On $I$-Statistical Convergence

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Abstract. In this paper we investigate the notion of $I$-statistical convergence and introduce $I$-st limit points and $I$-st cluster points of real number sequence and also studied some of its basic properties.

Keywords: $I$-limit point, $I$-cluster point, $I$-statistically Convergent.


1. Introduction

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of $I$-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal $I$ of subsets of the set $N$ to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of $I$-convergence was further extended to $I$-statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done.

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by Savas and Das [17], Debnath and Debnath [3], Mursaleen et.al [15], Et et al. [5] and many others [9, 10, 22, 23]. In [16], Savas and Das introduced the $I$-statistical convergence and $I$-$\lambda$-statistical convergence and the relation between them. Also they studied these concepts in the notion of $[V, \lambda]$-summability method.

In the present paper we return to the view of $I$-statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a $I$-statistical analogue of the set of limit points and cluster points of a real number sequence.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. [8] If $K$ is a subset of the positive integers $N$, then $K_n$ denotes the set \{ $k \in K : k \leq n$ \}. The natural density of $K$ is given by $D(K) = \lim_{n \to \infty} \frac{|K_n|}{n}$.

Definition 2.2. [8] A sequence $(x_n)$ is said to be statistically convergent to $x_0$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{ k \in N : d(x_k, x_0) \geq \varepsilon \}$ has natural density zero. $x_0$ is called the statistical limit of the sequence $(x_n)$ and we write $st\lim_{n \to \infty} x_n = x_0$.

Definition 2.3. [7] If $(x_{k(j)})$ be a subsequence of a sequence $x = (x_n)$ and density of $K = \{ k(j) : j \in N \}$ is zero then $(x_{k(j)})$ is called a thin subsequence. Otherwise $(x_{k(j)})$ is called a non-thin subsequence of $x$.

$x_0$ is said to be a statistical limit point of a sequence $(x_n)$, if there exist a non-thin subsequence of $(x_n)$ which converges to $x_0$.

Let $A_x$ denotes the set of all statistical limit points of the sequence $(x_n)$.

Definition 2.4. [7] $x_0$ is said to be a statistical cluster point of a sequence $x = (x_n)$, provided that for each $\varepsilon > 0$ the density of the set \{ $k \in N : d(x_k, x_0) < \varepsilon$ \} is not equal to 0.

Let $I_x$ denotes the set of all statistical cluster points of the sequence $(x_n)$.

Definition 2.5. [12] Let $X$ is a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal on $X$ if and only if

(i) $\emptyset \in I$;
(ii) for each $A, B \in I$ implies $A \cup B \in I$;
(iii) for each $A \in I$ and $B \subset A$ implies $B \in I$.

Definition 2.6. [12] Let $X$ is a non-empty set. A family of subsets $F \subset P(X)$ is called a filter on $X$ if and only if

(i) $\emptyset \notin F$;
(ii) for each $A, B \in F$ implies $A \cap B \in F$;
(iii) for each $A \in F$ and $B \supset A$ implies $B \in F$. 


An ideal $I$ is called non-trivial if $I \neq \emptyset$ and $X \notin I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal $I$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in $X$ if and only if $I \supset \{\{x\} : x \in X\}$.

**Definition 2.7.** [12] Let $I \subset P(N)$ be a non-trivial ideal on $N$. A sequence $(x_n)$ is said to be $I$-convergent to $x_0$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ belongs to $I$. $x_0$ is called the $I$-limit of the sequence $(x_n)$ and we write $I\lim_{n \to \infty} x_n = x_0$.

**Definition 2.8.** [12] $x_0$ is said to be $I$-limit point of a sequence $x = (x_n)$ provided that there is a subset $K = \{k_1 < k_2 < \ldots\} \subset N$ such that $K \notin I$ and $\lim x_{k_i} = x_0$.

Let $I(A_x)$ denotes the set of all $I$-limit points of the sequence $x$.

**Definition 2.9.** [12] $x_0$ is said to be $I$-cluster point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ the set $\{k \in N : d(x_k, x_0) < \varepsilon\} \notin I$.

Let $I(\Gamma_x)$ denotes the set of all $I$-cluster points of the sequence $x$.

**Definition 2.10.** [16] A sequence $x = (x_n)$ is said to be $I$-statistically convergent to $x_0$ if for every $\varepsilon > 0$ and every $\delta > 0$, \[ \{n \in N : \frac{1}{n} \sum_{k=1}^{n} [k \leq n : d(x_k, x_0) \geq \varepsilon] \geq \delta\} \in I. \]

$x_0$ is called $I$-statistical limit of the sequence $(x_n)$ and we write, $I-st\lim x_n = x_0$.

Throughout the paper we consider $I$ as an admissible ideal.

### 3. Main Results

**Theorem 3.1.** If $(x_n)$ be a sequence such that $I-st\lim x_n = x_0$, then $x_0$ determined uniquely.

**Proof.** If possible let the sequence $(x_n)$ be $I$-statistically convergent to two different numbers $x_0$ and $y_0$

i.e, for any $\varepsilon > 0$, $\delta > 0$ we have, 

$A_1 = \{n \in N : \frac{1}{n} \sum_{k=1}^{n} |k \leq n : d(x_k, x_0) \geq \varepsilon| < \delta\} \in \mathcal{F}(I)$

and $A_2 = \{n \in N : \frac{1}{n} \sum_{k=1}^{n} |k \leq n : d(x_k, y_0) \geq \varepsilon| < \delta\} \in \mathcal{F}(I)$

Therefore, $A_1 \cap A_2 \neq \emptyset$, since $A_1 \cap A_2 \in \mathcal{F}(I)$.

Let $m \in A_1 \cap A_2$ and take $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$

so, \[ \frac{1}{m} \sum_{k=1}^{m} |k \leq m : d(x_k, x_0) \geq \varepsilon| < \delta \]

and \[ \frac{1}{m} \sum_{k=1}^{m} |k \leq m : d(x_k, y_0) \geq \varepsilon| < \delta \]

i.e, for maximum $k \leq m$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus, we must have

$\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of $x_0$ and $y_0$ are disjoint.

Hence the theorem is proved. \[ \Box \]
Theorem 3.2. For any sequence \((x_n)\), \(\text{st-lim} x_n = x_0\) implies \(I\text{-st lim} x_n = x_0\).

Proof. Let \(\text{st-lim} x_n = x_0\).

Then for each \(\varepsilon > 0\), the set \(A(\varepsilon) = \{k \leq n : d(x_k, x_0) \geq \varepsilon\}\) has natural density zero.

i.e., \(\lim_{n \to \infty} \frac{1}{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} = 0\)

So for every \(\varepsilon > 0\) and \(\delta > 0\),

\(\{n \in N : \frac{1}{n} \{k \leq n : d(x_k, x_0) \geq \varepsilon\} \geq \delta\}\) is a finite set and therefore belongs to \(I\), as \(I\) is an admissible ideal.

Hence \(I\text{-st lim} x_n = x_0\). □

But the converse is not true. We can easily show this from example 3.5.

Example 3.3. Let \(I = \zeta\) be the class of \(A \subset N\) that intersect a finite number of \(\triangle_j\)'s where \(N = \bigcup_{j=1}^\infty \triangle_j\) and \(\triangle_i \cap \triangle_j = \emptyset\) for \(i \neq j\).

Let \(x_n = \frac{1}{n}\) and so \(\lim_{n \to \infty} d(x_n, 0) = 0\). Put \(\epsilon_n = d(x_n, 0)\) for \(n \in N\).

Now define a sequence \((y_n)\) by \(y_n = x_j\) if \(n \in \triangle_j\)

Let \(\eta > 0\). Choose \(\nu \in N\) such that \(\epsilon_\nu < \eta\). Then

\(A(\eta) = \{n : d(y_n, 0) \geq \eta\} \subset \triangle_1 \cup \ldots \cup \triangle_\nu \subset \zeta\).

Now, \(\{k \leq n : d(y_k, 0) \geq \eta\} \subset \{n \in N : d(y_n, 0) \geq \eta\}\)

i.e., \(\frac{1}{n} \{k \leq n : d(y_k, 0) \geq \eta\} \leq \{n \in N : d(y_n, 0) \geq \eta\}\)

so for any \(\delta > 0\),

\(\{n \in N : \frac{1}{n} \{k \leq n : d(y_k, 0) \geq \eta\} \geq \delta\} \subset \{n \in N : d(y_n, 0) \geq \eta\} \subset \zeta\).

Therefore \((y_n)\) is \(\zeta\)-statistically convergent to 0.

But \((y_n)\) is not a statistically convergent.

Theorem 3.4. For any sequence \((x_n)\), \(I\text{-lim} x_n = x_0\) implies \(I\text{-st lim} x_n = x_0\).

Proof. The proof is obvious. But the converse is not true. □

Example 3.5. If we take \(I = I_f\) the sequence \((x_n)\),

where \(x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & \text{otherwise} \end{cases}\)

is \(I\)-statistically convergent to 1. But \((x_n)\) is not \(I\)-convergent.

Theorem 3.6. If each subsequence of \((x_n)\) is \(I\)-statistically convergent to \(\xi\) then \((x_n)\) is also \(I\)-statistically convergent to \(\xi\).

Proof. Suppose \((x_n)\) is not \(I\)-statistically convergent to \(\xi\), then there exists \(\varepsilon > 0\) and \(\delta > 0\) such that

\(A = \{n \in N : \frac{1}{n} \{k \leq n : d(x_k, \xi) \geq \varepsilon\} \geq \delta\} \notin I\). Since \(I\) is admissible ideal so \(A\) must be an infinite set.

Let \(A = \{n_1 < n_2 < \ldots < n_m < \ldots\}\). Let \(y_m = x_{n_m}\) for \(m \in N\). Then \((y_m)_{m \in N}\) is a subsequence of \((x_n)\) which is not \(I\)-statistically convergent to \(\xi\), a contradiction. Hence the theorem is proved. □

But the converse is not true. We can easily show this from example 3.5.
Theorem 3.7. Let \((x_n)\) and \((y_n)\) be two sequences then
\(\text{(i) } \text{I-stlim } x_n = x_0 \text{ and } c \in \mathbb{R} \text{ implies I-stlim } cx_n = cx_0.\)
\(\text{(ii) } \text{I-stlim } x_n = x_0 \text{ and I-stlim } y_n = y_0 \text{ implies I-stlim } (x_n + y_n) = x_0 + y_0.\)

Proof. (i) If \(c = 0\), we have nothing to prove.

So we assume that \(c \neq 0\).

Now, \(\frac{1}{n} | \{ k \leq n : d(cx_k, cx_0) \geq \varepsilon \} | = \frac{1}{n} | \{ k \leq n : |c|d(x_k, x_0) \geq \varepsilon \} |
\leq \frac{1}{n} | \{ k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|} \} | < \delta \)

Therefore, \(\{ n \in N : \frac{1}{n} | \{ k \leq n : d(cx_k, cx_0) \geq \varepsilon \} | < \delta \} \in \mathcal{F}(I) \).

i.e, \(\text{I-stlim } cx_n = cx_0.\)

(ii) We have \(A_1 = \{ n \in N : \frac{1}{n} | \{ k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2} \} | < \frac{\delta}{2} \} \in \mathcal{F}(I)\)
and \(A_2 = \{ n \in N : \frac{1}{n} | \{ k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2} \} | < \frac{\delta}{2} \} \in \mathcal{F}(I)\).

Since \(A_1 \cap A_2 = \emptyset\), therefore for all \(n \in A_1 \cap A_2\) we have,
\(\frac{1}{n} | \{ k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon \} | < \delta.
\)
i.e, \(\{ n \in N : \frac{1}{n} | \{ k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon \} | < \delta \} \in \mathcal{F}(I)\).

Hence \(\text{I-stlim } (x_n + y_n) = (x_0 + y_0).\)

Definition 3.8. A sequence \(x = (x_n)_{n \in N}\) of elements of \(X\) is said to be I*-statistical convergent to \(\xi \in X\) if and only if there exists a set \(M = \{ m_1 < m_2 < \ldots < m_k < \ldots \} \in \mathcal{F}(I)\), such that \(\text{st-lim } d(x_{m_k}, \xi) = 0.\)

Theorem 3.9. If \(I^* - \text{stlim}_{n \to \infty} x_n = \xi\) then \(I^* - \text{stlim}_{n \to \infty} x_n = \xi.\)

Proof. Let \(I^* - \text{stlim}_{n \to \infty} x_n = \xi.\) By assumption there exist a set \(H \in I\) such that for \(M = N \setminus H = \{ m_1 < m_2 < \ldots < m_k < \ldots \}\) we have \(\text{st-lim } x_{m_k} = \xi,\)
i.e, \(\text{lim}_{n \to \infty} \frac{1}{n} | \{ m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon \} | = 0.\)

so for any \(\delta > 0\), \(\{ n \in N : \frac{1}{n} | \{ m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon \} | \geq \delta \} \in I\) since \(I\) is an admisible ideal.

Now, \(A(\varepsilon, \delta) = \{ n \in N : \frac{1}{n} | \{ k \leq n : d(x_k, \xi) \geq \varepsilon \} | \geq \delta \}
\subset H \cup \{ n \in N : \frac{1}{n} | \{ m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon \} | \geq \delta \} \in I\)
i.e, \(I^* - \text{stlim}_{n \to \infty} x_n = \xi.\)

But the converse may not be true.

From example 3.3, we have \(\zeta^* - \text{stlim}_{n \to \infty} y_n = 0.\)

Suppose that \(\zeta^* - \text{stlim}_{n \to \infty} y_n = 0.\) Then there exist a set \(H \in \zeta\) such that for \(M = N \setminus H = \{ m_1 < m_2 < \ldots < m_k < \ldots \}\) we have \(\text{st-lim } y_{m_k} = 0.\) By definition of \(\zeta\) there exist a \(p \in N\) such that \(H \subset \Delta_1 \cup \ldots \cup \Delta_p.\) But then \(\Delta_{p+1} \subset M,\) so for infinitely many \(m_k \in \Delta_{p+1},\)
\(D \{ m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta \} = 2^{-(p+1)} > 0 \text{ for } 0 < \eta < \frac{1}{p+1};\)
i.e, \(D \{ m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \geq \eta \} \neq 0,\) which is a contradicts \(\text{st-lim } y_{m_k} = 0.\)

Hence \(\zeta^* - \text{stlim}_{n \to \infty} y_n \neq 0.\)
Definition 3.10. An element \( x_0 \) is said to be an \( I \)-statistical limit point of a sequence \( x = (x_n) \) provided that for each \( \varepsilon > 0 \) there is a set \( M = \{m_1 < m_2 < \ldots \} \subset N \) such that \( M \notin I \) and st-lim \( x_{m_k} = x_0 \).

\( I - S (A_x) \) denotes the set of all \( I \)-statistical limit points of the sequence \((x_n)\).

Theorem 3.11. If \((x_n)\) be a sequence such that \( I - \lim x_n = x_0 \) then \( I - S (A_x) = \{x_0\} \).

Proof. Since \((x_n)\) is \( I \)-statistically convergent to \( x_0 \), so for each \( \varepsilon > 0 \) and \( \delta > 0 \) the set,

\[ A = \{n \in N : \frac{1}{n} \{k \leq n : d(x_k,x_0) \geq \varepsilon\} \geq \delta\} \in I, \text{ where } I \text{ is an admisible ideal.} \]

Suppose \( I - S (A_x) \) contains \( y_0 \) different from \( x_0 \), i.e, \( y_0 \in I - S (A_x) \).

So there exist a \( M \subset N \) such that \( M \notin I \) and st-lim \( X_{m_k} = y_0 \).

Let \( B = \{n \in M : \frac{1}{n} \{k \leq n : d(x_k,y_0) \geq \varepsilon\} \geq \delta\} \). So \( B \) is a finite set and therefore \( B \in I \) and so \( B^c = \{n \in M : \frac{1}{n} \{k \leq n : d(x_k,y_0) \geq \varepsilon\} < \delta\} \in \mathcal{F} (I) \).

Again let \( A_1 = \{n \in M : \frac{1}{n} \{k \leq n : d(x_k,x_0) \geq \varepsilon\} \geq \delta\} \). So \( A_1 \subset A \in I \).

i.e, \( A_1 \in \mathcal{F} (I) \). Therefore \( A_1 \cap B^c \neq \emptyset \), since \( A_1 \cap B^c \in \mathcal{F} (I) \)

Let \( p \in A_1 \cap B^c \) and take \( \varepsilon = \frac{d(x_k,y_0)}{p} > 0 \)

so \( \frac{1}{n} \{k \leq p : d(x_k,x_0) \geq \varepsilon\} \leq \delta \)

and \( \frac{1}{n} \{k \leq p : d(x_k,y_0) \geq \varepsilon\} \leq \delta \),

i.e, for maximum \( k \leq p \) will satisfy \( d(x_k,x_0) < \varepsilon \) and \( d(x_k,y_0) < \varepsilon \) for a very small \( \delta > 0 \).

Thus we must have,

\[ \{k \leq p : d(x_k,x_0) < \varepsilon\} \cap \{k \leq p : d(x_k,y_0) < \varepsilon\} \neq \emptyset \]

a contradiction, as the neighbourhood of \( x_0 \) and \( y_0 \) are disjoint.

Hence \( I - S (A_x) = \{x_0\} \).

Definition 3.12. [15] An element \( x_0 \) is said to be an \( I \)-statistical cluster point of a sequence \( x = (x_n) \) if for each \( \varepsilon > 0 \) and \( \delta > 0 \)

\[ \{n \in N : \frac{1}{n} \{k \leq n : d(x_k,x_0) \geq \varepsilon\} \leq \delta\} \notin I. \]

\( I - S (\Gamma_x) \) denotes the set of all \( I \)-statistical cluster points of the sequence \((x_n)\).

Theorem 3.13. For any sequence \( x = (x_n) \), \( I - S (\Gamma_x) \) is closed.

Proof. Let \( y_0 \) be a limit point of the set \( I - S (\Gamma_x) \) then for any \( \varepsilon > 0 \), \( I - S (\Gamma_x) \cap B (y_0, \varepsilon) \neq \emptyset \), where \( B (y_0, \varepsilon) = \{z \in R : d(z,y_0) < \varepsilon\} \)

Let \( z_0 \in I - S (\Gamma_x) \cap B (y_0, \varepsilon) \) and choose \( \varepsilon_1 > 0 \) such that \( B (z_0, \varepsilon_1) \subseteq B (y_0, \varepsilon) \).

Then we have \( \{k \leq n : d(x_k,z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d(x_k,y_0) \geq \varepsilon\} \)

\[ \Rightarrow \frac{1}{n} \{k \leq n : d(x_k,z_0) \geq \varepsilon_1\} \geq \frac{1}{n} \{k \leq n : d(x_k,y_0) \geq \varepsilon\} \]

Now for any \( \delta > 0 \),

\[ \{n \in N : \frac{1}{n} \{k \leq n : d(x_k,z_0) \geq \varepsilon_1\} < \delta\} \]
\[ \subseteq \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} < \delta \} \]

Since \( z_0 \in I-S(\Gamma_x) \) therefore, \( \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, y_0) \geq \varepsilon \} < \delta \} \notin I \).

i.e, \( y_0 \in I-S(\Gamma_x) \). Hence the theorem is proved. \( \square \)

**Theorem 3.14.** For any sequence \( x = (x_n) \), \( I-S(A_x) \subseteq I-S(\Gamma_x) \).

**Proof.** Let \( x_0 \in I-S(A_x) \). Then there exist a set \( M = \{ m_1 < m_2 < ... \} \notin I \) such that, st-lim \( x_m = x_0 \Rightarrow \lim_{k \to +\infty} \frac{1}{n} \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} = 0 \).

Take \( \delta > 0 \), so there exist \( k_0 \in N \) such that for \( n > k_0 \) we have,

\[ \frac{1}{n} \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} < \delta. \]

Let \( A = \{ n \in N : \frac{1}{n} \{ m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon \} < \delta \}. \)

Also, \( A \supseteq M/ \{ m_1 < m_2 < ... < m_{k_0} \} \). Since \( I \) is an admissible ideal and \( M \notin I \), therefore \( A \notin I \). So by definition of \( I \)-statistical cluster point \( x_0 \in I-S(\Gamma_x) \).

Hence the theorem is proved. \( \square \)

**Theorem 3.15.** If \( x = (x_n) \) and \( y = (y_n) \) be two sequences such that \( \{ n \in N : x_n \neq y_n \} \notin I \), then

(i) \( I-S(A_x) = I-S(A_y) \) and (ii) \( I-S(\Gamma_x) = I-S(\Gamma_y) \).

**Proof.** (i) Let \( x_0 \in I-S(A_x) \). So by definition there exist a set \( K = \{ k_1 < k_2 < k_3 < ... \} \) of \( N \) such that \( K \notin I \) and \( \text{st-lim } x_{k_n} = x_0 \).

Since \( \{ n \in K : x_n \neq y_n \} \subset \{ n \in N : x_n \neq y_n \} \notin I \), therefore \( K' = \{ n \in K : x_n = y_n \} \notin I \) and \( K' \subseteq K \).

So we have \( \text{st-lim } y_{k_n} = x_0 \).

This shows that \( x_0 \in I-S(A_y) \) and therefore \( I-S(A_x) \subseteq I-S(A_y) \).

By symmetry \( I-S(A_y) \subseteq I-S(A_x) \).

Hence \( I-S(A_y) = I-S(A_x) \).

(ii) Let \( x_0 \in I-S(\Gamma_x) \). So by definition for each \( \varepsilon > 0 \) the set,

\[ A = \{ n \in N : \frac{1}{n} \{ k \leq n : d(x_k, x_0) \geq \varepsilon \} < \delta \} \notin I. \]

Let \( B = \{ n \in N : \frac{1}{n} \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} < \delta \} \). We have to prove that \( B \notin I \).

Suppose \( B \in I \). So, \( B^c = \{ n \in N : \frac{1}{n} \{ k \leq n : d(y_k, x_0) \geq \varepsilon \} \geq \delta \} \in F(I) \).

By hypothesis the set \( C = \{ n \in N : x_n = y_n \} \in F(I) \).

Therefore \( B^c \cap C \in F(I) \). Also it is clear that \( B^c \cap C \subseteq A^c \in F(I) \), i.e, \( A \in I \), which is a contradiction.

Hence \( B \notin I \) and thus the result is proved. \( \square \)

**Theorem 3.16.** If \( g \) is a continuous function on \( X \) then it preserves \( I \)-statistical convergence in \( X \).

**Proof.** Let \( I \text{-st lim}_{n \to +\infty} x_n = \xi \).

Since \( g \) is continuous, then for each \( \varepsilon_1 > 0 \), there exist \( \varepsilon_2 > 0 \) such that if \( x \in B(\xi, \varepsilon_1) \) then \( g(x) \in B(g(\xi), \varepsilon_2) \).
Also we have,
\[ C(\varepsilon_1, \delta) = \left\{ n \in N : \frac{1}{n} \left| \left\{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \right\} \right| < \delta \right\} \in F(I) \]
Now, \( \left\{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \right\} \supseteq \left\{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \right\} \)
so, \( \frac{1}{n} \left| \left\{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \right\} \right| > \frac{1}{n} \left| \left\{ k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2 \right\} \right| \)
for \( \delta > 0, \left\{ n \in N : \frac{1}{n} \left| \left\{ k \leq n : d(x_k, \xi) \geq \varepsilon_1 \right\} \right| < \delta \right\} \)
since \( C(\varepsilon_1, \delta) \in F(I) \).
Hence the theorem is proved. \( \square \)

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