Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

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Abstract. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(u^2) \pm F(u^2)) = 0$ for all $u \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. $\text{char}(R) = 2$ and $R$ satisfies $s_4$;
4. $\text{char}(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

We also study the situations (i) $a(F(x^my^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, and (ii) $a(F(x^my^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$ in prime and semiprime rings.

Keywords: Prime ring, Generalized derivation, Utumi quotient ring.

1. Introduction

Let $R$ be an associative prime ring with center $Z(R)$ and $U$ the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \to R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, $d$ is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let $S$ be a nonempty subset of $R$ and $F : R \to R$ be an additive mapping. Then we say that $F$ acts as homomorphism or anti-homomorphism on $S$ if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping $F$ acts as a Jordan homomorphism on $S$ if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. Asma, Rehman, Shakir in [1] proved that if $d$ is a derivation of a 2-torsion free prime ring $R$ which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation $F$ acts as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring $R$ as well as associated map $F$ (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let $R$ be a prime ring of characteristic different from 2, $F$ a generalized derivation of $R$ associated to a derivation $d$, $L$ a Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$. If $F$ acts as a homomorphism or anti-homomorphism on $L$, then either $d = 0$ or $L$ is central in $R$. More recently in [9], Filippis studied the situation when generalized derivation $F$ acts as a Jordan homomorphism on a non-central Lie ideal $L$ of $R$.

Recently in [26], Rehman and Raza proved the following: Let $R$ be a prime ring of char $(R) \neq 2$, $Z$ the center of $R$, and $L$ a nonzero Lie ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ which acts as a homomorphism or as anti-homomorphism on $L$, then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map $F$.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map $F$.

The main results of this paper are as follows:
Theorem 1.1. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(u^2) \pm F(u)^2) = 0$ for all $u \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. $\text{char}(R) = 2$ and $R$ satisfies $s_4$;
4. $\text{char}(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

Theorem 1.2. Let $R$ be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $F$ a generalized derivation on $R$ and $0 \neq a \in R$.

1. If $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$ or $F(x) = \mp x$ for all $x \in R$.
2. If $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$.

Theorem 1.3. Let $R$ be a noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $F(x) = bx + d(x)$ a generalized derivation on $R$ associated to the derivation $d$ and $0 \neq a \in R$. If any one of the following holds:

1. $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$,
2. $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$,
then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and $e_3U$ is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let $R$ be a prime ring and $L$ a noncentral Lie ideal of $R$. If $\text{char}(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. If $\text{char}(R) = 2$ and $\dim_C RC > 4$, i.e., $\text{char}(R) = 2$ and $R$ does not satisfy $s_4$, then by [22, Theorem 13] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. Thus if either $\text{char}(R) \neq 2$ or $R$ does not satisfy $s_4$, then we may conclude that there exists a nonzero ideal $I$ of $R$ such that $[I, I] \subseteq L$.

Remark 1.5. We denote by $\text{Der}(U)$ the set of all derivations on $U$. By a derivation word $\Delta$ of $R$ we mean $\Delta = d_1 d_2 d_3 \ldots d_m$ for some derivations $d_i \in \text{Der}(U)$.

Let $D_{in}$ be the $C$-subspace of $\text{Der}(U)$ consisting of all inner derivations on $U$ and let $d$ be a non-zero derivation on $R$. By [21, Theorem 2] we have the following result:
If \( \Phi(x_1, x_2, \ldots, x_n, d(x_1), d(x_2) \cdots d(x_n)) \) is a differential identity on \( R \), then one of the following holds:

1. \( d \in D_{int} \);
2. \( R \) satisfies the generalized polynomial identity \( \Phi(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \).

**Remark 1.6.** In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping \( F : I \rightarrow U \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in I \), where \( I \) is a dense left ideal of \( R \) and \( d \) is a derivation from \( I \) into \( U \). Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of \( U \), and thus all generalized derivations of \( R \) will be implicitly assumed to be defined on the whole of \( U \). Lee obtained the following: every generalized derivation \( F \) on a dense left ideal of \( R \) can be uniquely extended to \( U \) and assumes the form \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \).

### 2. Proof of the Main Results

Now we begin with the following Lemmas:

**Lemma 2.1.** Let \( R = M_2(C) \) be the ring of all \( 2 \times 2 \) matrices over the field \( C \) of characteristic different from 2 and \( b, c \in R \). Suppose that there exists

\[
a(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2 = 0,
\]

for all \( x, y \in R \). Then \( c \in C \cdot I_2 \).

**Proof.** If \( c \in C \cdot I_2 \), then nothing to prove. Let \( c \notin C \cdot I_2 \). In this case \( R \) is a dense ring of \( C \)-linear transformations over a vector space \( V \). Assume that there exists \( 0 \neq v \in V \) such that \( \{v, cv\} \) is linearly \( C \)-independent. By density, there exist \( x, y \in R \) such that \( xv = v, xcv = 0; yv = 0, ycv = v \). Then \( [x, y]v = 0, [x, y]cv = v \) and hence \( a(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2 \) \( v = av \).

Of course for any \( u \in V \), \( \{u, v\} \) linearly \( C \)-dependent implies \( au = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and so \( \{w, v\} \) are linearly \( C \)-independent. Also \( a(w + v) = aw \neq 0 \) and \( a(w - v) = aw \neq 0 \). By the above argument, it follows that \( w \) and \( cw \) are linearly \( C \)-dependent, as are \( \{w + v, c(w + v)\} \) and \( \{w - v, c(w - v)\} \). Therefore there exist \( \alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C \) such that

\[
cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).
\]

In other words we have

\[
\alpha_w w + cw = \alpha_{w+v} w + \alpha_{w+v} v, \quad (2.1)
\]

and

\[
\alpha_w w - cw = \alpha_{w-v} w - \alpha_{w-v} v. \quad (2.2)
\]
By comparing (2.1) with (2.2) we get both
\[(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0\] (2.3)
and
\[2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.\] (2.4)
By (2.3), and since \(\{w, v\}\) are \(C\)-independent and \(\text{char}(R) \neq 2\), we have \(\alpha_w = \alpha_{w+v} = \alpha_{w-v}\). Thus by (2.4) it follows \(2cv = 2\alpha_w v\). This leads a contradiction with the fact that \(\{v, cv\}\) is linear \(C\)-independent.

In light of this, we may assume that for any \(v \in V\) there exists a suitable \(\alpha_v \in C\) such that \(cv = \alpha_v v\), and standard argument shows that there is \(a \in C\) such that \(cv = av\) for all \(v \in V\). Now let \(r \in R\), \(v \in V\). Since \(cv = av\),
\[[c, rv] = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.\]
Thus \([c, rv] = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\), a contradiction. \(\Box\)

**Lemma 2.2.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from 2 and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^m y^n - px^n py^m) = 0,\]
for all \(x, y \in R\). Then either \(ap = 0\) or \(p = 1\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). In this case \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = 0\). Then we get \(0 = a(px^m y^n - px^n py^m)v = apv\). Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), from \(ap = ap^2\), we get \(0 = a(p - 1)\). Since \(a \neq 0\), we conclude \(p = 1\). \(\Box\)

**Lemma 2.3.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from 2 and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^m y^n - py^m px^n) = 0,\]
for all \(x, y \in R\). Then \(ap = 0\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). Here \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = 0\). Then we have \(0 = a(px^m y^n - py^m px^n)v = -ap^2 v = -apv\). Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), by hypothesis, we get \(0 = a[x^m, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have
0 = a[x^m, y^n] = a[e_{11}, e_{11} + e_{12}] = ae_{12}. Again, for x = e_{22} and y = e_{22} + e_{21}, we have 0 = a[x^m, y^n] = a[e_{22}, e_{22} + e_{21}] = ae_{21}. These imply a = 0, a contradiction. □

**Lemma 2.4.** Let $R$ be a noncommutative prime ring with extended centroid $C$ and $b, c \in R$. Suppose that $0 \neq a \in R$ such that

$$a\{(b|x, y|^2 + [x, y]^2)c - (b|x, y| + [x, y]c)^2\} = 0$$

for all $x, y \in R$. Then one of the following holds:

1. $c \in C$ and $a(b + c) = 0$;
2. $b, c \in C$ and $b + c = 1$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and $c \in C$.

**Proof.** By assumption, $R$ satisfies the generalized polynomial identity (GPI)

$$f(x, y) = a\{(b|x, y|^2 + [x, y]^2)c - (b|x, y| + [x, y]c)^2\}.$$

By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by $U$. Now we consider the following two cases:

**Case-I.** $U$ does not satisfy any nontrivial GPI.

Let $T = U \ast_C C[x, y]$, the free product of $U$ and $C[x, y]$, the free $C$-algebra in noncommuting indeterminates $x$ and $y$. Thus

$$a\{[[x, y]^2c - (b|x, y| + [x, y]c)[x, y]c,]$$

which is

$$a\{[x, y] - b|x, y| - [x, y]c\}[x, y]c,$$

is zero in $T$. Again, since $c \notin C$, we have that $a[x, y]c[x, y]c$ is zero element in $T$, implying $a = 0$ or $c = 0$, a contradiction. Thus we conclude that $c \in C$. Then the identity reduces to

$$a\{(b + c)|x, y| - (b + c)[x, y](b + c)\}[x, y],$$

is zero element in $T$. Again, if $b + c \notin C$, then $a(b + c)|x, y|^2$ becomes zero element in $T$, implying $a(b + c) = 0$. If $b + c \in C$, then $a(b + c)(b + c - 1)|x, y|^2$ becomes zero element in $T$, implying $b + c = 0$ or $b + c = 1$. When $b + c = 0$, then $a(b + c) = 0$, which is our conclusion (1). When $b + c = 1$, then $b = 1 - c \in C$, which is our conclusion (2).

**Case-II.** $U$ satisfies a nontrivial GPI.
Thus we assume that
\[ a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0, \]
is a nontrivial GPI for \( U \). In case \( C \) is infinite, we have \( f(x, y) = 0 \) for all \( x, y \in U \otimes_C \overline{C} \), where \( \overline{C} \) is the algebraic closure of \( C \). Since both \( U \) and \( U \otimes_C \overline{C} \) are prime and centrally closed [17], we may replace \( R \) by \( U \) or \( U \otimes_C \overline{C} \) according to \( C \) finite or infinite. Thus we may assume that \( R \) centrally closed over \( C \) which either finite or algebraically closed and \( f(x, y) = 0 \) for all \( x, y \in R \).

By Martindale’s Theorem [25], \( R \) is then primitive ring having non-zero socle \( \text{soc}(R) \) with \( C \) as the associated division ring. Hence by Jacobson’s Theorem [20], \( R \) is isomorphic to a dense ring of linear transformations of a vector space \( V \) over \( C \). Since \( R \) is noncommutative, \( \dim_C V \geq 2 \). If \( \dim_C V = 2 \), then \( R \cong M_2(C) \). In this case by Lemma 2.1, either \( c \in C \) or \( \text{char } (R) = 2 \). This gives conclusions (3) and (4).

Let \( \dim_C V \geq 3 \). Let for some \( v \in V \), \( v \) and \( cv \) are linearly independent over \( C \). By density there exist \( x, y \in R \) such that
\[ xv = v, \quad xcv = 0; \]
\[ yv = 0, \quad ycv = v. \]

Then \( [x, y]v = 0, \) \( [x, y]cv = v \) and hence \( a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}v = av \).

This implies that if \( av \neq 0 \), then by contradiction we may conclude that \( v \) and \( cv \) are linearly \( C \)-dependent. Now choose \( v \in V \) such that \( v \) and \( cv \) are linearly \( C \)-independent. Set \( W = \text{Span}_C\{v, cv\} \). Then \( av = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and then \( a(v - w) = aw \neq 0 \). By the previous argument we have that \( w, cw \) are linearly \( C \)-dependent and \( (v - w), c(v - w) \) too. Thus there exist \( \alpha, \beta \in C \) such that \( cw = \alpha w \) and \( c(v - w) = \beta(v - w) \). Then \( cv = \beta(v - w) + cw = \beta(v - w) + \alpha w \) i.e., \((\alpha - \beta)w = cv - \beta v \in W \). Now \( \alpha = \beta \) implies that \( cv = \beta v \), a contradiction. Hence \( \alpha \neq \beta \) and so \( w \in W \). Again, if \( u \in V \) with \( au = 0 \) then \( a(u + w) \neq 0 \). So, \( w + u \in W \) forcing \( u \in W \). Thus it is observed that \( w \in V \) with \( aw \neq 0 \) implies \( w \in W \) and \( u \in V \) with \( au = 0 \) implies \( u \in W \). This implies that \( V = W \) i.e., \( \dim_C V = 2 \), a contradiction.

Hence, in any case, \( v \) and \( cv \) are linearly \( C \)-dependent for all \( v \in V \). Thus for each \( v \in V \), \( cv = \alpha_v v \) for some \( \alpha_v \in C \). It is very easy to prove that \( \alpha_v \) is independent of the choice of \( v \in V \). Thus we can write \( cv = \alpha v \) for all \( v \in V \) and \( \alpha \in C \) fixed. Now let \( r \in R, v \in V \). Since \( cv = \alpha v \),
\[ [c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0. \]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r]V = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\).

Thus our identity reduces to

\[a\{(b'[x, y]^2) - (b'[x, y])^2\} = 0,\]

for all \(x, y \in R\), where \(b' = b + c\).

Let for some \(v \in V\), \(v\) and \(b'v\) are linearly independent over \(C\). Since \(\dim_C V \geq 3\), there exists \(u \in V\) such that \(v, b'v, u\) are linearly independent over \(C\). By density there exist \(x, y \in R\) such that

\[xv = v, \quad xb'v = 0, \quad xu = v;\]

\[yv = 0, \quad yb'v = u, \quad yu = v.\]

Then \([x, y]v = 0, [x, y]b'v = v, [x, y]u = v\) and hence \(0 = a\{(b'[x, y]^2) - (b'[x, y])^2\}u = ab'v\). Then by same argument as before, we have either \(ab' = 0\) or \(v\) and \(b'v\) are linearly \(C\)-dependent for all \(v \in V\). In the first case, \(0 = ab' = a(b + c)\), which is conclusion (1). In the last case, again by standard argument, we have that \(b' \in C\). If \(b' = 0\), then also \(ab' = a(b + c) = 0\) which gives conclusion (1). So assume that \(0 \neq b' \in C\). Then our identity reduces to \(ab'(b' - 1)[x, y]^2 = 0\), for all \(x, y \in R\). This gives \(0 = ab'(b' - 1) = a(b' - 1)\). Since \(a \neq 0\), we get \(b' = 1\). This gives conclusion (2).

\[\Box\]

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when

\[a(F(u^2) - F(u)^2) = 0,\]

for all \(u \in L\). If \(\text{char } (R) = 2\) and \(R\) satisfies \(s_4\), then we have our conclusion (3). So we assume that either \(\text{char } (R) \neq 2\) or \(R\) does not satisfy \(s_4\). Since \(L\) is a noncentral by Remark 1.4, there exists a nonzero ideal \(I\) of \(R\) such that \([I, I] \subseteq L\). Thus by assumption \(I\) satisfies the differential identity

\[a(F([x, y]^2) - F([x, y])^2) = 0.\]

Now since \(R\) is a prime ring and \(F\) is a generalized derivation of \(R\), by Lee [23, Theorem 3], \(F(x) = bx + d(x)\) for some \(b \in U\) and derivation \(d\) on \(U\). Since \(I, R\) and \(U\) satisfy the same differential identities [24], without loss of generality, \(U\) satisfies

\[a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0.\]

(2.5)

Here we divide the proof into two cases:
Case 1. Let $d$ be inner derivation induced by element $c \in U$, that is, $d(x) = [c, x]$ for all $x \in U$. It follows that
\[
a(b[x, y]^2 + [c, [x, y]^2] - (b[x, y] + [c, [x, y]])^2) = 0,
\]
that is
\[
a((b + c)[x, y]^2 - [x, y]^2c - ((b + c)[x, y] - [x, y]c)^2) = 0,
\]
for all $x, y \in U$. Now by Lemma 2.4, one of the following holds:

1. $c \in C$ and $0 = a(b + c - c) = ab$. Thus $F(x) = bx$ for all $x \in R$, with $ab = 0$.

2. $b + c, c \in C$ and $b + c - c = 1$. Thus $F(x) = x$ for all $x \in R$.

3. $\text{char}(R) = 2$, $R$ satisfies $s_4$ and $c \in C$. Thus $F(x) = bx$ for all $x \in R$.

Case 2. Assume that $d$ is not inner derivation of $U$. We have from (2.5) that $U$ satisfies
\[
a(b[x, y]^2 + d([x, y])[[x, y] + [x, y]d([x, y])] - (b[x, y] + d([x, y]))^2) = 0,
\]
that is
\[
a(b[x, y]^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y][d(d(x), y) + [x, d(y)])
- (b[x, y] + [d(x), y] + [x, d(y)])^2 = 0.
\]
Then by Kharchenko’s Theorem [21], $U$ satisfies
\[
a(b[x, y]^2 + ([u, y] + [x, z])[x, y] + [x, y][(u, y) + [x, z]])
- (b[x, y] + [u, y] + [x, z])^2 = 0.
\]
Since $R$ is noncommutative, we may choose $q \in U$ such that $q \notin C$. Then replacing $u$ by $[q, x]$ and $z$ by $[y, y]$ in (2.6), we get
\[
a(b[x, y]^2 + ([q, x], y) + [x, [q, y]])[x, y] + [x, y][([q, x], y) + [x, [q, y]])
- (b[x, y] + ([q, x], y) + [x, [q, y]])^2 = 0,
\]
which is
\[
a(b[x, y]^2 + [q, [x, y]^2]) - (b[x, y] + [q, [x, y]^2]) = 0.
\]
Then by Lemma 2.4, we have $q \in C$, a contradiction.

Now replacing $F$ with $-F$ in the above result, we obtain the conclusion for the situation $a(F(a^2) + F(a)^2) = 0$ for all $a \in L$.

**Corollary 2.5.** Let $R$ be a prime ring with extended centroid $C$, $L$ a noncentral Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits the generalized derivation $F$ such that either $a(F(XY) \pm F(X)F(Y)) = 0$ for all $X, Y \in L$ or $a(F(XY) \pm F(Y)F(X)) = 0$ for all $X, Y \in L$, then one of the following holds:

1. There exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = bx$ for all $x \in R$;
3. $\text{char}(R) = 2$ and $R$ satisfies $s_4$;
4. $\text{char}(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$. 
Proof of Theorem 1.2. First consider the case when $a(F(x)F(y) - F(xm)F(yn)) = 0$ for all $x, y \in R$. Let $G_1$ be the additive subgroup of $R$ generated by the set $S_1 = \{xm | x \in R\}$ and $G_2$ be the additive subgroup of $R$ generated by the set $S_2 = \{x^n | x \in R\}$. Then by assumption
\[ a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in G_1, \quad \forall y \in G_2. \]

Then by [7], either $G_1 \subseteq Z(R)$ or char$(R) = 2$ and $R$ satisfies $s_4$, except when $G_1$ contains a noncentral Lie ideal $L_1$ of $R$. $G_1 \subseteq Z(R)$ implies that $x^m \in Z(R)$ for all $x \in R$. It is well known that in this case $R$ must be commutative, which is a contradiction. Since char$(R) \neq 2$, we are to consider the case when $G_1$ contains a noncentral Lie ideal $L_1$ of $R$. In this case by [4, Lemma 1], there exists a nonzero ideal $I_1$ of $R$ such that $[I_1, I_1] \subseteq L_1$.

Thus we have
\[ a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in G_2. \]

Analogously, we see that there exists a nonzero ideal $I_2$ of $R$ such that
\[ a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in [I_2, I_2]. \]

By Lee [23, Theorem 3], $F(x) = bx + d(x)$ for some $b \in U$ and derivations $d$ on $U$. Since $I_1, I_2, R$ and $U$ satisfy the same differential identities [24], without loss of generality,
\[ a(F(xy) - F(x)F(y)) = 0 \quad \forall x, y \in [R, R], \]

and in particular
\[ a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R]. \]

Then by Theorem 1.1, we get
(1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
(2) $F(x) = x$ for all $x \in R$;
(3) $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

In the last conclusion, $R$ satisfies polynomial identity and hence $R \subseteq M_2(C)$ for some field $C$ and $M_2(C)$ satisfies $a(bx^n y^m - bx^m y^n) = 0$. By lemma 2.2, we get either $ab = 0$ or $b = 1$. If $ab = 0$, then $F(x) = bx$ for all $x \in R$, with $ab = 0$, which is our conclusion (1). If $b = 1$ then $F(x) = x$ for all $x \in R$, which is our conclusion (2).

Now replacing $F$ with $-F$ in the hypothesis $a(F(x^m y^n) - F(x^m)F(y^n)) = 0$, we get $0 = a((-F)(x^m y^n) - (-F)(x^m)(-F)(y^n))$, that is $0 = a(F(x^m y^n) + F(x^m)F(y^n))$ for all $x, y \in R$ implies one of the following:
(1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
(2) $F(x) = -x$ for all $x \in R$;
Now consider the case when $a(F(x^m y^n) - F(y^n) F(x^m)) = 0$ for all $x, y \in R$. By similar argument as above we get
\[ a(F(xy) - F(y) F(x)) = 0 \quad \forall x, y \in [R, R], \]
and in particular
\[ a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R]. \]
Then by Theorem 1.1, we get
\[ (1) \text{ there exists } b \in U \text{ such that } F(x) = bx \text{ for all } x \in R, \text{ with } ab = 0; \]
\[ (2) F(x) = x \text{ for all } x \in R; \]
\[ (3) R \text{ satisfies s4 and there exists } b \in U \text{ such that } F(x) = bx \text{ for all } x \in R. \]
In the conclusion (3), $R$ satisfies polynomial identity and hence $R \subseteq M_2(C)$ for some field $C$ and $M_2(C)$ satisfies $a(bx^m y^n - by^n bx^m) = 0$. Then by Lemma 2.3, we have $ab = 0$, which is our conclusion (1).

Now replacing $F$ with $-F$ in the hypothesis $a(F(x^m y^n) - F(y^n) F(x^m)) = 0$, we get $0 = a((-F)(x^m y^n) - (-F)(y^n) (-F)(x^m))$. That is, $0 = a(F(x^m y^n) + F(y^n) F(x^m))$ for all $x, y \in R$. This implies that there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$ or $F(x) = -x$. This completes the proof.

In particular, we have the following corollary.

**Corollary 2.6.** Let $R$ be a prime ring of characteristic different from 2 and $0 \neq a \in R$. Suppose that $R$ admits the generalized derivation $F$ associated with a nonzero derivation $d$ of $R$. If any one of the following conditions is satisfied:

1. $a(F(x^m y^n) + F(x^m) F(y^n)) = 0$ for all $x, y \in R$;
2. $a(F(x^m y^n) - F(y^n) F(x^m)) = 0$ for all $x, y \in R$;
then $R$ is commutative.

**Proof of Theorem 1.3.** First we consider the case $a(F(x^m y^n) + F(x^m) F(y^n)) = 0$ for all $x, y \in R$. Other cases are similar. We know the fact that any derivation of a semiprime ring $R$ can be uniquely extended to a derivation of its left Utumi quotient ring $U$ and so any derivation of $R$ can be defined on the whole of $U$ [24, Lemma 2]. Moreover $R$ and $U$ satisfy the same GPIs as well as same differential identities. Thus
\[ a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0 \]
for all $x, y \in U$. Let $M(C)$ be the set of all maximal ideals of $C$ and $P \in M(C)$. Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have $P U$ is a prime ideal of $U$ invariant under all derivations of $U$. Moreover, $\bigcap \{P U \mid P \in M(C)\} = 0$. Set $\overline{U} = U/P U$. Then derivation $d$ canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,
\[ \overline{d}(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0 \]
for all \( \pi, \bar{\pi} \in U \). By the prime ring case of Corollary 2.6, we have either \( d = 0 \) or \( \{ U, U \} = 0 \) or \( a = 0 \). In any case we have \( ad(U)[U, U] \subseteq P U \) for all \( P \in M(C) \). Since \( \bigcap \{ PU \mid P \in M(C) \} = 0 \), \( ad(U)[U, U] = 0 \). In particular, \( ad(R)[R, R] = 0 \). This implies \( 0 = ad(R)[R^2, R] = ad(R)[R, R] + ad(R)[R, R]R = ad(R)[R, R] \). In particular, \( ad(R)[R, ad(R)] = 0 \). Therefore, \( ad(R)[R, ad(R)] = 0 \). Since \( R \) is semiprime, we obtain that \( ad(R) \subseteq Z(R) \). By Theorem 3.2 in [10], there exist orthogonal central idempotents \( e_1, e_2, e_3 \in U \) with \( e_1 + e_2 + e_3 = 1 \) such that \( d(e_1 U) = 0 \), \( e_2 a = 0 \), and \( e_3 U \) is commutative. Hence the theorem is proved.

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