Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

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Abstract. Let \( R \) be a prime ring with its Utumi ring of quotients \( U \), \( C = Z(U) \) the extended centroid of \( R \), \( L \) a non-central Lie ideal of \( R \) and \( 0 \neq a \in R \). If \( R \) admits a generalized derivation \( F \) such that \( a(F(u^2) \pm F(u)u^2) = 0 \) for all \( u \in L \), then one of the following holds:

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = \mp x \) for all \( x \in R \);
3. \( \text{char } (R) = 2 \) and \( R \) satisfies \( s_4 \);
4. \( \text{char } (R) \neq 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

We also study the situations (i) \( a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \), and (ii) \( a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \) in prime and semiprime rings.

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1. INTRODUCTION

Let $R$ be an associative prime ring with center $Z(R)$ and $U$ the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \to R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, $d$ is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let $S$ be a nonempty subset of $R$ and $F : R \to R$ be an additive mapping. Then we say that $F$ acts as homomorphism or anti-homomorphism on $S$ if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping $F$ acts as a Jordan homomorphism on $S$ if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. Asma, Rehman, Shakir in [1] proved that if $d$ is a derivation of a 2-torsion free prime ring $R$ which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation $F$ acts as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring $R$ as well as associated map $F$ (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let $R$ be a prime ring of characteristic different from 2, $F$ a generalized derivation of $R$ associated to a derivation $d$, $L$ a Lie ideal of $R$ such that $u^2 \in L$ for all $u \in L$. If $F$ acts as a homomorphism or anti-homomorphism on $L$, then either $d = 0$ or $L$ is central in $R$. More recently in [9], Filippis studied the situation when generalized derivation $F$ acts as a Jordan homomorphism on a non-central Lie ideal $L$ of $R$.

Recently in [26], Rehman and Raza proved the following: Let $R$ be a prime ring of char $(R) \neq 2$, $Z$ the center of $R$, and $L$ a nonzero Lie ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ which acts as a homomorphism or as anti-homomorphism on $L$, then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map $F$.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map $F$. The main results of this paper are as follows:
Theorem 1.1. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(u^2) \pm F(u)^2) = 0$ for all $u \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

Theorem 1.2. Let $R$ be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $F$ a generalized derivation on $R$ and $0 \neq a \in R$.

1. If $a(F(x^m) + F(y^n)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$ or $F(x) = \mp x$ for all $x \in R$.
2. $F(x) = bx$ for all $x \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$.

Theorem 1.3. Let $R$ be a noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $F(x) = bx + d(x)$ a generalized derivation on $R$ associated to the derivation $d$ and $0 \neq a \in R$. If any one of the following holds:

1. $a(F(x^m) + F(y^n)) = 0$ for all $x, y \in R$,
2. $a(F(x^m) + F(y^n)) = 0$ for all $x, y \in R$,

then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and $e_3U$ is commutative.

Remark 1.4. Let $R$ be a prime ring and $L$ a noncentral Lie ideal of $R$. If char$(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. If char$(R) = 2$ and dim$_C RC > 4$, i.e., char$(R) = 2$ and $R$ does not satisfy $s_4$, then by [22, Theorem 13] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. Thus if either char$(R) \neq 2$ or $R$ does not satisfy $s_4$, then we may conclude that there exists a nonzero ideal $I$ of $R$ such that $[I, I] \subseteq L$.

Remark 1.5. We denote by Der$(U)$ the set of all derivations on $U$. By a derivation word $\Delta$ of $R$ we mean $\Delta = d_1d_2d_3 \ldots d_m$ for some derivations $d_i \in$ Der$(U)$.

Let $D_{in}$ be the $C$-subspace of Der$(U)$ consisting of all inner derivations on $U$ and let $d$ be a non-zero derivation on $R$. By [21, Theorem 2] we have the following result:
If \( \Phi(x_1, x_2, \cdots, x_n, d(x_1), d(x_2) \cdots d(x_n)) \) is a differential identity on \( R \), then one of the following holds:

1. \( d \in D_{int} \);
2. \( R \) satisfies the generalized polynomial identity \( \Phi(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \).

**Remark 1.6.** In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping \( F : I \rightarrow U \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in I \), where \( I \) is a dense left ideal of \( R \) and \( d \) is a derivation from \( I \) into \( U \). Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of \( U \), and thus all generalized derivations of \( R \) will be implicitly assumed to be defined on the whole of \( U \). Lee obtained the following: every generalized derivation \( F \) on a dense left ideal of \( R \) can be uniquely extended to \( U \) and assumes the form \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \).

## 2. Proof of the Main Results

Now we begin with the following Lemmas:

**Lemma 2.1.** Let \( R = M_2(C) \) be the ring of all \( 2 \times 2 \) matrices over the field \( C \) of characteristic different from 2 and \( b, c \in R \). Suppose that there exists \( 0 \neq a \in R \) such that

\[
a(\{b[x, y]^2 + [x, y]^2 c\} - \{b[x, y] + [x, y]c\}^2) = 0,
\]

for all \( x, y \in R \). Then \( c \in C \cdot I_2 \).

**Proof.** If \( c \in C \cdot I_2 \), then nothing to prove. Let \( c \notin C \cdot I_2 \). In this case \( R \) is a dense ring of \( C \)-linear transformations over a vector space \( V \). Assume that there exists \( 0 \neq v \in V \) such that \( \{v, cv\} \) is linearly \( C \)-independent. By density, there exist \( x, y \in R \) such that \( xv = v, xcv = 0; yv = 0, ycv = v \). Then \( [x, y]v = 0, [x, y]cv = v \) and hence \( a(\{b[x, y]^2 + [x, y]^2 c\} - \{b[x, y] + [x, y]c\}^2) = 0 \).

Of course for any \( u \in V \), \( \{u, v\} \) linearly \( C \)-dependent implies \( au = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and so \( \{w, v\} \) are linearly \( C \)-independent. Also \( a(w + v) = aw \neq 0 \) and \( a(w - v) = aw \neq 0 \). By the above argument, it follows that \( w \) and \( cw \) are linearly \( C \)-dependent, as are \( \{w + v, c(w + v)\} \) and \( \{w - v, c(w - v)\} \). Therefore there exist \( \alpha_w, \alpha_{w+v}, \alpha_{w−v} \in C \) such that

\[
cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).
\]

In other words we have

\[
\alpha_w w + cw = \alpha_{w+v} w + \alpha_{w+v} v \quad \text{(2.1)}
\]

and

\[
\alpha_w w - cw = \alpha_{w-v} w - \alpha_{w-v} v. \quad \text{(2.2)}
\]
By comparing (2.1) with (2.2) we get both
\[(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0\] (2.3)
and
\[2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.\] (2.4)
By (2.3), and since \(\alpha \neq 0\) and \(v \neq 0\), we have \(\alpha_w = \alpha_{w+v} = \alpha_{w-v}\). Thus by (2.4) it follows \(2cv = 2\alpha_wv\). This leads to a contradiction with the fact that \(\{v, cv\}\) is linear \(C\)-independent.

In light of this, we may assume that for any \(v \in V\) there exists a suitable \(\alpha_v \in C\) such that \(cv = \alpha_v v\), and standard argument shows that there is \(\alpha \in C\) such that \(cv = \alpha v\) for all \(v \in V\). Now let \(r \in R, v \in V\). Since \(cv = \alpha v\),
\[[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.\]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\), a contradiction. \(\square\)

**Lemma 2.2.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from \(2\) and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^m y^n - px^m py^n) = 0,\]
for all \(x, y \in R\). Then either \(ap = 0\) or \(p = 1\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). In this case \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = 0\). Then we get \(0 = a(px^m y^n - px^m py^n)v = apv\). Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), from \(ap = ap^2\), we get \(0 = a(p-1)\). Since \(a \neq 0\), we conclude \(p = 1\). \(\square\)

**Lemma 2.3.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from \(2\) and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^m y^n - py^n px^m) = 0,\]
for all \(x, y \in R\). Then \(ap = 0\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). Here \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = 0\). Then we have \(0 = a(px^m y^n - py^n px^m)v = ap^2v = -apv\). Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), by hypothesis, we get \(0 = a[x^m, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have
0 = a[x^m, y^n] = a[e_{11}, e_{11} + e_{12}] = ae_{12}. Again, for \( x = e_{22} \) and \( y = e_{22} + e_{21} \), we have \( 0 = a[x^m, y^n] = a[e_{22}, e_{22} + e_{21}] = ae_{21} \). These imply \( a = 0 \), a contradiction. \( \square \)

**Lemma 2.4.** Let \( R \) be a noncommutative prime ring with extended centroid \( C \) and \( b, c \in R \). Suppose that \( 0 \neq a \in R \) such that
\[
a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0
\]
for all \( x, y \in R \). Then one of the following holds:
1. \( b \in C \) and \( a(b + c) = 0 \);
2. \( b, c \in C \) and \( b + c = 1 \);
3. \( \text{char} \( R \) = 2 \) and \( R \) satisfies \( s_4 \);
4. \( \text{char} \( R \) \neq 2 \), \( R \) satisfies \( s_4 \) and \( c \in C \).

**Proof.** By assumption, \( R \) satisfies the generalized polynomial identity (GPI)
\[
f(x, y) = a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}.
\]
By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by \( U \). Now we consider the following two cases:

**Case-I.** \( U \) does not satisfy any nontrivial GPI.

Let \( T = U \ast_C C\{x, y\} \), the free product of \( U \) and \( C\{x, y\} \), the free \( C \)-algebra in noncommuting indeterminates \( x \) and \( y \). Thus
\[
a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0
\]
is zero element in \( T = U \ast_C C\{x, y\} \). Let \( c \not\in C \). Then \( \{1, c\} \) is \( C \)-independent. Then from above
\[
a\{[x, y]^2c - (b[x, y] + [x, y]c)[x, y]c, \}
\]
which is
\[
a\{[x, y] - b[x, y] - [x, y]c\} [x, y]c,
\]
is zero in \( T \). Again, since \( c \not\in C \), we have that \( a[x, y]c[x, y]c \) is zero element in \( T \), implying \( a = 0 \) or \( c = 0 \), a contradiction. Thus we conclude that \( c \in C \). Then the identity reduces to
\[
a\{(b + c)[x, y] - (b + c)[x, y](b + c)\} [x, y],
\]
is zero element in \( T \). Again, if \( b + c \not\in C \), then \( a(b + c)[x, y]^2 \) becomes zero element in \( T \), implying \( a(b + c) = 0 \). If \( b + c \in C \), then \( a(b + c)(b + c - 1)[x, y]^2 \) becomes zero element in \( T \), implying \( b + c = 0 \) or \( b + c = 1 \). When \( b + c = 0 \), then \( a(b + c) = 0 \), which is our conclusion (1). When \( b + c = 1 \), then \( b = 1 - c \in C \), which is our conclusion (2).

**Case-II.** \( U \) satisfies a nontrivial GPI.
Thus we assume that
\[
\alpha \{(b[x,y]^2 + [x,y]^2 c) - (b[x,y] + [x,y]c)^2\} = 0,
\]
is a nontrivial GPI for \( U \). In case \( C \) is infinite, we have \( f(x,y) = 0 \) for all \( x, y \in U \otimes_C \overline{C} \), where \( \overline{C} \) is the algebraic closure of \( C \). Since both \( U \) and \( U \otimes_C \overline{C} \) are prime and centrally closed [17], we may replace \( R \) by \( U \) or \( U \otimes_C \overline{C} \) according to \( C \) finite or infinite. Thus we may assume that \( R \) centrally closed over \( C \) which either finite or algebraically closed and \( f(x,y) = 0 \) for all \( x, y \in R \).

By Martindale's Theorem [25], \( R \) is then primitive ring having non-zero socle \( \text{soc}(R) \) with \( C \) as the associated division ring. Hence by Jacobson's Theorem [20], \( R \) is isomorphic to a dense ring of linear transformations of a vector space \( V \) over \( C \). Since \( R \) is noncommutative, \( \dim_C V \geq 2 \). If \( \dim_C V = 2 \), then \( R \cong M_2(C) \). In this case by Lemma 2.1, either \( c \in C \) or \( \text{char } (R) = 2 \). This gives conclusions (3) and (4).

Let \( \dim_C V \geq 3 \). For some \( v \in V \), \( v \) and \( cv \) are linearly independent over \( C \). By density there exist \( x, y \in R \) such that
\[
xv = v, \quad xcv = 0;
\]
\[
yv = 0, \quad ycv = v.
\]

Then \( [x,y]v = 0 \), \([x,y]cv = v \) and hence \( \alpha \{(b[x,y]^2 + [x,y]^2 c) - (b[x,y] + [x,y]c)^2\} v = av \).

This implies that if \( av \neq 0 \), then by contradiction we may conclude that \( v \) and \( cv \) are linearly \( C \)-dependent. Now choose \( v \in V \) such that \( v \) and \( cv \) are linearly \( C \)-independent. Set \( W = \text{Span}_C \{v, cv\} \). Then \( av = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and then \( a(v-w) = aw \neq 0 \).

By the previous argument we have that \( w, cw \) are linearly \( C \)-dependent and \( (v-w), c(v-w) \) too. Thus there exist \( \alpha, \beta \in C \) such that \( cw = \alpha w \) and \( c(v-w) = \beta (v-w) \). Then \( cv = \beta (v-w) + cw = \beta (v-w) + \alpha w \) i.e., \( (\alpha - \beta)w = cv - \beta v \in W \). Now \( \alpha = \beta \) implies that \( cv = \beta v \), a contradiction. Hence \( \alpha \neq \beta \) and so \( w \in W \). Again, if \( u \in V \) with \( au = 0 \) then \( a(w+u) \neq 0 \).

So, \( w + u \in W \) forcing \( u \in W \). Thus it is observed that \( w \in V \) with \( aw \neq 0 \) implies \( w \in W \) and \( u \in V \) with \( au = 0 \) implies \( u \in W \). This implies that \( V = W \) i.e., \( \dim_C V = 2 \), a contradiction.

Hence, in any case, \( v \) and \( cv \) are linearly \( C \)-dependent for all \( v \in V \). Thus for each \( v \in V \), \( cv = \alpha_v v \) for some \( \alpha_v \in C \). It is very easy to prove that \( \alpha_v \) is independent of the choice of \( v \in V \). Thus we can write \( cv = \alpha v \) for all \( v \in V \) and \( \alpha \in C \) fixed. Now let \( r \in R, v \in V \). Since \( cv = \alpha v \),
\[
[r,v]v = (cr)v - (rc)v = (cr)v - r(cv) = \alpha (rv) - r(\alpha v) = 0.
\]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\).

Thus our identity reduces to
\[
a\{(b'[x, y])^2 - (b[x, y])^2\} = 0,
\]
for all \(x, y \in R\), where \(b' = b + c\).

Let for some \(v \in V\), \(v\) and \(b'v\) are linearly independent over \(C\). Since \(\dim_C V \geq 3\), there exists \(u \in V\) such that \(v, b'v, u\) are linearly independent over \(C\). By density there exist \(x, y \in R\) such that
\[
xv = v, \quad xb'v = 0, \quad xu = v;
\]
\[
yv = 0, \quad yb'v = u, \quad yu = v.
\]

Then \([x, y]v = 0\), \([x, y]b'v = v\), \([x, y]u = v\) and hence \(0 = a\{(b'[x, y])^2 - (b[x, y])^2\}u = ab'v\). Then by same argument as before, we have either \(ab' = 0\) or \(v\) and \(b'v\) are linearly \(C\)-dependent for all \(v \in V\). In the first case, \(0 = ab' = a(b + c)\), which is conclusion (1). In the last case, again by standard argument, we have that \(b' \in C\). If \(b' = 0\), then also \(ab' = a(b + c) = 0\) which gives conclusion (1). So assume that \(0 \neq b' \in C\). Then our identity reduces to
\[
ab'(b' - 1)[x, y]^2 = 0,
\]
for all \(x, y \in R\). This gives \(0 = ab'(b' - 1) = a(b' - 1)\). Since \(a \neq 0\), we get \(b' = 1\). This gives conclusion (2). \(\square\)

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when
\[
a(F(u^2) - F(u)^2) = 0,
\]
for all \(u \in L\). If \(\text{char } (R) = 2\) and \(R\) satisfies \(s_4\), then we have our conclusion (3). So we assume that either \(\text{char } (R) \neq 2\) or \(R\) does not satisfy \(s_4\). Since \(L\) is a noncentral by Remark 1.4, there exists a nonzero ideal \(I\) of \(R\) such that \([I, I] \subseteq L\). Thus by assumption \(I\) satisfies the differential identity
\[
a(F([x, y]^2) - F([x, y])^2) = 0.
\]

Now since \(R\) is a prime ring and \(F\) is a generalized derivation of \(R\), by Lee [23, Theorem 3], \(F(x) = bx + d(x)\) for some \(b \in U\) and derivation \(d\) on \(U\). Since \(I, R\) and \(U\) satisfy the same differential identities [24], without loss of generality, \(U\) satisfies
\[
a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0. \tag{2.5}
\]

Here we divide the proof into two cases:
Case 1. Let $d$ be inner derivation induced by element $c \in U$, that is, $d(x) = [c, x]$ for all $x \in U$. It follows that

$$a(b[x, y]^2 + [c, x, y]^2] - (b[x, y] + [c, x, y])^2) = 0,$$

that is

$$a((b + c)[x, y]^2 - [x, y]^2c - ((b + c)[x, y] - [x, y]c)^2) = 0,$$

for all $x, y \in U$. Now by Lemma 2.4, one of the following holds:

1. $c \in C$ and $0 = a(b + c - c) = ab$. Thus $F(x) = bx$ for all $x \in R$, with $ab = 0$.
2. $b + c, c \in C$ and $b + c - c = 1$. Thus $F(x) = x$ for all $x \in R$.
3. char $(R) \neq 2$, $R$ satisfies $s_4$ and $c \in C$. Thus $F(x) = bx$ for all $x \in R$.

Case 2. Assume that $d$ is not inner derivation of $U$. We have from (2.5) that $U$ satisfies

$$a(b[x, y]^2 + d([x, y])[x, y] + [x, y]d([x, y])) - (b[x, y] + d([x, y)])^2) = 0,$$

that is

$$a(b[x, y]^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y]([d(x), y] + [x, d(y)]) - (b[x, y] + [d(x), y] + [x, d(y)])^2 = 0.$$

Then by Kharchenko’s Theorem [21], $U$ satisfies

$$a(b[x, y]^2 + ([u, y] + [x, z])[x, y] + [x, y]([u, y] + [x, z])) - (b[x, y] + [u, y] + [x, z])^2 = 0. \tag{2.6}$$

Since $R$ is noncommutative, we may choose $q \in U$ such that $q \notin C$. Then replacing $u$ by $[q, x]$ and $z$ by $[y, y]$ in (2.6), we get

$$a(b[x, y]^2 + ([q, x], y) + [x, [q, y]])[x, y] + [x, y]([q, x], y) + [x, [q, y]]) - (b[x, y] + ([q, x], y) + [x, [q, y]])^2) = 0,$$

which is

$$a(b[x, y]^2 + [q, [x, y]^2]) - (b[x, y] + [q, [x, y]])^2) = 0.$$

Then by Lemma 2.4, we have $q \in C$, a contradiction.

Now replacing $F$ with $-F$ in the above result, we obtain the conclusion for the situation $a(F(a^2) + F(u)^2) = 0$ for all $u \in L$.

**Corollary 2.5.** Let $R$ be a prime ring with extended centroid $C$, $L$ a noncentral Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits the generalized derivation $F$ such that either $a(F(XY) + F(X)F(Y)) = 0$ for all $X, Y \in L$ or $a(F(XY) + F(Y)F(X)) = 0$ for all $X, Y \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = x$ for all $x \in R$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.
**Proof of Theorem 1.2.** First consider the case when \( a(F(xmyn) - F(xm)F(yn)) = 0 \) for all \( x, y \in R \). Let \( G_1 \) be the additive subgroup of \( R \) generated by the set \( S_1 = \{xm|x \in R\} \) and \( G_2 \) be the additive subgroup of \( R \) generated by the set \( S_2 = \{x^n|x \in R\} \). Then by assumption
\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in G_1, \forall y \in G_2.
\]
Then by [7], either \( G_1 \subseteq Z(R) \) or \( \text{char} \ (R) = 2 \) and \( R \) satisfies \( s_4 \), except when \( G_1 \) contains a noncentral Lie ideal \( L_1 \) of \( R \). \( G_1 \subseteq Z(R) \) implies that \( x^m \in Z(R) \) for all \( x \in R \). It is well known that in this case \( R \) must be commutative, which is a contradiction. Since \( \text{char} \ (R) \neq 2 \), we are to consider the case when \( G_1 \) contains a noncentral Lie ideal \( L_1 \) of \( R \). In this case by [4, Lemma 1], there exists a nonzero ideal \( I_1 \) of \( R \) such that \( [I_1, I_1] \subseteq L_1 \).

Thus we have
\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \forall y \in G_2.
\]
Analogously, we see that there exists a nonzero ideal \( I_2 \) of \( R \) such that
\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \forall y \in [I_2, I_2].
\]
By Lee [23, Theorem 3], \( F(x) = bx + d(x) \) for some \( b \in U \) and derivations \( d \) on \( U \). Since \( I_1, I_2, R \) and \( U \) satisfy the same differential identities [24], without loss of generality,
\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x, y \in [R, R],
\]
and in particular
\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].
\]
Then by Theorem 1.1, we get
\begin{enumerate}
\item there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
\item \( F(x) = x \) for all \( x \in R \);
\item \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).
\end{enumerate}

In the last conclusion, \( R \) satisfies polynomial identity and hence \( R \subseteq M_2(C) \) for some field \( C \) and \( M_2(C) \) satisfies \( a(bx^my^n - bx^nb^amy^n) = 0 \). By lemma 2.2, we get either \( ab = 0 \) or \( b = 1 \). If \( ab = 0 \), then \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \), which is our conclusion (1). If \( b = 1 \) then \( F(x) = x \) for all \( x \in R \), which is our conclusion (2).

Now replacing \( F \) with \( -F \) in the hypothesis \( a(F(x^my^n) - F(x^m)F(y^n)) = 0 \), we get \( 0 = a((-F)(x^my^n) - (-F)(x^m)(-F)(y^n)) \), that is \( 0 = a(F(x^my^n) + F(x^m)F(y^n)) \) for all \( x, y \in R \) implies one of the following:
\begin{enumerate}
\item there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
\item \( F(x) = -x \) for all \( x \in R \);
\end{enumerate}
Now consider the case when \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \).
By similar argument as above we get
\[
a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R, R],
\]
and in particular
\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].
\]
Then by Theorem 1.1, we get
(1) there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
(2) \( F(x) = x \) for all \( x \in R \);
(3) \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

In the conclusion (3), \( R \) satisfies polynomial identity and hence \( R \subseteq M_2(C) \) for some field \( C \) and \( M_2(C) \) satisfies \( a(bx^m y^n - by^n bx^m) = 0 \). Then by Lemma 2.3, we have \( ab = 0 \), which is our conclusion (1).

Now replacing \( F \) with \( -F \) in the hypothesis \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \), we get \( 0 = a((-F)(x^m y^n) - (-F)(y^n)(-F)(x^m)) \). That is, \( 0 = a(F(x^m y^n) + F(y^n)F(x^m)) \) for all \( x, y \in R \). This implies that there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \) with \( ab = 0 \) or \( F(x) = -x \). This completes the proof.

In particular, we have the following corollary.

**Corollary 2.6.** Let \( R \) be a prime ring of characteristic different from 2 and \( 0 \neq a \in R \). Suppose that \( R \) admits the generalized derivation \( F \) associated with a nonzero derivation \( d \) of \( R \). If any one of the following conditions is satisfied:
(1) \( a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \);
(2) \( a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \),
then \( R \) is commutative.

**Proof of Theorem 1.3.** First we consider the case \( a(F(x^m y^n) + F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \). Other cases are similar. We know the fact that any derivation of a semiprime ring \( R \) can be uniquely extended to a derivation of its left Utumi quotient ring \( U \) and so any derivation of \( R \) can be defined on the whole of \( U \) [24, Lemma 2]. Moreover \( R \) and \( U \) satisfy the same GPs as well as same differential identities. Thus
\[
a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0
\]
for all \( x, y \in U \). Let \( M(C) \) be the set of all maximal ideals of \( C \) and \( P \in M(C) \).

Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have \( PU \) is a prime ideal of \( U \) invariant under all derivations of \( U \). Moreover, \( \bigcap \{PU \mid P \in M(C) \} = 0 \). Set \( \overline{U} = U/PU \). Then derivation \( d \) canonically induces a derivation \( \overline{d} \) on \( \overline{U} \) defined by \( \overline{d}(\overline{x}) = \overline{d(x)} \) for all \( x \in U \).

Therefore,
\[
\overline{d}(b\overline{x}^m \overline{y}^n + d(\overline{x}^m \overline{y}^n) + (b\overline{x}^m + d(\overline{x}^m))(b\overline{y}^n + d(\overline{y}^n))) = 0
\]
for all $\pi, \sigma \in U$. By the prime ring case of Corollary 2.6, we have either $d = 0$ or $[U, U] = 0$ or $\sigma = 0$. In any case we have $ad(U)[U, U] \subseteq P(U)$ for all $P \in M(C)$. Since $\cap\{PU \mid P \in M(C)\} = 0$, $ad(U)[U, U] = 0$. In particular, $ad(R)[R, R] = 0$. This implies $0 = ad(R)[R^2, R] = ad(R)[R, R] - ad(R)[R, R]$. In particular, $ad(R)[R, ad(R)] = 0$. Therefore, $ad(R)[R, R] = 0$. Since $R$ is semiprime, we obtain that $ad(R) \subseteq Z(R)$. By Theorem 3.2 in [10], there exist orthogonal central idempotents $e_1$, $e_2$, $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1 U) = 0$, $e_2 a = 0$, and $e_3 U$ is commutative. Hence the theorem is proved.

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References


