st\lim y_{k_n} = x_0$.

This shows that $x_0 \in I$-S($A_y$) and therefore $I$-S($A_x$) $\subseteq$ I - S($A_y$).

By symmetry I - S($A_y$) $\subseteq$ I - S($A_x$).

Hence I - S($A_y$) $= I$ - S($A_x$).

(ii) Let $x_0 \in I$-S($\Gamma_x$). So by definition for each $\varepsilon > 0$ the set,

$A = \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, x_0) \geq \varepsilon \} < \delta \} \notin I$.

Let $B = \{ n \in N : \frac{1}{n} \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} < \delta \}$. We have to prove that $B \notin I$.

Suppose $B \in I$. So, $B^c = \{ n \in N : \frac{1}{n} \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} \geq \delta \} \in F(I)$.

By hypothesis the set $C = \{ n \in N : x_n = y_n \} \in F(I)$.

Therefore $B^c \cap C \in F(I)$. Also it is clear that $B^c \cap C \subseteq A^c \in F(I)$, i.e, $A \in I$, which is a contradiction.

Hence $B \notin I$ and thus the result is proved.

**Theorem 3.16.** If $g$ is a continuous function on $X$ then it preserves $I$-statistical convergence in $X$.

**Proof.** Let $I$-stlim$_{n \to \infty} x_n = \xi$.

Since $g$ is continuous, then for each $\varepsilon_1 > 0$, there exist $\varepsilon_2 > 0$ such that if $x \in B(\xi, \varepsilon_1)$ then $g(x) \in B(g(\xi), \varepsilon_2)$.
Also we have,  
\[ C(\varepsilon_1, \delta) = \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} < \delta \} \in \mathcal{F}(I) \]

Now, \( \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} \supseteq \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} \) so, \( \frac{1}{n} \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} \geq \frac{1}{n} \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} \) \[ \frac{1}{n} \{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \} \subseteq \{ n \in N : \frac{1}{n} \{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \} < \delta \} \in \mathcal{F}(I) \]

since \( C(\varepsilon_1, \delta) \in \mathcal{F}(I) \).

Hence the theorem is proved.  \( \square \)

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