

On I -Statistical Convergence

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ABSTRACT. In this paper we investigate the notion of I -statistical convergence and introduce I -st limit points and I -st cluster points of real number sequence and also studied some of its basic properties.

Keywords: I -limit point, I -cluster point, I -statistically Convergent.

2000 Mathematics subject classification: 40A35, 40D25.

1. INTRODUCTION

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of I -convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal I of subsets of the set N to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of I -convergence was further extended to I -statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done

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by Savas and Das [17], Debnath and Debnath [3], Mursaleen et.al [15], Et et al. [5] and many others [9, 10, 22, 23] . In [16], Savas and Das introduced the I -statistical convergence and I - λ -statistical convergence and the relation between them. Also they studied these concept in the notion of $[V, \lambda]$ - summability method.

In the present paper we return to the view of I -statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a I -statistical analogue of the set of limit points and cluster points of a real number sequence.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. [8] If K is a subset of the positive integers N , then K_n denotes the set $\{k \in K : k \leq n\}$. The natural density of K is given by $D(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$.

Definition 2.2. [8] A sequence (x_n) is said to be statistically convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ has natural density zero. x_0 is called the statistical limit of the sequence (x_n) and we write $st\text{-}\lim_{n \rightarrow \infty} x_n = x_0$.

Definition 2.3. [7] If $(x_{k(j)})$ be a subsequence of a sequence $x = (x_n)$ and density of $K = \{k(j) : j \in N\}$ is zero then $(x_{k(j)})$ is called a thin subsequence. Otherwise $(x_{k(j)})$ is called a non-thin subsequence of x .

x_0 is said to be a statistical limit point of a sequence (x_n) , if there exist a non-thin subsequence of (x_n) which converges to x_0 .

Let Λ_x denotes the set of all statistical limit points of the sequence (x_n) .

Definition 2.4. [7] x_0 is said to be a statistical cluster point of a sequence $x = (x_n)$, provided that for each $\varepsilon > 0$ the density of the set $\{k \in N : d(x_k, x_0) < \varepsilon\}$ is not equal to 0.

Let Γ_x denotes the set of all statistical cluster points of the sequence (x_n) .

Definition 2.5. [12] Let X is a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal on X if and only if

- (i) $\emptyset \in I$;
- (ii) for each $A, B \in I$ implies $A \cup B \in I$;
- (iii) for each $A \in I$ and $B \subset A$ implies $B \in I$.

Definition 2.6. [12] Let X is a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if

- (i) $\emptyset \notin \mathcal{F}$;
- (ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
- (iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal I . A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if $I \supset \{\{x\} : x \in X\}$

Definition 2.7. [12] Let $I \subset P(N)$ be a non-trivial ideal on N . A sequence (x_n) is said to be I -convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ belongs to I . x_0 is called the I -limit of the sequence (x_n) and we write $I\text{-}\lim_{n \rightarrow \infty} x_n = x_0$.

Definition 2.8. [12] x_0 is said to be I -limit point of a sequence $x = (x_n)$ provided that there is a subset $K = \{k_1 < k_2 < \dots\} \subset N$ such that $K \notin I$ and $\lim x_{k_i} = x_0$.

Let $I(A_x)$ denotes the set of all I -limit points of the sequence x .

Definition 2.9. [12] x_0 is said to be I -cluster point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ the set $\{k \in N : d(x_k, x_0) < \varepsilon\} \notin I$.

Let $I(I_x)$ denotes the set of all I -cluster points of the sequence x .

Definition 2.10. [16] A sequence $x = (x_n)$ is said to be I -statistically convergent to x_0 if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\right\} \in I.$$

x_0 is called I -statistical limit of the sequence (x_n) and we write, $I\text{-st}\lim x_n = x_0$.

Throughout the paper we consider I as an admissible ideal.

3. MAIN RESULTS

Theorem 3.1. If (x_n) be a sequence such that $I\text{-st}\lim x_n = x_0$, then x_0 determined uniquely.

Proof. If possible let the sequence (x_n) be I -statistically convergent to two different numbers x_0 and y_0

i.e, for any $\varepsilon > 0, \delta > 0$ we have,

$$A_1 = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$$

$$\text{and } A_2 = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$$

Therefore, $A_1 \cap A_2 \neq \emptyset$, since $A_1 \cap A_2 \in \mathcal{F}(I)$.

Let $m \in A_1 \cap A_2$ and take $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$

$$\text{so, } \frac{1}{m} |\{k \leq m : d(x_k, x_0) \geq \varepsilon\}| < \delta$$

$$\text{and } \frac{1}{m} |\{k \leq m : d(x_k, y_0) \geq \varepsilon\}| < \delta$$

i.e, for maximum $k \leq m$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus, we must have

$\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of x_0 and y_0 are disjoint.

Hence the theorem is proved. \square

Theorem 3.2. For any sequence (x_n) , $st\text{-}lim x_n = x_0$ implies $I\text{-}st\text{-}lim x_n = x_0$.

Proof. Let $st\text{-}lim x_n = x_0$.

Then for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \leq n : d(x_k, x_0) \geq \varepsilon\}$ has natural density zero.

i.e, $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| = 0$

So for every $\varepsilon > 0$ and $\delta > 0$,

$\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\}$ is a finite set and therefore belongs to I , as I is an admissible ideal.

Hence $I\text{-}st\text{-}lim x_n = x_0$. \square

But the converse is not true.

EXAMPLE 3.3. Let $I = \zeta$ be the class of $A \subset N$ that intersect a finite number of Δ_j 's where $N = \cup_{j=1}^{\infty} \Delta_j$ and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$.

Let $x_n = \frac{1}{n}$ and so $\lim_{n \rightarrow \infty} d(x_n, 0) = 0$. Put $\epsilon_n = d(x_n, 0)$ for $n \in N$.

Now define a sequence (y_n) by $y_n = x_j$ if $n \in \Delta_j$

Let $\eta > 0$. Choose $\nu \in N$ such that $\epsilon_\nu < \eta$. Then

$A(\eta) = \{n : d(y_n, 0) \geq \eta\} \subset \Delta_1 \cup \dots \cup \Delta_\nu \in \zeta$.

Now, $\{k \leq n : d(y_k, 0) \geq \eta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\}$

i.e, $\frac{1}{n} |\{k \leq n : d(y_k, 0) \geq \eta\}| \leq |\{n \in N : d(y_n, 0) \geq \eta\}|$

so for any $\delta > 0$,

$\{n \in N : \frac{1}{n} |\{k \leq n : d(y_k, 0) \geq \eta\}| \geq \delta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\} \in \zeta$.

Therefore (y_n) is ζ -statistically convergent to 0.

But (y_n) is not a statistically convergent.

Theorem 3.4. For any sequence (x_n) , $I\text{-}lim x_n = x_0$ implies $I\text{-}st\text{-}lim x_n = x_0$.

Proof. The proof is obvious. But the converse is not true. \square

EXAMPLE 3.5. If we take $I = I_f$ the sequence (x_n) ,

$$\text{where } x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & \text{otherwise} \end{cases}$$

is I -statistically convergent to 1. But (x_n) is not I -convergent.

Theorem 3.6. If each subsequence of (x_n) is I -statistically convergent to ξ then (x_n) is also I -statistically convergent to ξ .

Proof. Suppose (x_n) is not I -statistically convergent to ξ , then there exists $\varepsilon > 0$ and $\delta > 0$ such that

$A = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\} \notin I$. Since I is admissible ideal so A must be an infinite set.

Let $A = \{n_1 < n_2 < \dots < n_m < \dots\}$. Let $y_m = x_{n_m}$ for $m \in N$. Then $(y_m)_{m \in N}$ is a subsequence of (x_n) which is not I -statistically convergent to ξ , a contradiction. Hence the theorem is proved. \square

But the converse is not true. We can easily show this from example 3.5.

Theorem 3.7. Let (x_n) and (y_n) be two sequences then

(i) $I\text{-st } \lim x_n = x_0$ and $c \in R$ implies $I\text{-st } \lim cx_n = cx_0$.

(ii) $I\text{-st } \lim x_n = x_0$ and $I\text{-st } \lim y_n = y_0$ implies $I\text{-st } \lim (x_n + y_n) = x_0 + y_0$.

Proof. (i) If $c = 0$, we have nothing to prove.

So we assume that $c \neq 0$.

$$\begin{aligned} \text{Now, } \frac{1}{n} |\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| &= \frac{1}{n} |\{k \leq n : |c|d(x_k, x_0) \geq \varepsilon\}| \\ &\leq \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|}\}| < \delta \end{aligned}$$

Therefore, $\{n \in N : \frac{1}{n} |\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$.

i.e, $I\text{-st } \lim cx_n = cx_0$.

(ii) We have $A_1 = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)$

and $A_2 = \{n \in N : \frac{1}{n} |\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\} \in \mathcal{F}(I)$.

Since $A_1 \cap A_2 \neq \emptyset$, therefore for all $n \in A_1 \cap A_2$ we have,

$$\begin{aligned} \frac{1}{n} |\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| \\ \leq \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\}| + \frac{1}{n} |\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\}| < \delta. \end{aligned}$$

i.e, $\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$.

Hence $I\text{-st } \lim (x_n + y_n) = (x_0 + y_0)$. □

Definition 3.8. A sequence $x = (x_n)_{n \in N}$ of elements of X is said to be I^* -statistical convergent to $\xi \in X$ if and only if there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \in \mathcal{F}(I)$, such that $st\text{-lim } d(x_{m_k}, \xi) = 0$.

Theorem 3.9. If $I^*\text{-st } \lim_{n \rightarrow \infty} x_n = \xi$ then $I\text{-st } \lim_{n \rightarrow \infty} x_n = \xi$.

Proof. Let $I^*\text{-st } \lim_{n \rightarrow \infty} x_n = \xi$. By assumption there exist a set $H \in I$ such that for $M = N \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$ we have $st\text{-lim } x_{m_k} = \xi$

$$\text{i.e, } \lim_{n \rightarrow \infty} \frac{1}{n} |\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| = 0$$

so for any $\delta > 0$, $\{n \in N : \frac{1}{n} |\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\} \in I$ since I is an admissible ideal.

$$\begin{aligned} \text{Now, } A(\varepsilon, \delta) &= \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\} \\ &\subset H \cup \{n \in N : \frac{1}{n} |\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\} \in I \end{aligned}$$

i.e, $I\text{-st } \lim_{n \rightarrow \infty} x_n = \xi$. □

But the converse may not be true.

From example 3.3. we have $\zeta\text{-st } \lim_{n \rightarrow \infty} y_n = 0$.

Suppose that $\zeta^*\text{-st } \lim_{n \rightarrow \infty} y_n = 0$. Then there exist a set $H \in \zeta$ such that for $M = N \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$ we have $st\text{-lim } y_{m_k} = 0$. By definition of ζ there exist a $p \in N$ such that $H \subset \Delta_1 \cup \dots \cup \Delta_p$. But then $\Delta_{p+1} \subset M$, so for infinitely many $m_k \in \Delta_{p+1}$,

$$D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} = 2^{-(p+1)} > 0 \text{ for } 0 < \eta < \frac{1}{p+1}$$

i.e, $D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta\} \neq \emptyset$, which is a contradicts $st\text{-lim } y_{m_k} = 0$.

Hence $\zeta^*\text{-st } \lim_{n \rightarrow \infty} y_n \neq 0$.

Definition 3.10. An element x_0 is said to be an I -statistical limit point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < \dots\} \subset N$ such that $M \notin I$ and $st\text{-}lim x_{m_k} = x_0$.

$I\text{-}S(A_x)$ denotes the set of all I -statistical limit points of the sequence (x_n) .

Theorem 3.11. If (x_n) be a sequence such that $I\text{-}st\text{-}lim x_n = x_0$ then $I\text{-}S(A_x) = \{x_0\}$.

Proof. Since (x_n) is I -statistically convergent to x_0 , so for each $\varepsilon > 0$ and $\delta > 0$ the set,

$A = \{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\} \in I$, where I is an admissible ideal.

Suppose $I\text{-}S(A_x)$ contains y_0 different from x_0 . i.e, $y_0 \in I\text{-}S(A_x)$.

So there exist a $M \subset N$ such that $M \notin I$ and $st\text{-}lim X_{m_k} = y_0$.

Let $B = \{n \in M : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| \geq \delta\}$. So B is a finite set and therefore $B \in I$ and so $B^c = \{n \in M : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta\} \in \mathcal{F}(I)$.

Again let $A_1 = \{n \in M : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\}$. So $A_1 \subset A \in I$. i.e, $A_1^c \in \mathcal{F}(I)$. Therefore $A_1^c \cap B^c \neq \emptyset$, since $A_1^c \cap B^c \in \mathcal{F}(I)$

Let $p \in A_1^c \cap B^c$ and take $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$

so $\frac{1}{p} |\{k \leq p : d(x_k, x_0) \geq \varepsilon\}| < \delta$

and $\frac{1}{p} |\{k \leq p : d(x_k, y_0) \geq \varepsilon\}| < \delta$

i.e, for maximum $k \leq p$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus we must have,

$\{k \leq p : d(x_k, x_0) < \varepsilon\} \cap \{k \leq p : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of x_0 and y_0 are disjoint.

Hence $I\text{-}S(A_x) = \{x_0\}$. \square

Definition 3.12. [15] An element x_0 is said to be an I -statistical cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$

$\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\} \notin I$.

$I\text{-}S(\Gamma_x)$ denotes the set of all I -statistical cluster points of the sequence (x_n) .

Theorem 3.13. For any sequence $x = (x_n)$, $I\text{-}S(\Gamma_x)$ is closed.

Proof. Let y_0 be a limit point of the set $I\text{-}S(\Gamma_x)$ then for any $\varepsilon > 0$, $I\text{-}S(\Gamma_x) \cap B(y_0, \varepsilon) \neq \emptyset$, where $B(y_0, \varepsilon) = \{z \in R : d(z, y_0) < \varepsilon\}$

Let $z_0 \in I\text{-}S(\Gamma_x) \cap B(y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B(z_0, \varepsilon_1) \subseteq B(y_0, \varepsilon)$.

Then we have $\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d(x_k, y_0) \geq \varepsilon\}$

$\Rightarrow \frac{1}{n} |\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\}| \geq \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}|$

Now for any $\delta > 0$,

$\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\}| < \delta\}$

$$\subseteq \left\{ n \in N : \frac{1}{n} \left| \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} \right| < \delta \right\}$$

Since $z_0 \in I\text{-}S(\Gamma_x)$ therefore, $\left\{ n \in N : \frac{1}{n} \left| \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} \right| < \delta \right\} \notin I$.
i.e, $y_0 \in I\text{-}S(\Gamma_x)$. Hence the theorem is proved. \square

Theorem 3.14. For any sequence $x = (x_n)$, $I\text{-}S(\Lambda_x) \subseteq I\text{-}S(\Gamma_x)$.

Proof. Let $x_0 \in I\text{-}S(\Lambda_x)$. Then there exist a set $M = \{m_1 < m_2 < \dots\} \notin I$ such that, $st\text{-}lim x_{m_k} = x_0 \Rightarrow \lim_{k \rightarrow \infty} \frac{1}{k} \left| \{ m_i \leq k : d(x_{m_i}, x_0) \geq \varepsilon \} \right| = 0$.

Take $\delta > 0$, so there exist $k_0 \in N$ such that for $n > k_0$ we have,

$$\frac{1}{n} \left| \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} \right| < \delta.$$

$$\text{Let } A = \left\{ n \in N : \frac{1}{n} \left| \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} \right| < \delta \right\}.$$

Also, $A \supset M / \{m_1 < m_2 < \dots < m_{k_0}\}$. Since I is an admissible ideal and $M \notin I$, therefore $A \notin I$. So by definition of I -statistical cluster point $x_0 \in I\text{-}S(\Gamma_x)$.

Hence the theorem is proved. \square

Theorem 3.15. If $x = (x_n)$ and $y = (y_n)$ be two sequences such that

$\{n \in N : x_n \neq y_n\} \in I$, then

(i) $I\text{-}S(\Lambda_x) = I\text{-}S(\Lambda_y)$ and (ii) $I\text{-}S(\Gamma_x) = I\text{-}S(\Gamma_y)$.

Proof. (i) Let $x_0 \in I\text{-}S(\Lambda_x)$. So by definition there exist a set

$K = \{k_1 < k_2 < k_3 < \dots\}$ of N such that $K \notin I$ and $st\text{-}lim x_{k_n} = x_0$.

Since $\{n \in K : x_n \neq y_n\} \subset \{n \in N : x_n \neq y_n\} \in I$,

therefore $K' = \{n \in K : x_n = y_n\} \notin I$ and $K' \subseteq K$.

So we have $st\text{-}lim y_{k'_n} = x_0$.

This shows that $x_0 \in I\text{-}S(\Lambda_y)$ and therefore $I\text{-}S(\Lambda_x) \subseteq I\text{-}S(\Lambda_y)$.

By symmetry $I\text{-}S(\Lambda_y) \subseteq I\text{-}S(\Lambda_x)$.

Hence $I\text{-}S(\Lambda_y) = I\text{-}S(\Lambda_x)$.

(ii) Let $x_0 \in I\text{-}S(\Gamma_x)$. So by definition for each $\varepsilon > 0$ the set,

$$A = \left\{ n \in N : \frac{1}{n} \left| \{ k \leq n : d(x_k, x_0) \geq \varepsilon \} \right| < \delta \right\} \notin I.$$

Let $B = \left\{ n \in N : \frac{1}{n} \left| \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} \right| < \delta \right\}$. We have to prove that $B \notin I$.

Suppose $B \in I$. So, $B^c = \left\{ n \in N : \frac{1}{n} \left| \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} \right| \geq \delta \right\} \in \mathcal{F}(I)$.

By hypothesis the set $C = \{n \in N : x_n = y_n\} \in \mathcal{F}(I)$.

Therefore $B^c \cap C \in \mathcal{F}(I)$. Also it is clear that $B^c \cap C \subset A^c \in \mathcal{F}(I)$,

i.e, $A \in I$, which is a contradiction.

Hence $B \notin I$ and thus the result is proved. \square

Theorem 3.16. If g is a continuous function on X then it preserves I -statistical convergence in X .

Proof. Let $I\text{-}st \lim_{n \rightarrow \infty} x_n = \xi$.

Since g is continuous, then for each $\varepsilon_1 > 0$, there exist $\varepsilon_2 > 0$ such that if $x \in B(\xi, \varepsilon_1)$ then $g(x) \in B(g(\xi), \varepsilon_2)$.

Also we have,

$$C(\varepsilon_1, \delta) = \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| < \delta \right\} \in \mathcal{F}(I)$$

$$\text{Now, } \{k \leq n : d(x_k, \xi) \geq \varepsilon_1\} \supseteq \{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}$$

$$\text{so, } \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| \geq \frac{1}{n} |\{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}|$$

$$\text{for } \delta > 0, \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| < \delta \right\}$$

$$\subseteq \left\{ n \in N : \frac{1}{n} |\{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}| < \delta \right\} \in \mathcal{F}(I)$$

since $C(\varepsilon_1, \delta) \in \mathcal{F}(I)$.

Hence the theorem is proved. \square

ACKNOWLEDGMENTS

The authors are grateful to CSIR for their financial support to carry out this research work through the Project No. 25(0236)/14/EMR-II.

REFERENCES

1. R. C. Buck, The measure theoretic approach to density, *Amer. J. Math.*, **68**, (1946), 560-580.
2. R. C. Buck, Generalized asymptotic density, *Amer. J. Math.*, **75**, (1953), 335-346.
3. S. Debnath, J. Debnath, On I -statistically convergent sequence spaces defined by sequences of Orlicz functions using matrix transformation, *Proyecciones J. Math.*, **33**(3), (2014), 277-285.
4. K. Dems, On I -Cauchy sequences, *Real Anal. Exchange*, **30**(1), (2004/2005), 123-128.
5. M. Et, A. Alotaibi, S. A. Mohiuddine, On (Δ^m, I) -statistical convergence of order α , *The Scientific world journal*, vol 2014.
6. H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2**, (1951), 241-244.
7. J. A. Fridy, Statistical limit points, *Proc. Amer. Math. Soc.*, **4**, (1993), 1187-1192.
8. J. A. Fridy, On statistical convergence, *Analysis*, **5**, (1985), 301-313.
9. M. Gurdal, H. Sari, Extremal A -statistical limit points via ideals, *J. Egyptian Math. Soc.*, **22**(1), (2014), 55-58.
10. M. Gurdal, M. O. Ozgur, A generalized statistical convergence via moduli, *Electronic J. Math. Anal. & Appl.*, **3**(2), (2015), 173-178.
11. P. Kostyrko, M. Macaj, T. Salat, M. Slezziak, I -convergence and extremal I -limit points, *Math. Slov.*, **4**, (2005), 443-464.
12. P. Kostyrko, T. Salat, W. Wilczynski, I -convergence, *Real Anal. Exchange*, **26**(2), (2000/2001), 669-686.
13. M. Mamedov, S. Pehlivan, Statistical cluster points and turnpike theorem in non convex problems, *J. Math. Anal. Appl.*, **256**, (2001), 686-693.
14. D. S. Mitrinovic, J. Sandor, B. Crstici, *Handbook of Number Theory*, Kluwer Acad. Publ. Dordrecht-Boston-London, 1996.
15. M. Mursaleen, S. Debnath, D. Rakshit, I -statistical limit superior and I -statistical limit inferior, *Filomat*, **31**(7), (2017), 2103-2108.
16. E. Savas, P. Das, A generalized statistical convergence via ideals, *App. Math. Lett.*, **24**, (2011), 826-830.
17. E. Savas, P. Das, On I -statistically pre-Cauchy sequences, *Taiwanese J. Math.*, **18**(1), (2014), 115-126.
18. H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.*, **2**, (1951), 73-74.

19. T. Salat, On statistically convergent sequences of real numbers, *Math. Slov.*, **30**, (1980), 139-150.
20. T. Salat, B. Tripathy, M. Ziman, On some properties of I -convergence, *Tatra. Mt. Math. Publ.*, **28**, (2004), 279-286.
21. B. C. Tripathy, On statistically convergent and statistically bounded sequences, *Bull. Malaysian. Math. Soc.*, **20**, (1997), 31-33.
22. U. Yamanci, M. Gurdal, I -statistical convergence in 2-normed space, *Arab J. Math. Sci.*, **20**(1), (2014), 41-47.
23. U. Yamanci, M. Gurdal, I -statistically pre-Cauchy double sequences, *Global J. Math. Anal.*, **2**(4), (2014), 297-303.