Let \( R \) be a prime ring with its Utumi ring of quotients \( U \), \( C = Z(U) \) the extended centroid of \( R \), \( L \) a non-central Lie ideal of \( R \) and \( 0 \neq a \in R \). If \( R \) admits a generalized derivation \( F \) such that \( a(F(u^2) \pm F(u))^2 = 0 \) for all \( u \in L \), then one of the following holds:

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = \mp x \) for all \( x \in R \);
3. \( \text{char } (R) = 2 \) and \( R \) satisfies \( s_4 \);
4. \( \text{char } (R) \neq 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

We also study the situations (i) \( a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \), and (ii) \( a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \) in prime and semiprime rings.

Keywords: Prime ring, Generalized derivation, Utumi quotient ring.


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1. Introduction

Let $R$ be an associative prime ring with center $Z(R)$ and $U$ the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \to R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, $d$ is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let $S$ be a nonempty subset of $R$ and $F : R \to R$ be an additive mapping. Then we say that $F$ acts as homomorphism or anti-homomorphism on $S$ if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping $F$ acts as a Jordan homomorphism on $S$ if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. Asma, Rehman, Shakir in [1] proved that if $d$ is a derivation of a 2-torsion free prime ring $R$ which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$, then $d \equiv 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation $F$ acts as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring $R$ as well as associated map $F$ (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let $R$ be a prime ring of characteristic different from 2, $F$ a generalized derivation of $R$ associated to a derivation $d$, $L$ a Lie ideal of $R$ such that $u^2 \in L$ for all $u \in L$. If $F$ acts as a homomorphism or anti-homomorphism on $L$, then either $d \equiv 0$ or $L \subseteq Z$. More recently in [9], Filippis studied the situation when generalized derivation $F$ acts as a Jordan homomorphism on a non-central Lie ideal $L$ of $R$.

Recently in [26], Rehman and Raza proved the following: Let $R$ be a prime ring of char $(R) \neq 2$, $Z$ the center of $R$, and $L$ a nonzero Lie ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ which acts as a homomorphism or as anti-homomorphism on $L$, then either $d \equiv 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map $F$.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map $F$. The main results of this paper are as follows:
Theorem 1.1. Let \( R \) be a prime ring with its Utumi ring of quotients \( U, C = Z(U) \) the extended centroid of \( R, L \) a non-central Lie ideal of \( R \) and \( 0 \neq a \in R \). If \( R \) admits a generalized derivation \( F \) such that \( a(F(u^2) \pm F(u)^2) = 0 \) for all \( u \in L \), then one of the following holds:

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = \mp x \) for all \( x \in R \);
3. \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \);
4. \( \text{char}(R) \neq 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

Theorem 1.2. Let \( R \) be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients \( U, C = Z(U) \) the extended centroid of \( R, F \) a generalized derivation on \( R \) and \( 0 \neq a \in R \).

1. if \( a(F(x^m y^n) \pm F(x^m) F(y^n)) = 0 \) for all \( x, y \in R \), then there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \) or \( F(x) = \mp x \) for all \( x \in R \).
2. if \( a(F(x^m y^n) \pm F(y^n) F(x^m)) = 0 \) for all \( x, y \in R \), then there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \).

Theorem 1.3. Let \( R \) be a noncommutative 2-torsion free semiprime ring, \( U \) the left Utumi quotient ring of \( R, C = Z(U) \) the extended centroid of \( R, F(x) = bx + d(x) \) a generalized derivation on \( R \) associated to the derivation \( d \) and \( 0 \neq a \in R \). If any one of the following holds:

1. \( a(F(x^m y^n) \pm F(x^m) F(y^n)) = 0 \) for all \( x, y \in R \),
2. \( a(F(x^m y^n) \pm F(y^n) F(x^m)) = 0 \) for all \( x, y \in R \),
then there exist orthogonal central idempotents \( e_1, e_2, e_3 \in U \) with \( e_1 + e_2 + e_3 = 1 \) such that \( d(e_1 U) = 0, e_2 a = 0 \), and \( e_3 U \) is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let \( R \) be a prime ring and \( L \) a noncentral Lie ideal of \( R \). If \( \text{char}(R) \neq 2 \), by [4, Lemma 1] there exists a nonzero ideal \( I \) of \( R \) such that \( 0 \neq [I, R] \subseteq L \). If \( \text{char}(R) = 2 \) and \( \dim_C RC > 4 \), i.e., \( \text{char}(R) = 2 \) and \( R \) does not satisfy \( s_4 \), then by [22, Theorem 13] there exists a nonzero ideal \( I \) of \( R \) such that \( 0 \neq [I, R] \subseteq L \). Thus if either \( \text{char}(R) \neq 2 \) or \( R \) does not satisfy \( s_4 \), then we may conclude that there exists a nonzero ideal \( I \) of \( R \) such that \( [I, I] \subseteq L \).

Remark 1.5. We denote by \( \text{Der}(U) \) the set of all derivations on \( U \). By a derivation word \( \Delta \) of \( R \) we mean \( \Delta = d_1 d_2 d_3 \ldots d_m \) for some derivations \( d_i \in \text{Der}(U) \).

Let \( D_{in} \) be the \( C \)-subspace of \( \text{Der}(U) \) consisting of all inner derivations on \( U \) and let \( d \) be a non-zero derivation on \( R \). By [21, Theorem 2] we have the following result:
If \( \Phi(x_1, x_2, \ldots, x_n, d(x_1), d(x_2) \cdots d(x_n)) \) is a differential identity on \( R \), then one of the following holds:

1. \( d \in D_{int} \);
2. \( R \) satisfies the generalized polynomial identity \( \Phi(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \).

**Remark 1.6.** In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping \( F : I \to U \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in I \), where \( I \) is a dense left ideal of \( R \) and \( d \) is a derivation from \( I \) into \( U \). Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of \( U \), and thus all generalized derivations of \( R \) will be implicitly assumed to be defined on the whole of \( U \). Lee obtained the following: every generalized derivation \( F \) on a dense left ideal of \( R \) can be uniquely extended to \( U \) and assumes the form \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \).

2. **Proof of the Main Results**

Now we begin with the following Lemmas:

**Lemma 2.1.** Let \( R = M_2(C) \) be the ring of all \( 2 \times 2 \) matrices over the field \( C \) of characteristic different from 2 and \( b, c \in R \). Suppose that there exists \( 0 \neq a \in R \) such that

\[
\alpha \{(b[x, y]^2 + [x, y]^2) - (b[x, y] + [x, y]c)^2\} = 0,
\]

for all \( x, y \in R \). Then \( c \in C \cdot I_2 \).

**Proof.** If \( c \in C \cdot I_2 \), then nothing to prove. Let \( c \notin C \cdot I_2 \). In this case \( R \) is a dense ring of \( C \)-linear transformations over a vector space \( V \). Assume that there exists \( 0 \neq v \in V \) such that \( \{v, cv\} \) is linearly \( C \)-independent. By density, there exist \( x, y \in R \) such that \( xv = v, xcv = 0; yv = 0, ycv = v \). Then \( [x, y]v = 0, [x, y]cv = v \) and hence \( \alpha \{(b[x, y]^2 + [x, y]^2) - (b[x, y] + [x, y]c)^2\}v = av \).

Of course for any \( u \in V \), \( \{u, v\} \) linearly \( C \)-independent implies \( au = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and so \( \{w, v\} \) are linearly \( C \)-independent. Also \( a(w + v) = aw \neq 0 \) and \( a(w - v) = aw \neq 0 \). By the above argument, it follows that \( w \) and \( cw \) are linearly \( C \)-dependent, as are \( \{w + v, c(w + v)\} \) and \( \{w - v, c(w - v)\} \). Therefore there exist \( \alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C \) such that

\[
cw = \alpha_w w, \quad c(w+v) = \alpha_{w+v}(w+v), \quad c(w-v) = \alpha_{w-v}(w-v).
\]

In other words we have

\[
\alpha_w w + cw = \alpha_{w+v} w + \alpha_{w+v} v \tag{2.1}
\]

and

\[
\alpha_w w - cw = \alpha_{w-v} w - \alpha_{w-v} v. \tag{2.2}
\]
By comparing (2.1) with (2.2) we get both
\[(2a_w - a_{w+v} - a_{w-v})w + (a_{w-v} - a_{w+v})v = 0 \tag{2.3}\]
and
\[2cv = (a_{w+v} - a_{w-v})w + (a_{w+v} + a_{w-v})v. \tag{2.4}\]
By (2.3), and since \(\{w, v\}\) are \(C\)-independent and \(\text{char } (R) \neq 2\), we have
\[a_w = a_{w+v} = a_{w-v}.\]
Thus by (2.4) it follows \(2cv = 2a_wv\). This leads a contradiction with the fact that \(v, cv\) is linear \(C\)-independent.

In light of this, we may assume that for any \(v \in V\) there exists a suitable \(\alpha_v \in C\) such that \(cv = \alpha_vv\), and standard argument shows that there is \(\alpha \in C\) such that \(cv = \alpha v\) for all \(v \in V\). Now let \(r \in R, v \in V\). Since \(cv = \alpha v\),
\[[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.\]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r]\) = 0 for all \(r \in R\). Therefore, \(c \in Z(R)\), a contradiction.

**Lemma 2.2.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from 2 and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^my^n - px^mpy^n) = 0,\]
for all \(x, y \in R\). Then either \(ap = 0\) or \(p = 1\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). In this case \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = 0\). Then we get
\[0 = a(px^my^n - px^mpy^n)v = apv.\]
Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), from \(ap = ap^2\), we get \(0 = a(p - 1)\). Since \(a \neq 0\), we conclude \(p = 1\).

**Lemma 2.3.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from 2 and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^my^n - py^npx^m) = 0,\]
for all \(x, y \in R\). Then \(ap = 0\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). Here \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = pv\). Then we have
\[0 = a(px^my^n - py^npx^m)v = -ap^2v = -apv.\]
Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), by hypothesis, we get \(0 = a[x^m, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have
0 = a(x^m, y^n) = a[e_{11}, e_{11} + e_{12}] = ae_{12}. Again, for \( x = e_{22} \) and \( y = e_{22} + e_{21} \), we have \( 0 = a(x^m, y^n) = a[e_{22}, e_{22} + e_{21}] = ae_{21} \). These imply \( a = 0 \), a contradiction. □

**Lemma 2.4.** Let \( R \) be a noncommutative prime ring with extended centroid \( C \) and \( b, c \in R \). Suppose that \( 0 \neq a \in R \) such that

\[
0 = a\{(b[x, y]^2 + [x, y]^2)c - (b[x, y] + [x, y]c)^2\} = 0
\]

for all \( x, y \in R \). Then one of the following holds:

1. \( c \in C \) and \( a(b + c) = 0 \);
2. \( b, c \in C \) and \( b + c = 1 \);
3. \( \text{char} (R) = 2 \) and \( R \) satisfies \( s_4 \);
4. \( \text{char} (R) \neq 2 \), \( R \) satisfies \( s_4 \) and \( c \in C \).

**Proof.** By assumption, \( R \) satisfies the generalized polynomial identity (GPI)

\[
f(x, y) = a\{(b[x, y]^2 + [x, y]^2)c - (b[x, y] + [x, y]c)^2\}.
\]

By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by \( U \). Now we consider the following two cases:

**Case-I.** \( U \) does not satisfy any nontrivial GPI.

Let \( T = U \ast C \{x, y\} \), the free product of \( U \) and \( C \{x, y\} \), the free \( C \)-algebra in noncommuting indeterminates \( x \) and \( y \). Thus

\[
a\{(b[x, y]^2 + [x, y]^2)c - (b[x, y] + [x, y]c)^2\}
\]

is zero element in \( T = U \ast C \{x, y\} \). Let \( c \notin C \). Then \( \{1, c\} \) is \( C \)-independent. Then from above

\[
a\{x, y\}^2c - (b[x, y] + [x, y]c)x, y\}
\]

which is

\[
a\{x, y\} - b[x, y] - [x, y]c\}
\]

is zero in \( T \). Again, since \( c \notin C \), we have that \( a[x, y]c[x, y]c \) is zero element in \( T \), implying \( a = 0 \) or \( c = 0 \), a contradiction. Thus we conclude that \( c \in C \). Then the identity reduces to

\[
a\{(b + c)[x, y] - (b + c)[x, y](b + c)\}
\]

is zero element in \( T \). Again, if \( b + c \notin C \), then \( a(b + c)[x, y]^2 \) becomes zero element in \( T \), implying \( a(b + c) = 0 \). If \( b + c \in C \), then \( a(b + c)(b + c - 1)[x, y]^2 \) becomes zero element in \( T \), implying \( b + c = 0 \) or \( b + c = 1 \). When \( b + c = 0 \), then \( a(b + c) = 0 \), which is our conclusion (1). When \( b + c = 1 \), then \( b = 1 - c \in C \), which is our conclusion (2).

**Case-II.** \( U \) satisfies a nontrivial GPI.
Thus we assume that
\[ a \{ (b[x, y]^2 + [x, y]^2 c) - (b[x, y] + [x, y]c)^2 \} = 0, \]
is a nontrivial GPI for \( U \). In case \( C \) is infinite, we have \( f(x, y) = 0 \) for all \( x, y \in U \otimes_C \overline{C} \), where \( \overline{C} \) is the algebraic closure of \( C \). Since both \( U \) and \( U \otimes_C \overline{C} \) are prime and centrally closed [17], we may replace \( R \) by \( U \) or \( U \otimes_C \overline{C} \) according to \( C \) finite or infinite. Thus we may assume that \( R \) centrally closed over \( C \) which either finite or algebraically closed and \( f(x, y) = 0 \) for all \( x, y \in R \).

By Martindale’s Theorem [25], \( R \) is then primitive ring having non-zero socle \( soc(R) \) with \( C \) as the associated division ring. Hence by Jacobson’s Theorem [20], \( R \) is isomorphic to a dense ring of linear transformations of a vector space \( V \) over \( C \). Since \( R \) is noncommutative, \( \dim_C V \geq 2 \). If \( \dim_C V = 2 \), then \( R \cong M_2(C) \). In this case by Lemma 2.1, either \( c \in C \) or \( char \,(R) = 2 \). This gives conclusions (3) and (4).

Let \( \dim_C V \geq 3 \). Let for some \( v \in V \), \( v \) and \( cv \) are linearly independent over \( C \). By density there exist \( x, y \in R \) such that
\[ xv = v, \quad xcv = 0; \]
\[ yv = 0, \quad ycv = v. \]

Then \( [x, y]v = 0 \), \( [x, y]cv = v \) and hence \( a \{ (b[x, y]^2 + [x, y]^2 c) - (b[x, y] + [x, y]c)^2 \} v = av. \)

This implies that if \( av \neq 0 \), then by contradiction we may conclude that \( v \) and \( cv \) are linearly \( C \)-dependent. Now choose \( v \in V \) such that \( v \) and \( cv \) are linearly \( C \)-independent. Set \( W = \text{Span}_C \{ v, cv \} \). Then \( av = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and then \( a(v - w) = aw \neq 0 \). By the previous argument we have that \( w, cw \) are linearly \( C \)-dependent and \( (v - w), c(v - w) \) too. Thus there exist \( \alpha, \beta \in C \) such that \( cw = \alpha w \) and \( c(v - w) = \beta(v - w) \). Then \( cv = \beta(v - w) + cw = \beta(v - w) + aw \) i.e., \( (\alpha - \beta)w = cv - \beta v \in W \). Now \( \alpha = \beta \) implies that \( cv = \beta v \), a contradiction. Hence \( \alpha \neq \beta \) and so \( w \in W \). Again, if \( u \in V \) with \( au = 0 \) then \( a(w + u) \neq 0 \). So, \( w + u \in W \) forcing \( u \in W \). Thus it is observed that \( w \in V \) with \( aw \neq 0 \) implies \( w \in W \) and \( u \in V \) with \( au = 0 \) implies \( u \in W \). This implies that \( V = W \) i.e., \( \dim_C V = 2 \), a contradiction.

Hence, in any case, \( v \) and \( cv \) are linearly \( C \)-dependent for all \( v \in V \). Thus for each \( v \in V \), \( cv = \alpha_v v \) for some \( \alpha_v \in C \). It is very easy to prove that \( \alpha_v \) is independent of the choice of \( v \in V \). Thus we can write \( cv = \alpha v \) for all \( v \in V \) and \( \alpha \in C \) fixed. Now let \( r \in R, v \in V \). Since \( cv = \alpha v \),
\[ [c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0. \]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\).

Thus our identity reduces to

\[a((b'[x, y]^2) - (b'[x, y])^2) = 0,\]

for all \(x, y \in R\), where \(b' = b + c\).

Let for some \(v \in V\), \(v\) and \(b'v\) are linearly independent over \(C\). Since \(\dim_C V \geq 3\), there exists \(u \in V\) such that \(v, b'v, u\) are linearly independent over \(C\). By density there exist \(x, y \in R\) such that

\[xv = v, \quad xb'v = 0, \quad xu = v;\]

\[yv = 0, \quad yb'v = u, \quad yu = v.\]

Then \([x, y]v = 0, [x, y]b'v = v, [x, y]u = v\) and hence \(0 = a((b'[x, y]^2) - (b'[x, y])^2)u = ab'v\). Then by same argument as before, we have either \(ab' = 0\) or \(v\) and \(b'v\) are linearly dependent for all \(v \in V\). In the first case, \(0 = ab' = a(b + c)\), which is conclusion (1). In the last case, again by standard argument, we have that \(b' \in C\). If \(b' = 0\), then also \(ab' = a(b + c) = 0\) which gives conclusion (1). So assume that \(0 \neq b' \in C\). Then our identity reduces to

\[ab'(b' - 1)[x, y]^2 = 0,\]

for all \(x, y \in R\). This gives \(0 = ab'(b' - 1) = a(b' - 1)\). Since \(a \neq 0\), we get \(b' = 1\). This gives conclusion (2). \(\square\)

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when

\[a(F(u^2) - F(u)^2) = 0,\]

for all \(u \in L\). If char \((R) = 2\) and \(R\) satisfies \(s_4\), then we have our conclusion (3). So we assume that either char \((R) \neq 2\) or \(R\) does not satisfy \(s_4\). Since \(L\) is a noncentral by Remark 1.4, there exists a nonzero ideal \(I\) of \(R\) such that \([I, I] \subseteq L\). Thus by assumption \(I\) satisfies the differential identity

\[a(F([x, y]^2) - F([x, y])^2) = 0.\]

Now since \(R\) is a prime ring and \(F\) is a generalized derivation of \(R\), by Lee \[23, Theorem 3\], \(F(x) = bx + d(x)\) for some \(b \in U\) and \(b'v\) for some \(b \in U\) and \(d\) on \(U\). Since \(I, R\) and \(U\) satisfy the same differential identities \[24\], without loss of generality, \(U\) satisfies

\[a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0.\] \quad (2.5)

Here we divide the proof into two cases:
Case 1. Let $d$ be inner derivation induced by element $c \in U$, that is, $d(x) = [c,x]$ for all $x \in R$. It follows that
\[ a(b(x,y))^2 + [c,[x,y]]^2 - (b(x,y) + [c,[x,y]])^2 = 0, \]
that is
\[ a((b+c)[x,y]^2 - [x,y]^2c - ((b+c)[x,y] - [x,y]c)^2 = 0, \]
for all $x, y \in U$. Now by Lemma 2.4, one of the following holds:
(1) $c \in C$ and $0 = a(b+c-c) = ab$. Thus $F(x) = bx$ for all $x \in R$, with $ab = 0$.
(2) $b + c, c \in C$ and $b + c - c = 1$. Thus $F(x) = x$ for all $x \in R$.
(3) char $(R) \neq 2$, $R$ satisfies $s_4$ and $c \in C$. Thus $F(x) = bx$ for all $x \in R$.

Case 2. Assume that $d$ is not inner derivation of $U$. We have from (2.5) that $U$ satisfies
\[ a(b(x,y))^2 + d([x,y])[x,y] + [x,y]d([x,y]) - (b(x,y) + d([x,y]))^2 = 0, \]
that is
\[ a(b(x,y))^2 + ([d(x),y] + [x,d(y)])[x,y] + [x, y](d([x,y])) - (b(x,y) + [x,d(y)])^2 = 0. \]
Then by Kharchenko’s Theorem [21], $U$ satisfies
\[ a(b(x,y))^2 + ([u,y] + [x,z])[x,y] + [x,y]([u,y] + [x,z]) - (b(x,y) + [u,y] + [x,z])^2 = 0. \]
Since $R$ is noncommutative, we may choose $q \in U$ such that $q \notin C$. Then replacing $u$ by $[q,x]$ and $z$ by $[q,y]$ in (2.6), we get
\[ a(b(x,y))^2 + ([q,x],y) + [x,[q,y]]=[x,y]([q,x] + [x,[q,y]] - (b(x,y) + [q,y])^2 = 0, \]
which is
\[ a(b(x,y))^2 + [q,[x,y]^2] - (b(x,y) + [q,[x,y]^2])^2 = 0. \]
Then by Lemma 2.4, we have $q \in C$, a contradiction.

Now replacing $F$ with $-F$ in the above result, we obtain the conclusion for the situation $a(F(a^2)) + F(a^2) = 0$ for all $a \in L$.

**Corollary 2.5.** Let $R$ be a prime ring with extended centroid $C$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits the generalized derivation $F$ such that either $a(F(XY) \pm F(X)F(Y)) = 0$ for all $X, Y \in L$ or $a(F(XY) \pm F(Y)F(X)) = 0$ for all $X, Y \in L$, then one of the following holds:
(1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
(2) $F(x) = \mp x$ for all $x \in R$;
(3) char $(R) = 2$ and $R$ satisfies $s_4$;
(4) char $(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$. 
Proof of Theorem 1.2. First consider the case when $a(F(xy) - F(x)F(y)) = 0$ for all $x, y \in R$. Let $G_1$ be the additive subgroup of $R$ generated by the set $S_1 = \{x^m | x \in R\}$ and $G_2$ be the additive subgroup of $R$ generated by the set $S_2 = \{x^n | x \in R\}$. Then by assumption

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in G_1, \quad \forall y \in G_2.$$ 

Then by [7], either $G_1 \subseteq Z(R)$ or $\text{char}(R) = 2$ and $R$ satisfies $s_4$, except when $G_1$ contains a noncentral Lie ideal $L_1$ of $R$. $G_1 \subseteq Z(R)$ implies that $x^m \in Z(R)$ for all $x \in R$. It is well known that in this case $R$ must be commutative, which is a contradiction. Since $\text{char}(R) \neq 2$, we are to consider the case when $G_1$ contains a noncentral Lie ideal $L_1$ of $R$. In this case by [4, Lemma 1], there exists a nonzero ideal $I_1$ of $R$ such that $[I_1, I_1] \subseteq L_1$.

Thus we have

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in G_2.$$ 

Analogously, we see that there exists a nonzero ideal $I_2$ of $R$ such that

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in [I_2, I_2].$$ 

By Lee [23, Theorem 3], $F(x) = bx + d(x)$ for some $b \in U$ and derivations $d$ on $U$. Since $I_1, I_2, R$ and $U$ satisfy the same differential identities [24], without loss of generality,

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x, y \in [R, R],$$ 

and in particular

$$a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].$$ 

Then by Theorem 1.1, we get

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = x$ for all $x \in R$;
3. $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

In the last conclusion, $R$ satisfies polynomial identity and hence $R \subseteq M_2(C)$ for some field $C$ and $M_2(C)$ satisfies $a(bx^m y^n - bx^n y^m) = 0$. By lemma 2.2, we get either $ab = 0$ or $b = 1$. If $ab = 0$, then $F(x) = bx$ for all $x \in R$, with $ab = 0$, which is our conclusion (1). If $b = 1$ then $F(x) = x$ for all $x \in R$, which is our conclusion (2).

Now replacing $F$ with $-F$, in the hypothesis $a(F(xy^n) - F(x)F(y^n)) = 0$, we get $0 = a((-F)(x^m y^n) - (-F)(x^m)(-F)(y^n))$, that is $0 = a(F(x^m y^n) + F(x^m)F(y^n))$ for all $x, y \in R$ implies one of the following:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = -x$ for all $x \in R$;
Now consider the case when \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \).
By similar argument as above we get
\[
a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R, R],
\]
and in particular
\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].
\]
Then by Theorem 1.1, we get

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = x \) for all \( x \in R \);
3. \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

In the conclusion (3), \( R \) satisfies polynomial identity and hence \( R \subseteq M_2(C) \) for some field \( C \) and \( M_2(C) \) satisfies \( a(bx^m y^n - by^m bx^m) = 0 \). Then by Lemma 2.3, we have \( ab = 0 \), which is our conclusion (1).

Now replacing \( F \) with \( -F \) in the hypothesis \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \), we get \( 0 = a(-F(x^m y^n) - (-F)(y^n)(-F)(x^m)) \). That is, \( 0 = a(F(x^m y^n) + F(y^n)F(x^m)) \) for all \( x, y \in R \). This implies that there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \) with \( ab = 0 \) or \( F(x) = -x \). This completes the proof.

In particular, we have the following corollary.

**Corollary 2.6.** Let \( R \) be a prime ring of characteristic different from 2 and \( 0 \neq a \in R \). Suppose that \( R \) admits the generalized derivation \( F \) associated with a nonzero derivation \( d \) of \( R \). If any one of the following conditions is satisfied:

1. \( a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \);
2. \( a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \),

then \( R \) is commutative.

**Proof of Theorem 1.3.** First we consider the case \( a(F(x^m y^n) + F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \). Other cases are similar. We know the fact that any derivation of a semiprime ring \( R \) can be uniquely extended to a derivation of its left Utumi quotient ring \( U \) and so any derivation of \( R \) can be defined on the whole of \( U \) [24, Lemma 2]. Moreover \( R \) and \( U \) satisfy the same GPIs as well as same differential identities. Thus
\[
a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0
\]
for all \( x, y \in U \). Let \( M(C) \) be the set of all maximal ideals of \( C \) and \( P \in M(C) \).
Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have \( PU \) is a prime ideal of \( U \) invariant under all derivations of \( U \). Moreover, \( \bigcap \{PU \mid P \in M(C) \} = 0 \). Set \( \mathcal{U} = U/PU \). Then derivation \( d \) canonically induces a derivation \( \overline{d} \) on \( U \) defined by \( \overline{d}(\pi) = \overline{d}(x) \) for all \( x \in U \).
Therefore,
\[
\overline{d}(b\overline{x}^m \overline{y}^n + d(\overline{x}^m \overline{y}^n) + (b\overline{x}^m + d(\overline{x}^m))(b\overline{y}^n + d(\overline{y}^n))) = 0
\]
for all $\pi, \overline{\pi} \in U$. By the prime ring case of Corollary 2.6, we have either $\overline{d} = 0$ or $[U, U] = 0$ or $\pi = 0$. In any case we have $\text{ad}(U)[U, U] \subseteq PU$ for all $P \in M(C)$. Since $\bigcap\{PU \mid P \in M(C)\} = 0$, $\text{ad}(U)[U, U] = 0$. In particular, $\text{ad}(R)[R, R] = 0$. This implies $0 = \text{ad}(R)[R^2, R] = \text{ad}(R)[R, R] + \text{ad}(R)[R, R][R, R]$. In particular, $\text{ad}(R)[R, \text{ad}(R)] = 0$. Therefore, $\text{ad}(R), [R, \text{ad}(R), R] = 0$. Since $R$ is semiprime, we obtain that $\text{ad}(R) \subseteq Z(R)$. By Theorem 3.2 in [10], there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1 U) = 0$, $e_2 a = 0$, and $e_3 U$ is commutative. Hence the theorem is proved.

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**References**


