Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

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Abstract. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(u^2) \pm F(u^2)) = 0$ for all $u \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

We also study the situations (i) $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, and (ii) $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$ in prime and semiprime rings.

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1. Introduction

Let $R$ be an associative prime ring with center $Z(R)$ and $U$ the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \to R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, $d$ is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let $S$ be a nonempty subset of $R$ and $F : R \to R$ be an additive mapping. Then we say that $F$ acts as homomorphism or anti-homomorphism on $S$ if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping $F$ acts as a Jordan homomorphism on $S$ if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. Asma, Rehman, Shakir in [1] proved that if $d$ is a derivation of a 2-torsion free prime ring $R$ which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation $F$ acts as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring $R$ as well as associated map $F$ (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let $R$ be a prime ring of characteristic different from 2, $F$ a generalized derivation of $R$ associated to a derivation $d$, $L$ a Lie ideal of $R$ such that $u^2 \in L$ for all $u \in L$. If $F$ acts as a homomorphism or anti-homomorphism on $L$, then either $d = 0$ or $L$ is central in $R$. More recently in [9], Filippis studied the situation when generalized derivation $F$ acts as a Jordan homomorphism on a non-central Lie ideal $L$ of $R$.

Recently in [26], Rehman and Raza proved the following: Let $R$ be a prime ring of char $(R) \neq 2$, $Z$ the center of $R$, and $L$ a nonzero Lie ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ which acts as a homomorphism or as anti-homomorphism on $L$, then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map $F$.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map $F$. The main results of this paper are as follows:
Theorem 1.1. Let \( R \) be a prime ring with its Utumi ring of quotients \( U \), \( C = Z(U) \) the extended centroid of \( R \), \( L \) a non-central Lie ideal of \( R \) and \( 0 \neq a \in R \). If \( R \) admits a generalized derivation \( F \) such that \( a(F(u^2) ± F(u)^3) = 0 \) for all \( u \in L \), then one of the following holds:

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = ±x \) for all \( x \in R \);
3. \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \);
4. \( \text{char}(R) ≠ 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

Theorem 1.2. Let \( R \) be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients \( U \), \( C = Z(U) \) the extended centroid of \( R \), \( F \) a generalized derivation on \( R \) and \( 0 \neq a \in R \).

1. If \( a(F(x^my^n) ± F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \), then there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \) or \( F(x) = ±x \) for all \( x \in R \).
2. If \( a(F(x^my^n) ± F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \), then there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \).

Theorem 1.3. Let \( R \) be a noncommutative 2-torsion free semiprime ring, \( U \) the left Utumi quotient ring of \( R \), \( C = Z(U) \) the extended centroid of \( R \), \( F(x) = bx + d(x) \) a generalized derivation on \( R \) associated to the derivation \( d \) and \( 0 \neq a \in R \). If any one of the following holds:

1. \( a(F(x^my^n) ± F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \),
2. \( a(F(x^my^n) ± F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \),
then there exist orthogonal central idempotents \( e_1, e_2, e_3 \in U \) with \( e_1 + e_2 + e_3 = 1 \) such that \( d(e_1U) = 0 \), \( e_2a = 0 \), and \( e_3U \) is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let \( R \) be a prime ring and \( L \) a noncentral Lie ideal of \( R \). If \( \text{char}(R) ≠ 2 \), by [4, Lemma 1] there exists a nonzero ideal \( I \) of \( R \) such that \( 0 \neq I, I \subseteq L \). If \( \text{char}(R) = 2 \) and \( \dim_C RC > 4 \), i.e., \( \text{char}(R) = 2 \) and \( R \) does not satisfy \( s_4 \), then by [22, Theorem 13] there exists a nonzero ideal \( I \) of \( R \) such that \( 0 \neq I, I \subseteq L \). Thus if either \( \text{char}(R) ≠ 2 \) or \( R \) does not satisfy \( s_4 \), then we may conclude that there exists a nonzero ideal \( I \) of \( R \) such that \( I, I \subseteq L \).

Remark 1.5. We denote by \( \text{Der}(U) \) the set of all derivations on \( U \). By a derivation word \( \Delta \) of \( R \) we mean \( \Delta = d_1d_2d_3 \ldots d_m \) for some derivations \( d_i \in \text{Der}(U) \).

Let \( D_{\text{int}} \) be the \( C \)-subspace of \( \text{Der}(U) \) consisting of all inner derivations on \( U \) and let \( d \) be a non-zero derivation on \( R \). By [21, Theorem 2] we have the following result:
If $\Phi(x_1, x_2, \cdots, x_n, d(x_1), d(x_2) \cdots d(x_n))$ is a differential identity on $R$, then one of the following holds:

1. $d \in D_{int}$;
2. $R$ satisfies the generalized polynomial identity $\Phi(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$.

**Remark 1.6.** In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \rightarrow U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where $I$ is a dense left ideal of $R$ and $d$ is a derivation from $I$ into $U$. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of $U$, and thus all generalized derivations of $R$ will be implicitly assumed to be defined on the whole of $U$. Lee obtained the following: every generalized derivation $F$ on a dense left ideal of $R$ can be uniquely extended to $U$ and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation $d$ on $U$.

### 2. Proof of the Main Results

Now we begin with the following Lemmas:

**Lemma 2.1.** Let $R = M_2(C)$ be the ring of all $2 \times 2$ matrices over the field $C$ of characteristic different from 2 and $b, c \in R$. Suppose that there exists $0 \neq a \in R$ such that

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0,$$

for all $x, y \in R$. Then $c \in C \cdot I_2$.

**Proof.** If $c \in C \cdot I_2$, then nothing to prove. Let $c \notin C \cdot I_2$. In this case $R$ is a dense ring of $C$-linear transformations over a vector space $V$. Assume that there exists $0 \neq v \in V$ such that $\{v, cv\}$ is linearly $C$-independent. By density, there exist $x, y \in R$ such that $xv = v, xcv = 0; yv = 0, ycv = v$. Then $[x, y]v = 0$, $[x, y]cv = v$ and hence $a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}v = av$.

Of course for any $u \in V, \{u, v\}$ linearly $C$-dependent implies $au = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $\{w, v\}$ are linearly $C$-independent. Also $a(w + v) = aw \neq 0$ and $a(w - v) = av \neq 0$. By the above argument, it follows that $w$ and $cw$ are linearly $C$-dependent, as are $\{w + v, c(w + v)\}$ and $\{w - v, c(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).$$

In other words we have

$$\alpha_w w + cw = \alpha_{w+v} w + \alpha_{w+v} v \quad (2.1)$$

and

$$\alpha_w w - cw = \alpha_{w-v} w - \alpha_{w-v} v. \quad (2.2)$$
By comparing (2.1) with (2.2) we get both
\[(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0\] (2.3)
and
\[2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.\] (2.4)

By (2.3), and since \(\{w, v\}\) are \(C\)-independent and \(\text{char } (R) \neq 2\), we have
\[\alpha_w = \alpha_{w+v} = \alpha_{w-v}.\]
Thus by (2.4) it follows \(2cv = 2\alpha_w v\). This leads a contradiction with the fact that \(\{v, cv\}\) is linear \(C\)-independent.

In light of this, we may assume that for any \(v \in V\) there exists a suitable \(\alpha_v \in C\) such that \(cv = \alpha_v v\), and standard argument shows that there is \(\alpha \in C\) such that \(cv = \alpha v\) for all \(v \in V\). Now let \(r \in R, v \in V\). Since \(cv = \alpha v\),
\[[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.\]
Thus \([c, r]v = 0\) for all \(v \in V\), i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\), a contradiction.

\[\text{Lemma 2.2.}\] Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from \(2\) and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^my^n - px^mpy^n) = 0,\]
for all \(x, y \in R\). Then either \(ap = 0\) or \(p = 1\).

\[\text{Proof.}\] Putting \(x = y = I_2\), we get \(ap = ap^2\). In this case \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = 0\). Then we get
\[0 = a(px^my^n - px^m py^n)v = apv.\]
Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), from \(ap = ap^2\), we get \(0 = a(p - 1)\). Since \(a \neq 0\), we conclude \(p = 1\). \(\square\)

\[\text{Lemma 2.3.}\] Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from \(2\) and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^my^n - py^mx^n) = 0,\]
for all \(x, y \in R\). Then \(ap = 0\).

\[\text{Proof.}\] Putting \(x = y = I_2\), we get \(ap = ap^2\). Here \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = pv\). Then we have
\[0 = a(px^my^n - py^mx^n)v = -ap^2v = -apv.\]
Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), by hypothesis, we get \(0 = a[x^m, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have
\[0 = a[x^m, y^n] = a[e_{11}, e_{11} + e_{12}] = ae_{12}.\] Again, for \(x = e_{22}\) and \(y = e_{22} + e_{21}\), we have \(0 = a[x^m, y^n] = a[e_{22}, e_{22} + e_{21}] = ae_{21}.\) These imply \(a = 0\), a contradiction. \(\square\)

**Lemma 2.4.** Let \(R\) be a noncommutative prime ring with extended centroid \(C\) and \(b, c \in R\). Suppose that \(0 \neq a \in R\) such that
\[a\{(b|x, y|^2 + [x, y]^2)c - (b|x, y| + [x, y]|c|^2)\} = 0\]
for all \(x, y \in R\). Then one of the following holds:

1. \(c \in C\) and \(a(b + c) = 0;\)
2. \(b, c \in C\) and \(b + c = 1;\)
3. \(\text{char}(R) = 2\) and \(R\) satisfies \(s_4;\)
4. \(\text{char}(R) \neq 2\), \(R\) satisfies \(s_4\) and \(c \in C\).

**Proof.** By assumption, \(R\) satisfies the generalized polynomial identity (GPI)
\[f(x, y) = a\{(b|x, y|^2 + [x, y]^2)c - (b|x, y| + [x, y]|c|^2)\}.\]
By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by \(U\). Now we consider the following two cases:

**Case-I.** \(U\) does not satisfy any nontrivial GPI.

Let \(T = U \ast C C \{x, y\}\), the free product of \(U\) and \(C \{x, y\}\), the free \(C\)-algebra in noncommuting indeterminates \(x\) and \(y\). Thus
\[a\{(b|x, y|^2 + [x, y]^2)c - (b|x, y| + [x, y]|c|^2)\}\]
is zero element in \(T = U \ast C C \{x, y\}\). Let \(c \notin C\). Then \(\{1, c\}\) is \(C\)-independent. Then from above
\[a\{[x, y]^2c - (b|x, y| + [x, y]|c|)[x, y]c,\}\]
which is
\[a\{[x, y] - b|x, y| - [x, y]|c|\}][x, y]c,
is zero in \(T\). Again, since \(c \notin C\), we have that \(a[x, y]|c[x, y]|c\) is zero element in \(T\), implying \(a = 0\) or \(c = 0\), a contradiction. Thus we conclude that \(c \in C\).
Then the identity reduces to
\[a\{(b + c)[x, y] - (b + c)|x, y|(b + c)|x, y|\},\]
is zero element in \(T\). Again, if \(b + c \notin C\), then \(a(b + c)|x, y|^2\) becomes zero element in \(T\), implying \(a(b + c) = 0\). If \(b + c \in C\), then \(a(b + c)(b + c - 1)|x, y|^2\) becomes zero element in \(T\), implying \(b + c = 0\) or \(b + c = 1\). When \(b + c = 0\), then \(a(b + c) = 0\), which is our conclusion (1). When \(b + c = 1\), then \(b = 1 - c \in C\), which is our conclusion (2).

**Case-II.** \(U\) satisfies a nontrivial GPI.
Thus we assume that
\[
a\{b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0,
\]
is a nontrivial GPI for $U$. In case $C$ is infinite, we have $f(x, y) = 0$ for all $x, y \in U \otimes_{C} \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $U$ and $U \otimes_{C} \overline{C}$ are prime and centrally closed [17], we may replace $R$ by $U$ or $U \otimes_{C} \overline{C}$ according to $C$ finite or infinite. Thus we may assume that $R$ centrally closed over $C$ which either finite or algebraically closed and $f(x, y) = 0$ for all $x, y \in R$. By Martindale’s Theorem [25], $R$ is then primitive ring having non-zero socle $soc(R)$ with $C$ as the associated division ring. Hence by Jacobson’s Theorem [20], $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. Since $R$ is noncommutative, $dim_{C}V \geq 2$. If $dim_{C}V = 2$, then $R \cong M_{2}(C)$. In this case by Lemma 2.1, either $c \in C$ or $char(R) = 2$. This gives conclusions (3) and (4).

Let $dim_{C}V \geq 3$. Let for some $v \in V$, $v$ and $cv$ are linearly independent over $C$. By density there exist $x, y \in R$ such that
\[
xv = v, \quad xcv = 0;
\]
\[
yv = 0, \quad ycv = v.
\]
This implies that if $av \neq 0$, then by contradiction we may conclude that $v$ and $cv$ are linearly $C$-dependent. Now choose $v \in V$ such that $v$ and $cv$ are linearly $C$-independent. Set $W = SpanC\{v, cv\}$. Then $av = 0$. Since $a \neq 0$, there exists $w \in V$ such that $av \neq 0$ and then $a(v - w) = av \neq 0$. By the previous argument we have that $w, cw$ are linearly $C$-dependent and $(v - w), c(v - w)$ too. Thus there exist $\alpha, \beta \in C$ such that $cw = \alpha w$ and $c(v - w) = \beta(v - w)$. Then $cv = \beta(v - w) + cw = \beta w + \alpha w = (\alpha - \beta)w + \beta v \in W$. Now $\alpha = \beta$ implies that $cv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$. Again, if $u \in V$ with $au = 0$ then $a(w + u) \neq 0$. So, $w + u \in W$ forcing $u \in W$. Thus it is observed that $w \in V$ with $av \neq 0$ implies $w \in W$ and $u \in V$ with $au = 0$ implies $u \in W$. This implies that $V = W$ i.e., $dim_{C}V = 2$, a contradiction.

Hence, in any case, $v$ and $cv$ are linearly $C$-dependent for all $v \in V$. Thus for each $v \in V$, $cv = \alpha_{v}v$ for some $\alpha_{v} \in C$. It is very easy to prove that $\alpha_{v}$ is independent of the choice of $v \in V$. Thus we can write $cv = \alpha v$ for all $v \in V$ and $\alpha \in C$ fixed. Now let $r \in R, v \in V$. Since $cv = \alpha v$,
\[
[r, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.
\]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r]\) = 0 for all \(r \in R\). Therefore, \(c \in Z(R)\).

Thus our identity reduces to
\[
a\{(b'[x, y]^2) - (b'[x, y])^2\} = 0,
\]
for all \(x, y \in R\), where \(b' = b + c\).

Let for some \(v \in V\), \(v\) and \(b'v\) are linearly independent over \(C\). Since \(\dim_C V \geq 3\), there exists \(u \in V\) such that \(v, b'v, u\) are linearly independent over \(C\). By density there exist \(x, y \in R\) such that
\[
xv = v, \quad xb'v = 0, \quad xu = v;
\]
\[
yv = 0, \quad yb'v = u, \quad yu = v.
\]

Then \([x, y]v = 0, [x, y]b'v = v, [x, y]u = v\) and hence \(0 = a\{(b'[x, y]^2) - (b'[x, y])^2\}u = ab'v\). Then by same argument as before, we have either \(ab' = 0\) or \(v\) and \(b'v\) are linearly \(C\)-dependent for all \(v \in V\). In the first case, \(0 = ab' = a(b + c)\), which is conclusion (1). In the last case, again by standard argument, we have that \(b' \in C\). If \(b' = 0\), then also \(ab' = a(b + c) = 0\) which gives conclusion (1). So assume that \(0 \neq b' \in C\). Then our identity reduces to
\[
ab'(b' - 1)[x, y]^2 = 0,
\]
for all \(x, y \in R\). This gives \(0 = ab'(b' - 1) = a(b' - 1)\). Since \(a \neq 0\), we get \(b' = 1\). This gives conclusion (2). \(\Box\)

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when
\[
a(F(u^2) - F(u)^2) = 0,
\]
for all \(u \in L\). If \(\text{char } (R) = 2\) and \(R\) satisfies \(s_4\), then we have our conclusion (3). So we assume that either \(\text{char } (R) \neq 2\) or \(R\) does not satisfy \(s_4\). Since \(L\) is a noncentral by Remark 1.4, there exists a nonzero ideal \(I\) of \(R\) such that \([I, I] \subseteq L\). Thus by assumption \(I\) satisfies the differential identity
\[
a(F([x, y]^2) - F([x, y])^2) = 0.
\]
Now since \(R\) is a prime ring and \(F\) is a generalized derivation of \(R\), by Lee [23, Theorem 3], \(F(x) = bx + d(x)\) for some \(b \in U\) and derivation \(d\) on \(U\). Since \(I, R\) and \(U\) satisfy the same differential identities [24], without loss of generality, \(U\) satisfies
\[
ab([x, y]^2) + d([x, y]^2) - (b([x, y] + d([x, y]))^2) = 0. \quad (2.5)
\]
Here we divide the proof into two cases:
Case 1. Let $d$ be inner derivation induced by element $c \in U$, that is, $d(x) = [c, x]$ for all $x \in U$. It follows that
\[
a(b[x, y]^2 + [c, [x, y]^2]) - (b[x, y] + [c, [x, y]])^2 = 0,
\]
that is
\[
a((b + c)[x, y]^2 - [x, y]^2c - ((b + c)[x, y] - [x, y](c^2)) = 0,
\]
for all $x, y \in U$. Now by Lemma 2.4, one of the following holds:
1. $c \in C$ and $0 = a(b + c - c) = ab$. Thus $F(x) = bx$ for all $x \in R$, with $ab = 0$.
2. $b + c, c \in C$ and $b + c - c = 1$. Thus $F(x) = x$ for all $x \in R$.
3. char $(R) \neq 2$, $R$ satisfies $s_4$ and $c \in C$. Thus $F(x) = bx$ for all $x \in R$.

Case 2. Assume that $d$ is not inner derivation of $U$. We have from (2.5) that $U$ satisfies
\[
a(b[x, y]^2 + d([x, y])[x, y] + [x, y]d([x, y])) - (b[x, y] + d([x, y]))^2 = 0,
\]
that is
\[
a(b[x, y]^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y][d(x), y] + [x, d(y)]) - (b[x, y] + [d(x), y] + [x, d(y)])^2 = 0.
\]
Then by Kharchenko’s Theorem [21], $U$ satisfies
\[
a(b[x, y]^2 + ([u, y] + [x, z])[x, y] + [x, y]([u, y] + [x, z]) - (b[x, y] + [u, y] + [x, z])^2 = 0.
\]
Since $R$ is noncommutative, we may choose $q \in U$ such that $q \notin C$. Then replacing $u$ by $[q, x]$ and $z$ by $[g, y]$ in (2.6), we get
\[
a(b[x, y]^2 + ([q, x], y] + [x, [q, y]])[x, y] + [x, y][[q, x], y] + [x, [q, y]]) - (b[x, y] + [[[q, x], y] + [x, [q, y]])^2 = 0,
\]
which is
\[
a(b[x, y]^2 + [q, [x, y]^2]) - (b[x, y] + [q, [x, y]^2])^2 = 0.
\]
Then by Lemma 2.4, we have $q \in C$, a contradiction.

Now replacing $F$ with $-F$ in the above result, we obtain the conclusion for the situation $a(F(u^2) + F(u)^2) = 0$ for all $u \in L$.

**Corollary 2.5.** Let $R$ be a prime ring with extended centroid $C$, $L$ a noncentral Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits the generalized derivation $F$ such that either $a(F(XY) + F(X)F(Y)) = 0$ for all $X, Y \in L$ or $a(F(XY) + F(Y)F(X)) = 0$ for all $X, Y \in L$, then one of the following holds:
1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \pm x$ for all $x \in R$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$. 

Proof of Theorem 1.2. First consider the case when \( a(F(x^m y^n) - F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \). Let \( G_1 \) be the additive subgroup of \( R \) generated by the set \( S_1 = \{ x^m | x \in R \} \) and \( G_2 \) be the additive subgroup of \( R \) generated by the set \( S_2 = \{ x^n | x \in R \} \). Then by assumption

\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in G_1, \quad \forall y \in G_2.
\]

Then by [7], either \( G_1 \subseteq Z(R) \) or \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \), except when \( G_1 \) contains a noncentral Lie ideal \( L_1 \) of \( R \). \( G_1 \subseteq Z(R) \) implies that \( x^m \in Z(R) \) for all \( x \in R \). It is well known that in this case \( R \) must be commutative, which is a contradiction. Since \( \text{char}(R) \neq 2 \), we are to consider the case when \( G_1 \) contains a noncentral Lie ideal \( L_1 \) of \( R \). In this case by [4, Lemma 1], there exists a nonzero ideal \( I_1 \) of \( R \) such that \( [I_1, I_1] \subseteq L_1 \).

Thus we have

\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in G_2.
\]

Analogously, we see that there exists a nonzero ideal \( I_2 \) of \( R \) such that

\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in [I_2, I_2].
\]

By Lee [23, Theorem 3], \( F(x) = bx + d(x) \) for some \( b \in U \) and derivations \( d \) on \( U \). Since \( I_1, I_2, R \) and \( U \) satisfy the same differential identities [24], without loss of generality,

\[
a(F(xy) - F(x)F(y)) = 0 \quad \forall x, y \in [R, R],
\]

and in particular

\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].
\]

Then by Theorem 1.1, we get

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = x \) for all \( x \in R \);
3. \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

In the last conclusion, \( R \) satisfies polynomial identity and hence \( R \subseteq M_2(C) \) for some field \( C \) and \( M_2(C) \) satisfies \( a(bx^my^n - bx^ny^m) = 0 \). By lemma 2.2, we get either \( ab = 0 \) or \( b = 1 \). If \( ab = 0 \), then \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \), which is our conclusion (1). If \( b = 1 \) then \( F(x) = x \) for all \( x \in R \), which is our conclusion (2).

Now replacing \( F \) with \( -F \) in the hypothesis \( a(F(x^m y^n) - F(x^m)F(y^n)) = 0 \), we get

\[
0 = a((-F)(x^m y^n) - (-F)(x^m)(-F)(y^n)),
\]

that is

\[
0 = a(F(x^m y^n) + F(x^m)F(y^n)) \quad \forall x, y \in R
\]

implies one of the following:

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = -x \) for all \( x \in R \);
Now consider the case when \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \).
By similar argument as above we get
\[
a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R, R],
\]
and in particular
\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].
\]
Then by Theorem 1.1, we get
\[
\text{(1) there exists } b \in U \text{ such that } F(x) = bx \text{ for all } x \in R, \text{ with } ab = 0;
\]
\[
\text{(2) } F(x) = x \text{ for all } x \in R;
\]
\[
\text{(3) } R \text{ satisfies } s_4 \text{ and there exists } b \in U \text{ such that } F(x) = bx \text{ for all } x \in R.
\]
In the conclusion (3), \( R \) satisfies polynomial identity and hence \( R \subseteq M_2(C) \)
for some field \( C \) and \( M_2(C) \) satisfies \( a(bx^m y^n - by^n bx^m) = 0 \). Then by Lemma
2.3, we have \( ab = 0 \), which is our conclusion (1).

Now replacing \( F \) with \( -F \) in the hypothesis \( a(F(x^m y^n) + F(y^n)F(x^m)) = 0 \),
we get \( 0 = a((-F)(x^m y^n) - (F)(y^n)(-F)(x^m)) \). That is, \( 0 = a(F(x^m y^n) + F(y^n)F(x^m)) \)
for all \( x, y \in R \). This implies that there exists \( b \in U \) such that
\( F(x) = bx \) for all \( x \in R \) with \( ab = 0 \) or \( F(x) = -x \). This completes the proof.

In particular, we have the following corollary.

**Corollary 2.6.** Let \( R \) be a prime ring of characteristic different from 2 and \( 0 \neq a \in R \). Suppose that \( R \) admits the generalized derivation \( F \) associated with
a nonzero derivation \( d \) of \( R \). If any one of the following conditions is satisfied:

(1) \( a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \);
(2) \( a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \),
then \( R \) is commutative.

**Proof of Theorem 1.3.** First we consider the case \( a(F(x^m y^n) + F(x^m)F(y^n)) = 0 \)
for all \( x, y \in R \). Other cases are similar. We know the fact that any derivation
of a semiprime ring \( R \) can be uniquely extended to a derivation of its left
Utumi quotient ring \( U \) and so any derivation of \( R \) can be defined on the whole
of \( U \) [24, Lemma 2]. Moreover \( R \) and \( U \) satisfy the same GPs as well as same
differential identities. Thus
\[
a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0
\]
for all \( x, y \in U \). Let \( M(C) \) be the set of all maximal ideals of \( C \) and \( P \in M(C) \).
Now by the standard theory of orthogonal completions for semiprime rings (see
[24, p.31-32]), we have \( PU \) is a prime ideal of \( U \) invariant under all derivations
of \( U \). Moreover, \( \bigcap \{PU \mid P \in M(C) \} = 0 \). Set \( \overline{U} = U/PU \). Then derivation \( d \)
canonically induces a derivation \( \overline{d} \) on \( \overline{U} \) defined by \( \overline{d}(x) = \overline{d}(x) \) for all \( x \in U \).
Therefore,
\[
\overline{d}(b\overline{x}^m \overline{y}^n + d(\overline{x}^m \overline{y}^n) + (b\overline{x}^m + d(\overline{x}^m))(b\overline{y}^n + d(\overline{y}^n))) = 0
\]
for all $\pi, \overline{\pi} \in \mathcal{U}$. By the prime ring case of Corollary 2.6, we have either $d = 0$ or $[\mathcal{U}, \mathcal{U}] = 0$ or $\pi = 0$. In any case we have $\text{ad}(\mathcal{U})[\mathcal{U}, \mathcal{U}] \subseteq P\mathcal{U}$ for all $P \in M(\mathcal{C})$. Since $\bigcap\{PU \mid P \in M(\mathcal{C})\} = 0$, $\text{ad}(\mathcal{U})[\mathcal{U}, \mathcal{U}] = 0$. In particular, $\text{ad}(\mathcal{R})[\mathcal{R}, \mathcal{R}] = 0$. This implies $0 = \text{ad}(\mathcal{R})[\mathcal{R}^2, \mathcal{R}] = \text{ad}(\mathcal{R})[\mathcal{R}, \mathcal{R}] + \text{ad}(\mathcal{R})[\mathcal{R}, \mathcal{R}][\mathcal{R}, \mathcal{R}]$. In particular, $\text{ad}(\mathcal{R})[\mathcal{R}, \text{ad}(\mathcal{R})] = 0$. Therefore, $[\text{ad}(\mathcal{R}), \mathcal{R}][\mathcal{R}, \mathcal{R}] = 0$. Since $\mathcal{R}$ is semiprime, we obtain that $\text{ad}(\mathcal{R}) \subseteq Z(\mathcal{R})$. By Theorem 3.2 in [10], there exist orthogonal central idempotents $e_1, e_2, e_3 \in \mathcal{U}$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1 \mathcal{U}) = 0$, $e_2 \alpha = 0$, and $e_3 \mathcal{U}$ is commutative. Hence the theorem is proved.

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**REFERENCES**