Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

Basudeb Dhara\textsuperscript{a,*}, Krishna Gopal Pradhan\textsuperscript{b} and Shailesh Kumar Tiwari\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Belda College, Belda, Paschim Medinipur, 721424, W.B., India.

\textsuperscript{b}Department of Mathematics, Midnapore City College, Bhadutala, Paschim Medinipur, W.B.- 721129 India.

\textsuperscript{c}Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi-110016, India.

E-mail: basu dhara@yahoo.com

E-mail: kgp.math@gmail.com

E-mail: shaileshiitd84@gmail.com

Abstract. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(u^2) \pm F(u^2)) = 0$ for all $u \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

We also study the situations (i) $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, and (ii) $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$ in prime and semiprime rings.

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*Corresponding Author

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1. Introduction

Let $R$ be an associative prime ring with center $Z(R)$ and $U$ the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \rightarrow R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, $d$ is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \rightarrow R$ is called generalized derivation, if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Then we say that $F$ acts as homomorphism or anti-homomorphism on $S$ if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping $F$ acts as a Jordan homomorphism on $S$ if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. Asma, Rehman, Shakir in [1] proved that if $d$ is a derivation of a 2-torsion free prime ring $R$ which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation $F$ acts as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring $R$ as well as associated map $F$ (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let $R$ be a prime ring of characteristic different from 2, $F$ a generalized derivation of $R$ associated to a derivation $d$, $L$ a Lie ideal of $R$ such that $u^2 \in L$ for all $u \in L$. If $F$ acts as a homomorphism or anti-homomorphism on $L$, then either $d = 0$ or $L$ is central in $R$. More recently in [9], Filippis studied the situation when generalized derivation $F$ acts as a Jordan homomorphism on a non-central Lie ideal $L$ of $R$.

Recently in [26], Rehman and Raza proved the following: Let $R$ be a prime ring of char $(R) \neq 2, Z$ the center of $R$, and $L$ a nonzero Lie ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ which acts as a homomorphism or as anti-homomorphism on $L$, then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map $F$.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map $F$. The main results of this paper are as follows:
Theorem 1.1. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(x^2) + F(ax^3)) = 0$ for all $x \in R$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. $\text{char } (R) = 2$ and $R$ satisfies $s_4$;
4. $\text{char } (R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

Theorem 1.2. Let $R$ be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $F$ a generalized derivation on $R$ and $0 \neq a \in R$.

1. If $a(F(x^m y^n) + F(x^n)F(y^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$ or $F(x) = \mp x$ for all $x \in R$.
2. If $a(F(x^m y^n) + F(y^n)F(x^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$.

Theorem 1.3. Let $R$ be a noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $F(x) = bx + d(x)$ a generalized derivation on $R$ associated to the derivation $d$ and $0 \neq a \in R$. If any one of the following holds:

1. $a(F(x^m y^n) + F(x^n)F(y^m)) = 0$ for all $x, y \in R$,
2. $a(F(x^m y^n) + F(y^n)F(x^m)) = 0$ for all $x, y \in R$,
then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1 U) = 0$, $e_2 a = 0$, and $e_3 U$ is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let $R$ be a prime ring and $L$ a noncentral Lie ideal of $R$. If $\text{char}(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. If $\text{char}(R) = 2$ and $\dim C_R C > 4$, i.e., $\text{char}(R) = 2$ and $R$ does not satisfy $s_4$, then by [22, Theorem 13] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. Thus if either $\text{char}(R) \neq 2$ or $R$ does not satisfy $s_4$, then we may conclude that there exists a nonzero ideal $I$ of $R$ such that $[I, I] \subseteq L$.

Remark 1.5. We denote by $\text{Der}(U)$ the set of all derivations on $U$. By a derivation word $\Delta$ of $R$ we mean $\Delta = d_1d_2d_3 \ldots d_m$ for some derivations $d_i \in \text{Der}(U)$.

Let $D_{\text{int}}$ be the $C$-subspace of $\text{Der}(U)$ consisting of all inner derivations on $U$ and let $d$ be a non-zero derivation on $R$. By [21, Theorem 2] we have the following result:
If \( \Phi(x_1, x_2, \ldots, x_n, d(x_1), d(x_2) \cdots d(x_n)) \) is a differential identity on \( R \), then one of the following holds:

1. \( d \in D_{int} \);
2. \( R \) satisfies the generalized polynomial identity \( \Phi(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \).

Remark 1.6. In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping \( F : I \to U \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in I \), where \( I \) is a dense left ideal of \( R \) and \( d \) is a derivation from \( I \) into \( U \). Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of \( U \), and thus all generalized derivations of \( R \) will be implicitly assumed to be defined on the whole of \( U \). Lee obtained the following: every generalized derivation \( F \) on a dense left ideal of \( R \) can be uniquely extended to \( U \) and assumes the form \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \).

2. Proof of the Main Results

Now we begin with the following Lemmas:

**Lemma 2.1.** Let \( R = M_2(C) \) be the ring of all \( 2 \times 2 \) matrices over the field \( C \) of characteristic different from 2 and \( b, c \in R \). Suppose that there exists \( 0 \neq a \in R \) such that

\[
a\{(b[x, y]^2 + [x, y]^2)c - (b[x, y] + [x, y]c)^2\} = 0,
\]

for all \( x, y \in R \). Then \( c \in C \cdot I_2 \).

**Proof.** If \( c \notin C \cdot I_2 \), then nothing to prove. Let \( c \in C \cdot I_2 \). In this case \( R \) is a dense ring of \( C \)-linear transformations over a vector space \( V \). Assume that there exists \( 0 \neq v \in V \) such that \( \{v, cv\} \) is linearly \( C \)-independent. By density, there exist \( x, y \in R \) such that \( xv = v, xcv = 0; yv = 0, ycv = v \). Then \( [x, y]v = 0, [x, y]cv = v \) and hence \( a\{(b[x, y]^2 + [x, y]^2)c - (b[x, y] + [x, y]c)^2\}v = av \).

Of course for any \( u \in V \), \( \{u, v\} \) linearly \( C \)-dependent implies \( au = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and so \( \{w, v\} \) are linearly \( C \)-independent. Also \( a(w + v) = aw \neq 0 \) and \( a(w - v) = aw \neq 0 \). By the above argument, it follows that \( w \) and \( cw \) are linearly \( C \)-dependent, as are \( \{w + v, c(w + v)\} \) and \( \{w - v, c(w - v)\} \). Therefore there exist \( \alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C \) such that

\[
cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v} (w + v), \quad c(w - v) = \alpha_{w-v} (w - v).
\]

In other words we have

\[
\alpha_w w + cw = \alpha_{w+v} w + \alpha_{w+v} v \quad (2.1)
\]

and

\[
\alpha_w w - cw = \alpha_{w-v} w - \alpha_{w-v} v. \quad (2.2)
\]
By comparing (2.1) with (2.2) we get both

\[(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \quad (2.3)\]

and

\[2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \quad (2.4)\]

By (2.3), and since \{w, v\} are \(C\)-independent and \(\text{char}(R) \neq 2\), we have

\[\alpha_w = \alpha_{w+v} = \alpha_{w-v}.\]

Thus by (2.4) it follows \(2cv = 2\alpha_wv\). This leads a contradiction with the fact that \(\{v, cv\}\) is linear \(C\)-independent.

In light of this, we may assume that for any \(v \in V\) there exists a suitable \(\alpha_v \in C\) such that \(cv = \alpha_vv\), and standard argument shows that there is \(\alpha \in C\) such that \(cv = \alpha v\) for all \(v \in V\). Now let \(r \in R, v \in V\). Since \(cv = \alpha v\),

\[[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha rv - r(\alpha v) = 0.\]

Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r)V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\), a contradiction. \(\square\)

**Lemma 2.2.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from 2 and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that

\[a(px^my^n - px^mpy^n) = 0,\]

for all \(x, y \in R\). Then either \(ap = 0\) or \(p = 1\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). In this case \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0, yv = v, ypv = 0\). Then we get \(0 = a(px^my^n - px^mpy^n)v = apv\). Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), from \(ap = ap^2\), we get \(0 = a(p - 1)\). Since \(a \neq 0\), we conclude \(p = 1\). \(\square\)

**Lemma 2.3.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from 2 and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that

\[a(px^my^n - py^npx^m) = 0,\]

for all \(x, y \in R\). Then \(ap = 0\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). Here \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0, yv = v, ypv = 0\). Then we have \(0 = a(px^my^n - py^npx^m)v = -ap^2v = -apv\). Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), by hypothesis, we get \(0 = a[x^n, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have
Lemma 2.4. Let $R$ be a noncommutative prime ring with extended centroid $C$ and $b, c \in R$. Suppose that $0 \neq a \in R$ such that
\[a(\{b[x,y]^2 + [x,y]^2c\} - \{b[x,y] + [x,y]c\}^2) = 0\]
for all $x, y \in R$. Then one of the following holds:
1. $c \in C$ and $a(b + c) = 0$;
2. $b, c \in C$ and $b + c = 1$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and $c \in C$.

Proof. By assumption, $R$ satisfies the generalized polynomial identity (GPI)
\[f(x,y) = a(\{b[x,y]^2 + [x,y]^2c\} - \{b[x,y] + [x,y]c\}^2).\]
By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by $U$. Now we consider the following two cases:

Case-I. $U$ does not satisfy any nontrivial GPI.

Let $T = U \ast_C C[x,y]$, the free product of $U$ and $C[x,y]$, the free $C$-algebra in noncommuting indeterminates $x$ and $y$. Thus
\[a(\{b[x,y]^2 + [x,y]^2c\} - \{b[x,y] + [x,y]c\}^2)\]
is zero element in $T = U \ast_C C[x,y]$. Let $c \notin C$. Then \{1, c\} is $C$-independent. Then from above
\[a(\{x,y\}^2c - \{b[x,y] + [x,y]c\}[x,y]c)\]
which is
\[a(\{x,y\} - b[x,y] - [x,y]c\})[x,y]c.\]
is zero in $T$. Again, since $c \notin C$, we have that $a[x,y]c[x,y]c$ is zero element in $T$, implying $a = 0$ or $c = 0$, a contradiction. Thus we conclude that $c \in C$.
Then the identity reduces to
\[a(\{b + c\}[x,y] - (b + c)[x,y]^{b + c}\})[x,y],\]
is zero element in $T$. Again, if $b + c \notin C$, then $a(b + c)[x,y]^{b + c}$ becomes zero element in $T$, implying $a(b + c) = 0$. If $b + c \in C$, then $a(b + c)(b + c - 1)[x,y]^2$ becomes zero element in $T$, implying $b + c = 0$ or $b + c = 1$. When $b + c = 0$, then $a(b + c) = 0$, which is our conclusion (1). When $b + c = 1$, then $b = 1 - c \in C$, which is our conclusion (2).

Case-II. $U$ satisfies a nontrivial GPI.
Thus we assume that
\[ a\{(b[x,y]^2 + [x,y]^2c) - (b[x,y] + [x,y]c)^2\} = 0, \]
is a nontrivial GPI for \( U \). In case \( C \) is infinite, we have \( f(x,y) = 0 \) for all \( x, y \in U \otimes_C \overline{C} \), where \( \overline{C} \) is the algebraic closure of \( C \). Since both \( U \) and \( U \otimes_C \overline{C} \) are prime and centrally closed \cite{17}, we may replace \( R \) by \( U \) or \( U \otimes_C \overline{C} \) according to \( C \) finite or infinite. Thus we may assume that \( R \) centrally closed over \( C \) which either finite or algebraically closed and \( f(x,y) = 0 \) for all \( x, y \in R \). By Martindale’s Theorem \cite{25}, \( R \) is then primitive ring having non-zero socle \( soc(R) \) with \( C \) as the associated division ring. Hence by Jacobson’s Theorem \cite{20}, \( R \) is isomorphic to a dense ring of linear transformations of a vector space \( V \) over \( C \). Since \( R \) is noncommutative, \( \dim_C V \geq 2 \). If \( \dim_C V = 2 \), then \( R \cong M_2(C) \). In this case by Lemma 2.1, either \( c \in C \) or \( \text{char } (R) = 2 \). This gives conclusions (3) and (4).

Let \( \dim_C V \geq 3 \). Let for some \( v \in V \), \( v \) and \( cv \) are linearly independent over \( C \). By density there exist \( x, y \in R \) such that
\[ xv = v, \quad xcv = 0; \]
\[ yv = 0, \quad ycv = v. \]

Then \( [x,y]v = 0, \ [x,y]cv = v \) and hence \( a\{(b[x,y]^2 + [x,y]^2c) - (b[x,y] + [x,y]c)^2\} = av \).

This implies that if \( av \neq 0 \), then by contradiction we may conclude that \( v \) and \( cv \) are linearly \( C \)-dependent. Now choose \( v \in V \) such that \( v \) and \( cv \) are linearly \( C \)-independent. Set \( W = \text{Span}_C \{v, cv\} \). Then \( av = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and then \( a(v-w) = aw \neq 0 \). By the previous argument we have that \( w, cw \) are linearly \( C \)-dependent and \( (v-w), c(v-w) \) too. Thus there exist \( \alpha, \beta \in C \) such that \( cw = \alpha w \) and \( c(v-w) = \beta(v-w) \). Then \( cv = \beta(v-w) + cw = \beta(v-w) + \alpha w \) i.e., \( (\alpha - \beta)w = cv - \beta v \in W \). Now \( \alpha = \beta \) implies that \( cv = \beta v \), a contradiction. Hence \( \alpha \neq \beta \) and so \( w \in W \). Again, if \( u \in V \) with \( au = 0 \) then \( a(w+u) \neq 0 \). So, \( w+u \in W \) forcing \( u \in W \). Thus it is observed that \( w \in V \) with \( aw \neq 0 \) implies \( w \in W \) and \( u \in V \) with \( au = 0 \) implies \( u \in W \). This implies that \( V = W \) i.e., \( \dim_C V = 2 \), a contradiction.

Hence, in any case, \( v \) and \( cv \) are linearly \( C \)-dependent for all \( v \in V \). Thus for each \( v \in V \), \( cv = \alpha_v v \) for some \( \alpha_v \in C \). It is very easy to prove that \( \alpha_v \) is independent of the choice of \( v \in V \). Thus we can write \( cv = \alpha_v \) for all \( v \in V \) and \( \alpha \in C \) fixed. Now let \( r \in R, v \in V \). Since \( cv = \alpha_v \),
\[ [c,r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0. \]
Thus $[c, r] v = 0$ for all $v \in V$ i.e., $[c, r] V = 0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space $V$, $[c, r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$.

Thus our identity reduces to

$$a\{(b'[x, y]^2) - (b'[x, y])^2\} = 0,$$

for all $x, y \in R$, where $b' = b + c$.

Let for some $v \in V$, $v$ and $b' v$ are linearly independent over $C$. Since $\dim_C V \geq 3$, there exists $u \in V$ such that $v, b' v, u$ are linearly independent over $C$. By density there exist $x, y \in R$ such that

$$x v = v, \quad x b' v = 0, \quad x u = v;$$

$$y v = 0, \quad y b' v = u, \quad y u = v.$$

Then $[x, y] v = 0$, $[x, y] b' v = v$, $[x, y] u = v$ and hence $0 = a\{(b'[x, y]^2) - (b'[x, y])^2\} u = ab' v$. Then by same argument as before, we have either $ab' = 0$ or $v$ and $b' v$ are linearly $C$-dependent for all $v \in V$. In the first case, $0 = ab' = a(b + c)$, which is conclusion (1). In the last case, again by standard argument, we have that $b' \in C$. If $b' = 0$, then also $ab' = a(b + c) = 0$ which gives conclusion (1). So assume that $0 \neq b' \in C$. Then our identity reduces to

$$ab'(b' - 1)[x, y]^2 = 0,$$

for all $x, y \in R$. This gives $0 = ab'(b' - 1) = a(b' - 1)$. Since $a \neq 0$, we get $b' = 1$. This gives conclusion (2). \hfill \Box

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when

$$a(F(u^2) - F(u)^2) = 0,$$

for all $u \in L$. If char $(R) = 2$ and $R$ satisfies $s_4$, then we have our conclusion (3). So we assume that either char $(R) \neq 2$ or $R$ does not satisfy $s_4$. Since $L$ is a noncentral by Remark 1.4, there exists a nonzero ideal $I$ of $R$ such that $[I, I] \subseteq L$. Thus by assumption $I$ satisfies the differential identity

$$a(F([x, y]^2) - F([x, y])^2) = 0.$$

Now since $R$ is a prime ring and $F$ is a generalized derivation of $R$, by Lee [23, Theorem 3], $F(x) = bx + d(x)$ for some $b \in U$ and derivation $d$ on $U$. Since $I, R$ and $U$ satisfy the same differential identities [24], without loss of generality, $U$ satisfies

$$a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0. \quad (2.5)$$

Here we divide the proof into two cases:
Case 1. Let \( d \) be inner derivation induced by element \( c \in U \), that is, \( d(x) = [c, x] \) for all \( x \in U \). It follows that
\[
 a(b[x, y]^2 + [c, [x, y]^2]) - (b[x, y] + [c, [x, y]])^2) = 0, 
\]
that is
\[
 a((b + c)[x, y]^2 - [x, y]^2 c - ((b + c)[x, y] - [x, y]c)^2 = 0, 
\]
for all \( x, y \in U \). Now by Lemma 2.4, one of the following holds:
\begin{enumerate}
  \item \( c \in C \) and \( 0 = a(b + c - c) = ab \). Thus \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \).
  \item \( b + c, c \in C \) and \( b + c - c = 1 \). Thus \( F(x) = x \) for all \( x \in R \).
  \item \( \text{char} (R) \neq 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).
\end{enumerate}

Case 2. Assume that \( d \) is not inner derivation of \( U \). We have from (2.5) that \( U \) satisfies
\[
 a(b[x, y]^2 + d([x, y])[x, y] + [x, y]d([x, y])) - (b[x, y] + d([x, y]))^2) = 0, 
\]
that is
\[
 a(b[x, y]^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y]([d(x), y] + [x, d(y)]))
\]
\[
 - (b[x, y] + [d(x), y] + [x, d(y)])^2) = 0. 
\]
Then by Kharchenko’s Theorem [21], \( U \) satisfies
\[
 a(b[x, y]^2 + ([u, y] + [x, z])[x, y] + [x, y]([u, y] + [x, z]))
\]
\[
 - (b[x, y] + [u, y] + [x, z])^2) = 0. 
\]
(2.6)

Since \( R \) is noncommutative, we may choose \( q \in U \) such that \( q \notin C \). Then replacing \( u \) by \([q, x] \) and \( z \) by \([y, y] \) in (2.6), we get
\[
 a(b[x, y]^2 + ([q, x], y] + [x, [q, y]][x, y] + [x, y]([q, x], y] + [x, [q, y]]))
\]
\[
 - (b[x, y] + ([q, x], y] + [x, [q, y]])^2) = 0, 
\]
which is
\[
 a(b[x, y]^2 + [q, [x, y]^2]) - (b[x, y] + [q, [x, y]]^2) = 0. 
\]
Then by Lemma 2.4, we have \( q \in C \), a contradiction.

Now replacing \( F \) with \(-F \) in the above result, we obtain the conclusion for the situation \( a(F(a^2) + F(a)^2) = 0 \) for all \( a \in L \).

Corollary 2.5. Let \( R \) be a prime ring with extended centroid \( C \), \( L \) a noncentral Lie ideal of \( R \) and \( 0 \neq a \in R \). If \( R \) admits the generalized derivation \( F \) such that either \( a(F(XY) + F(X)F(Y)) = 0 \) for all \( X, Y \in L \) or \( a(F(XY) + F(Y)F(X)) = 0 \) for all \( X, Y \in L \), then one of the following holds:
\begin{enumerate}
  \item there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
  \item \( F(x) = ±x \) for all \( x \in R \);
  \item \( \text{char} (R) = 2 \) and \( R \) satisfies \( s_4 \);
  \item \( \text{char} (R) \neq 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).
\end{enumerate}
Proof of Theorem 1.2. First consider the case when \(a(F(xymn) − F(xm)F(yn)) = 0\) for all \(x, y \in R\). Let \(G_1\) be the additive subgroup of \(R\) generated by the set \(S_1 = \{x^m | x \in R\}\) and \(G_2\) be the additive subgroup of \(R\) generated by the set \(S_2 = \{x^n | x \in R\}\). Then by assumption

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x \in G_1, \quad \forall y \in G_2.
\]

Then by [7], either \(G_1 \subseteq Z(R)\) or \(\text{char } (R) = 2\) and \(R\) satisfies \(s_4\), except when \(G_1\) contains a noncentral Lie ideal \(L_1\) of \(R\). \(G_1 \subseteq Z(R)\) implies that \(x^m \in Z(R)\) for all \(x \in R\). It is well known that in this case \(R\) must be commutative, which is a contradiction. Since \(\text{char } (R) \neq 2\), we are to consider the case when \(G_1\) contains a noncentral Lie ideal \(L_1\) of \(R\). In this case by [4, Lemma 1], there exists a nonzero ideal \(I_1\) of \(R\) such that \([I_1, I_1] \subseteq L_1\).

Thus we have

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in G_2.
\]

Analogously, we see that there exists a nonzero ideal \(I_2\) of \(R\) such that

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in [I_2, I_2].
\]

By Lee [23, Theorem 3], \(F(x) = bx + d(x)\) for some \(b \in U\) and derivations \(d\) on \(U\). Since \(I_1, I_2, R\) and \(U\) satisfy the same differential identities [24], without loss of generality,

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x, y \in [R, R],
\]

and in particular

\[
a(F(x^2) − F(x)^2) = 0 \quad \forall x \in [R, R].
\]

Then by Theorem 1.1, we get

1. there exists \(b \in U\) such that \(F(x) = bx\) for all \(x \in R\), with \(ab = 0\);
2. \(F(x) = x\) for all \(x \in R\);
3. \(R\) satisfies \(s_4\) and there exists \(b \in U\) such that \(F(x) = bx\) for all \(x \in R\).

In the last conclusion, \(R\) satisfies polynomial identity and hence \(R \subseteq M_2(C)\) for some field \(C\) and \(M_2(C)\) satisfies \(a(bx^my^n − bx^my^n) = 0\). By lemma 2.2, we get either \(ab = 0\) or \(b = 1\). If \(ab = 0\), then \(F(x) = bx\) for all \(x \in R\), with \(ab = 0\), which is our conclusion (1). If \(b = 1\) then \(F(x) = x\) for all \(x \in R\), which is our conclusion (2).

Now replacing \(F\) with \(-F\) in the hypothesis \(a(F(x^my^n) − F(x^m)F(y^n)) = 0\), we get \(0 = a((−F)(x^my^n) − (−F)(x^m)(−F)(y^n))\), that is \(0 = a(F(x^my^n) + F(x^m)F(y^n))\) for all \(x, y \in R\) implies one of the following:

1. there exists \(b \in U\) such that \(F(x) = bx\) for all \(x \in R\), with \(ab = 0\);
2. \(F(x) = −x\) for all \(x \in R\);
Now consider the case when \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \).

By similar argument as above we get
\[
a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R, R],
\]
and in particular
\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].
\]

Then by Theorem 1.1, we get
\[
(1) \text{ there exists } b \in U \text{ such that } F(x) = bx \text{ for all } x \in R, \text{ with } ab = 0;
(2) F(x) = x \text{ for all } x \in R;
(3) R \text{ satisfies } s_4 \text{ and there exists } b \in U \text{ such that } F(x) = bx \text{ for all } x \in R.
\]

In conclusion, if \( R \) is a prime ring of characteristic different from 2 and

\[
a \neq 0 \text{ in } R. \quad \text{Suppose that } R \text{ admits the generalized derivation } F \text{ associated with a nonzero derivation } d \text{ of } R.
\]

If any one of the following conditions is satisfied:

\[
\begin{align*}
(1) & \quad a(F(x^m y^n) - F(x^m)F(y^n)) = 0 \text{ for all } x, y \in R; \\
(2) & \quad a(F(x^m y^n) + F(y^n)F(x^m)) = 0 \text{ for all } x, y \in R,
\end{align*}
\]

then \( R \) is commutative.

**Proof of Theorem 1.3.** First we consider the case
\[
a(F(x^m y^n) + F(x^m)F(y^n)) = 0 \quad \forall x, y \in R.
\]

Other cases are similar. We know the fact that any derivation of a semiprime ring \( R \) can be uniquely extended to a derivation of its left Utumi quotient ring \( U \) and so any derivation of \( R \) can be defined on the whole of \( U \) [24, Lemma 2]. Moreover \( R \) and \( U \) satisfy the same GPIs as well as same differential identities. Thus
\[
a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0
\]
for all \( x, y \in U \). Let \( M(C) \) be the set of all maximal ideals of \( C \) and \( P \in M(C) \).

Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have \( PU \) is a prime ideal of \( U \) invariant under all derivations of \( U \). Moreover, \( \bigcap \{ PU \mid P \in M(C) \} = 0 \). Set \( \overline{U} = U/PU \). Then derivation \( d \) canonically induces a derivation \( \overline{d} \) on \( \overline{U} \) defined by \( \overline{d}(x) = \overline{d}(x) \) for all \( x \in U \).

Therefore,
\[
\overline{d}(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0
\]
for all \( \pi, \eta \in U \). By the prime ring case of Corollary 2.6, we have either 
\( d = 0 \) or 
\( [U, U] = 0 \) or 
\( a = 0 \). In any case we have 
\( ad(U)[U, U] \subseteq P U \) for all 
\( P \in M(C) \). Since 
\( \bigcap \{ PU \mid P \in M(C) \} = 0 \), 
\( ad(U)[U, U] = 0 \). In particular, 
\( ad(R)[R, R] = 0 \). This implies 
\( 0 = ad(R)[R^2, R] = ad(R)[R, R] + ad(R)[R, R^2] \). In particular, 
\( ad(R)[R, ad(R)] = 0 \). Therefore, 
\( ad(R)[R, R] = 0 \). Since \( R \) is semiprime, we obtain that 
\( ad(R) \subseteq Z(R) \). By Theorem 3.2 in [10], there exist orthogonal central idempotents 
\( e_1, e_2, e_3 \in U \) with 
\( e_1 + e_2 + e_3 = 1 \) such that 
\( d(e_1 U) = 0 \), 
\( e_2 a = 0 \), and 
\( e_3 U \) is commutative. Hence the theorem is proved.

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