Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

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Abstract. Let \( R \) be a prime ring with its Utumi ring of quotients \( U \), \( C = Z(U) \) the extended centroid of \( R \), \( L \) a non-central Lie ideal of \( R \) and \( 0 \neq a \in R \). If \( R \) admits a generalized derivation \( F \) such that \( a(F(u^2) \pm F(u)^2) = 0 \) for all \( u \in L \), then one of the following holds:

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = \mp x \) for all \( x \in R \);
3. \( \text{char } (R) = 2 \) and \( R \) satisfies \( s_4 \);
4. \( \text{char } (R) \neq 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

We also study the situations (i) \( a(F(x^my^n) \pm F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \), and (ii) \( a(F(x^my^n) \pm F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \) in prime and semiprime rings.

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1. Introduction

Let $R$ be an associative prime ring with center $Z(R)$ and $U$ the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \rightarrow R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, $d$ is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \rightarrow R$ is called generalized derivation, if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let $S$ be a nonempty subset of $R$ and $F : R \rightarrow R$ be an additive mapping. Then we say that $F$ acts as homomorphism or anti-homomorphism on $S$ if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping $F$ acts as a Jordan homomorphism on $S$ if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. Asma, Rehman, Shakir in [1] proved that if $d$ is a derivation of a 2-torsion free prime ring $R$ which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation $F$ acts as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring $R$ as well as associated map $F$ (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26]). In [18] Golbasi and Kaya proved the following: Let $R$ be a prime ring of characteristic different from $2$, $F$ a generalized derivation of $R$ associated to a derivation $d$, $L$ a Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$. If $F$ acts as a homomorphism or anti-homomorphism on $L$, then either $d = 0$ or $L$ is central in $R$. More recently in [9], Filippis studied the situation when generalized derivation $F$ acts as a Jordan homomorphism on a non-central Lie ideal $L$ of $R$.

Recently in [26], Rehman and Raza proved the following: Let $R$ be a prime ring of char $(R) \neq 2$, $Z$ the center of $R$, and $L$ a nonzero Lie ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ which acts as a homomorphism or as anti-homomorphism on $L$, then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map $F$.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map $F$.

The main results of this paper are as follows:
Theorem 1.1. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(u^2) ± F(u)^2) = 0$ for all $u \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. $\text{char}(R) = 2$ and $R$ satisfies $s_4$;
4. $\text{char}(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

Theorem 1.2. Let $R$ be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $F$ a generalized derivation on $R$ and $0 \neq a \in R$.

1. If $a(F(x^m y^n) ± F(x^m)F(y^n)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$ or $F(x) = \mp x$ for all $x \in R$.
2. If $a(F(x^m y^n) ± F(y^n)F(x^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$.

Theorem 1.3. Let $R$ be a noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $F(x) = bx + d(x)$ a generalized derivation on $R$ associated to the derivation $d$ and $0 \neq a \in R$. If any one of the following holds:

1. $a(F(x^m y^n) ± F(x^m)F(y^n)) = 0$ for all $x, y \in R$,
2. $a(F(x^m y^n) ± F(y^n)F(x^m)) = 0$ for all $x, y \in R$,

then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and $e_3U$ is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let $R$ be a prime ring and $L$ a noncentral Lie ideal of $R$. If $\text{char}(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. If $\text{char}(R) = 2$ and $\dim_C RC > 4$, i.e., $\text{char}(R) = 2$ and $R$ does not satisfy $s_4$, then by [22, Theorem 13] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. Thus if either $\text{char}(R) \neq 2$ or $R$ does not satisfy $s_4$, then we may conclude that there exists a nonzero ideal $I$ of $R$ such that $[I, L] \subseteq L$.

Remark 1.5. We denote by $\text{Der}(U)$ the set of all derivations on $U$. By a derivation word $\Delta$ of $R$ we mean $\Delta = d_1 d_2 d_3 \ldots d_m$ for some derivations $d_i \in \text{Der}(U)$.

Let $D_{int}$ be the $C$-subspace of $\text{Der}(U)$ consisting of all inner derivations on $U$ and let $d$ be a non-zero derivation on $R$. By [21, Theorem 2] we have the following result:
If $\Phi(x_1, x_2, \cdots, x_n, d(x_1), d(x_2) \cdots d(x_n))$ is a differential identity on $R$, then one of the following holds:

1. $d \in D_{inz}$;
2. $R$ satisfies the generalized polynomial identity $\Phi(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$.

Remark 1.6. In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive map $F : I \rightarrow U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where $I$ is a dense left ideal of $R$ and $d$ is a derivation from $I$ into $U$. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of $U$, and thus all generalized derivations of $R$ will be implicitly assumed to be defined on the whole of $U$. Lee obtained the following: every generalized derivation $F$ on a dense left ideal of $R$ can be uniquely extended to $U$ and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation $d$ on $U$.

2. Proof of the Main Results

Now we begin with the following Lemmas:

Lemma 2.1. Let $R = M_2(C)$ be the ring of all $2 \times 2$ matrices over the field $C$ of characteristic different from 2 and $b, c \in R$. Suppose that there exists $0 \neq a \in R$ such that

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0,$$

for all $x, y \in R$. Then $c \in C \cdot I_2$.

Proof. If $c \in C \cdot I_2$, then nothing to prove. Let $c \notin C \cdot I_2$. In this case $R$ is a dense ring of $C$-linear transformations over a vector space $V$. Assume that there exists $0 \neq v \in V$ such that $\{v, cv\}$ is linearly $C$-independent. By density, there exist $x, y \in R$ such that $xv = v, xcv = 0; yv = 0, ycv = v$. Then $[x, y]^2v = 0, [x, y]cv = v$ and hence $a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}v = av$.

Of course for any $u \in V$, $\{u, v\}$ linearly $C$-dependent implies $au = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $\{w, v\}$ are linearly $C$-independent. Also $a(w + v) = aw \neq 0$ and $a(w - v) = aw \neq 0$. By the above argument, it follows that $w$ and $cw$ are linearly $C$-dependent, as are $\{w + v, c(w + v)\}$ and $\{w - v, c(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v} (w + v), \quad c(w - v) = \alpha_{w-v} (w - v).$$

In other words we have

$$\alpha_w w + cw = \alpha_{w+v} w + \alpha_{w+v} v \quad (2.1)$$

and

$$\alpha_w w - cw = \alpha_{w-v} w - \alpha_{w-v} v. \quad (2.2)$$
By comparing (2.1) with (2.2) we get both

\[(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0\]  
(2.3)

and

\[2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.\]  
(2.4)

By (2.3), and since \(\alpha_w \neq 0\) and \(\alpha_v \neq 0\), we get \(0 = a[x^m, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have

\[\text{by (2.3), and since } \alpha_w \neq 0 \text{ and } \alpha_v \neq 0, \text{ we get } 0 = a[x^m, y^n].\]

Thus \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have \(0 = \alpha_v \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), by hypothesis, we get \(0 = a[x^m, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have
0 = a[x^m, y^n] = a[e_{11}, e_{11} + e_{12}] = ae_{12}. Again, for x = e_{22} and y = e_{22} + e_{21}, we have 0 = a[x^m, y^n] = ae_{22}, e_{22} + e_{21} = ae_{21}. These imply a = 0, a contradiction. □

Lemma 2.4. Let R be a noncommutative prime ring with extended centroid C and b, c ∈ R. Suppose that 0 ≠ a ∈ R such that
\[ a(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2 = 0 \]
f for all x, y ∈ R. Then one of the following holds:
1. c ∈ C and a(b + c) = 0;
2. b, c ∈ C and b + c = 1;
3. char (R) = 2 and R satisfies s_{4};
4. char (R) ≠ 2, R satisfies s_{4} and c ∈ C.

Proof. By assumption, R satisfies the generalized polynomial identity (GPI)
\[ f(x, y) = a\{b[x, y]^2 + [x, y]^2c\} - (b[x, y] + [x, y]c)^2 \]
By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by U. Now we consider the following two cases:

Case-I. U does not satisfy any nontrivial GPI.

Let T = U \ast_{C} C\{x, y\}, the free product of U and C\{x, y\}, the free C-algebra in noncommuting indeterminates x and y. Thus
\[ a\{b[x, y]^2 + [x, y]^2c\} - (b[x, y] + [x, y]c)^2 \]
is zero element in T = U \ast_{C} C\{x, y\}. Let c \notin C. Then \{1, c\} is C-independent. Then from above
\[ a\{[x, y]^2c - (b[x, y] + [x, y]c)[x, y]c, \} \]
which is
\[ a\{[x, y] - b[x, y] - [x, y]c\}[x, y]c, \]
is zero in T. Again, since c \notin C, we have that a[x, y]c[x, y]c is zero element in T, implying a = 0 or c = 0, a contradiction. Thus we conclude that c ∈ C. Then the identity reduces to
\[ a\{(b + c)[x, y] - (b + c)[x, y][b + c]\}[x, y], \]
is zero element in T. Again, if b + c \notin C, then a(b + c)[x, y]^2 becomes zero element in T, implying a(b + c) = 0. If b + c ∈ C, then a(b + c)(b + c - 1)[x, y]^2 becomes zero element in T, implying b + c = 0 or b + c = 1. When b + c = 0, then a(b + c) = 0, which is our conclusion (1). When b + c = 1, then b = 1 - c ∈ C, which is our conclusion (2).

Case-II. U satisfies a nontrivial GPI.
Thus we assume that
\[
a\{(b[x,y]^2 + [x,y]^2c) - (b[x,y] + [x,y]c)^2\} = 0,
\]
is a nontrivial GPI for \(U\). In case \(C\) is infinite, we have \(f(x,y) = 0\) for all \(x,y \in U \otimes_C \overline{C}\), where \(\overline{C}\) is the algebraic closure of \(C\). Since both \(U\) and \(U \otimes_C \overline{C}\) are prime and centrally closed [17], we may replace \(R\) by \(U\) or \(U \otimes_C \overline{C}\) according to \(C\) finite or infinite. Thus we may assume that \(R\) centrally closed over \(C\) which either finite or algebraically closed and \(f(x,y) = 0\) for all \(x,y \in R\).

By Martindale’s Theorem [25], \(R\) is then primitive ring having non-zero socle \(\text{soc}(R)\) with \(C\) as the associated division ring. Hence by Jacobson’s Theorem [20], \(R\) is isomorphic to a dense ring of linear transformations of a vector space \(V\) over \(C\). Since \(R\) is noncommutative, \(\dim_C V \geq 2\). If \(\dim_C V = 2\), then \(R \cong M_2(C)\). In this case by Lemma 2.1, either \(c \in C\) or \(\text{char}(R) = 2\). This gives conclusions (3) and (4).

Let \(\dim_C V \geq 3\). Let for some \(v \in V\), \(v\) and \(cv\) are linearly independent over \(C\). By density there exist \(x, y \in R\) such that
\[
xv = v, \quad xcv = 0;
\]
\[
yv = 0, \quad ycv = v.
\]

Then \([x,y]v = 0, [x,y]cv = v\) and hence \(a\{(b[x,y]^2 + [x,y]^2c) - (b[x,y] + [x,y]c)^2\}v = av\).

This implies that if \(av \neq 0\), then by contradiction we may conclude that \(v\) and \(cv\) are linearly \(C\)-dependent. Now choose \(v \in V\) such that \(v\) and \(cv\) are linearly \(C\)-independent. Set \(W = \text{Span}_C\{v, cv\}\). Then \(av = 0\). Since \(a \neq 0\), there exists \(w \in V\) such that \(aw \neq 0\) and then \(a(v-w) = aw \neq 0\).

By the previous argument we have that \(w, cw\) are linearly \(C\)-dependent and \((v-w), c(v-w)\) too. Thus there exist \(\alpha, \beta \in C\) such that \(cw = aw\) and \(c(v-w) = \beta(v-w)\). Then \(cv = \beta(v-w) + cw = \beta(v-w) + aw\) i.e., \((\alpha-\beta)w = cv - \beta v \in W\). Now \(\alpha = \beta\) implies that \(cv = \beta v\), a contradiction.

Hence \(a \neq 0\) and so \(w \in W\). Again, if \(u \in V\) with \(au = 0\) then \(a(w+u) \neq 0\).

So, \(w+u \in W\) forcing \(u \in W\). Thus it is observed that \(w \in W\) with \(aw \neq 0\) implies \(w \in W\) and \(u \in W\) with \(au = 0\) implies \(u \in W\). This implies that \(V = W\) i.e., \(\dim_C V = 2\), a contradiction.

Hence, in any case, \(v\) and \(cv\) are linearly \(C\)-dependent for all \(v \in V\). Thus for each \(v \in V\), \(cv = \alpha_v v\) for some \(\alpha_v \in C\). It is very easy to prove that \(\alpha_v\) is independent of the choice of \(v \in V\). Thus we can write \(cv = \alpha v\) for all \(v \in V\) and \(\alpha \in C\) fixed. Now let \(r \in R\), \(v \in V\). Since \(cv = \alpha v\),
\[
[e, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.
\]
Thus \( [c, r] v = 0 \) for all \( v \in V \) i.e., \( [c, r] V = 0 \). Since \( [c, r] \) acts faithfully as a linear transformation on the vector space \( V \), \( [c, r] = 0 \) for all \( r \in R \). Therefore, \( c \in Z(R) \).

Thus our identity reduces to
\[
a\{ (b'[x, y]^2) - (b'[x, y])^2 \} = 0,
\]
for all \( x, y \in R \), where \( b' = b + c \).

Let for some \( v \in V \), \( v \) and \( b'v \) are linearly independent over \( C \). Since \( \dim_C V \geq 3 \), there exists \( u \in V \) such that \( v, b'v, u \) are linearly independent over \( C \). By density there exist \( x, y \in R \) such that
\[
xv = v, \quad xb'v = 0, \quad xu = v;
\]
\[
yv = 0, \quad yb'v = u, \quad yu = v.
\]

Then \( [x, y]v = 0, \ [x, y]b'v = v, \ [x, y]u = v \) and hence \( 0 = a\{ (b'[x, y]^2) - (b'[x, y])^2 \} u = ab'v \). Then by same argument as before, we have either \( ab = 0 \) or \( v \) and \( b'v \) are linearly \( C \)-dependent for all \( v \in V \). In the first case, \( 0 = ab = a(b + c) \), which is conclusion (1). In the last case, again by standard argument, we have that \( b' \in C \). If \( b' = 0 \), then also \( ab' = a(b + c) = 0 \) which gives conclusion (1). So assume that \( 0 \neq b' \in C \). Then our identity reduces to
\[
ab'(b' - 1)[x, y]^2 = 0,
\]
for all \( x, y \in R \). This gives \( 0 = ab'(b' - 1) = a(b' - 1) \). Since \( a \neq 0 \), we get \( b' = 1 \). This gives conclusion (2).

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when
\[
a(F(u^2) - F(u^2)) = 0,
\]
for all \( u \in L \). If \( \text{char} \ (R) = 2 \) and \( R \) satisfies \( s_4 \), then we have our conclusion (3). So we assume that either \( \text{char} \ (R) \neq 2 \) or \( R \) does not satisfy \( s_4 \). Since \( L \) is a noncentral by Remark 1.4, there exists a nonzero ideal \( I \) of \( R \) such that \( [I, I] \subseteq L \). Thus by assumption \( I \) satisfies the differential identity
\[
a(F([x, y]^2) - F([x, y])^2) = 0.
\]
Now since \( R \) is a prime ring and \( F \) is a generalized derivation of \( R \), by Lee [23, Theorem 3], \( F(x) = bx + d(x) \) for some \( b \in U \) and derivation \( d \) on \( U \). Since \( I, R \) and \( U \) satisfy the same differential identities [24], without loss of generality, \( U \) satisfies
\[
a(b[x, y]^2 + d([x, y]^2)) - (b[x, y] + d([x, y]))^2 = 0. \quad (2.5)
\]
Here we divide the proof into two cases:


Case 1. Let $d$ be inner derivation induced by element $c \in U$, that is, $d(x) = [c, x]$ for all $x \in U$. It follows that

$$a(b(x, y)^2 + [c, [x, y]^2] - (b[x, y] + [c, [x, y]^2]) = 0,$$

that is

$$a((b + c)[x, y]^2 - [x, y]^2c - ((b + c)[x, y] - [x, y][c, x, y]^2) = 0,$$

for all $x, y \in U$. Now by Lemma 2.4, one of the following holds:

1. $c \in C$ and $0 = a(b + c - c) = ab$. Thus $F(x) = bx$ for all $x \in R$, with $ab = 0$.
2. $b + c \in C$ and $b + c - c = 1$. Thus $F(x) = x$ for all $x \in R$.
3. char $(R) \neq 2$, $R$ satisfies $s_4$ and $c \in C$. Thus $F(x) = bx$ for all $x \in R$.

Case 2. Assume that $d$ is not inner derivation of $U$. We have from (2.5) that $U$ satisfies

$$a(b(x, y)^2 + d([x, y])[x, y] + [x, y]d([x, y])) - (b[x, y] + d([x, y])) = 0,$$

that is

$$a(b(x, y)^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y][d(x), y] + [x, d(y)]) - (b[x, y] + [d(x), y] + [x, d(y)]) = 0.$$

Then by Kharchenko’s Theorem [21], $U$ satisfies

$$a(b(x, y)^2 + ([u, y] + [x, z])[x, y] + [x, y][u, y] + [x, z]) - (b[x, y] + [u, y] + [x, z])^2 = 0. \tag{2.6}$$

Since $R$ is noncommutative, we may choose $q \in U$ such that $q \notin C$. Then replacing $u$ by $[q, x]$ and $z$ by $[q, y]$ in (2.6), we get

$$a(b(x, y)^2 + ([[q, x], y] + [x, [q, y]])[x, y] + [x, y][[[q, x], y] + [x, [q, y]]) - (b[x, y] + [[[q, x], y] + [x, [q, y]])^2 = 0,$$

which is

$$a(b(x, y)^2 + [q, [x, y]^2] - (b[x, y] + [q, [x, y]^2]) = 0.$$

Then by Lemma 2.4, we have $q \in C$, a contradiction.

Now replacing $F$ with $-F$ in the above result, we obtain the conclusion for the situation $a(F(a^2) + F(u^2) = 0$ for all $u \in L$.

**Corollary 2.5.** Let $R$ be a prime ring with extended centroid $C$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits the generalized derivation $F$ such that either $aF(XY) + F(X)F(Y)) = 0$ for all $X, Y \in L$ or $a(F(XY) + F(Y))F(X) = 0$ for all $X, Y \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$. 


Proof of Theorem 1.2. First consider the case when \( a(F(xmyn) − F(xm)F(yn)) = 0 \) for all \( x, y \in R \). Let \( G_1 \) be the additive subgroup of \( R \) generated by the set \( S_1 = \{xm|x \in R\} \) and \( G_2 \) be the additive subgroup of \( R \) generated by the set \( S_2 = \{xn|x \in R\} \). Then by assumption

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x \in G_1, \forall y \in G_2.
\]

Then by [7], either \( G_1 \subseteq Z(R) \) or char \( (R) = 2 \) and \( R \) satisfies \( s_4 \), except when \( G_1 \) contains a noncentral Lie ideal \( L_1 \) of \( R \). \( G_1 \subseteq Z(R) \) implies that \( x^m \in Z(R) \) for all \( x \in R \). It is well known that in this case \( R \) must be commutative, which is a contradiction. Since char \( (R) \neq 2 \), we are to consider the case when \( G_1 \) contains a noncentral Lie ideal \( L_1 \) of \( R \). In this case by [4, Lemma 1], there exists a nonzero ideal \( I \) of \( R \) such that \( [I, I] \subseteq L_1 \).

Thus we have

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x \in [I, I], \forall y \in G_2.
\]

Analogously, we see that there exists a nonzero ideal \( I_2 \) of \( R \) such that

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \forall y \in [I_2, I_2].
\]

By Lee [23, Theorem 3], \( F(x) = bx + d(x) \) for some \( b \in U \) and derivations \( d \) on \( U \). Since \( I_1, I_2, R \) and \( U \) satisfy the same differential identities [24], without loss of generality,

\[
a(F(xy) − F(x)F(y)) = 0 \quad \forall x, y \in [R, R],
\]

and in particular

\[
a(F(x^2) − F(x)^2) = 0 \quad \forall x \in [R, R].
\]

Then by Theorem 1.1, we get

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = x \) for all \( x \in R \);
3. \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

In the last conclusion, \( R \) satisfies polynomial identity and hence \( R \subseteq M_2(C) \) for some field \( C \) and \( M_2(C) \) satisfies \( a(bmx^n − bx^mby^n) = 0 \). By lemma 2.2, we get either \( ab = 0 \) or \( b = 1 \). If \( ab = 0 \), then \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \), which is our conclusion (1). If \( b = 1 \) then \( F(x) = x \) for all \( x \in R \), which is our conclusion (2).

Now replacing \( F \) with \( −F \) in the hypothesis \( a(F(xmyn) − F(xm)F(yn)) = 0 \), we get

\[
0 = a((−F)(x^myn) − (−F)(xm)(−F)(yn)),
\]

that is \( 0 = a(F(xmyn) + F(x^m)F(yn)) \) for all \( x, y \in R \) implies one of the following:

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = −x \) for all \( x \in R \);
Now consider the case when $a(F(x^m y^n) - F(y^n)F(x^m)) = 0$ for all $x, y \in R$. By similar argument as above we get

$$a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R, R],$$

and in particular

$$a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].$$

Then by Theorem 1.1, we get

1. $\exists b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = x$ for all $x \in R$;
3. $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

In the conclusion (3), $R$ satisfies polynomial identity and hence $R \subseteq M_2(C)$ for some field $C$ and $M_2(C)$ satisfies $a(bx^m y^n - by^n bx^m) = 0$. Then by Lemma 2.3, we have $ab = 0$, which is our conclusion (1).

Now replacing $F$ with $-F$ in the hypothesis $a(F(x^m y^n) - F(y^n)F(x^m)) = 0$, we get $0 = a(-F)(x^m y^n) - (-F)(y^n)(-F)(x^m)$. That is, $0 = a(F(x^m y^n) + F(y^n)F(x^m))$ for all $x, y \in R$. This implies that there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$ or $F(x) = -x$. This completes the proof.

In particular, we have the following corollary.

**Corollary 2.6.** Let $R$ be a prime ring of characteristic different from 2 and $0 \neq a \in R$. Suppose that $R$ admits the generalized derivation $F$ associated with a nonzero derivation $d$ of $R$. If any one of the following conditions is satisfied:

1. $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$;
2. $a(F(x^m y^n) \mp F(y^n)F(x^m)) = 0$ for all $x, y \in R$,

then $R$ is commutative.

**Proof of Theorem 1.3.** First we consider the case $a(F(x^m y^n) + F(x^m)F(y^n)) = 0$ for all $x, y \in R$. Other cases are similar. We know the fact that any derivation of a semiprime ring $R$ can be uniquely extended to a derivation of its left Utumi quotient ring $U$ and so any derivation of $R$ can be defined on the whole of $U$ [24, Lemma 2]. Moreover $R$ and $U$ satisfy the same GPIs as well as same differential identities. Thus

$$a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0$$

for all $x, y \in U$. Let $M(C)$ be the set of all maximal ideals of $C$ and $P \in M(C)$. Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have $PU$ is a prime ideal of $U$ invariant under all derivations of $U$. Moreover, $\bigcap \{ PU \mid P \in M(C) \} = 0$. Set $\overline{U} = U/PU$. Then derivation $d$ canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,

$$\overline{d}(b\overline{x}^m y^n + d(\overline{x}^m y^n) + (b\overline{x}^m + d(\overline{x}^m))(b\overline{y}^n + d(\overline{y}^n))) = 0$$
for all $\pi, \eta \in U$. By the prime ring case of Corollary 2.6, we have either $d = 0$ or $[U, U] = 0$ or $a = 0$. In any case we have $ad(U)[U, U] \subseteq P U$ for all $P \in M(C)$. Since $\bigcap \{PU \mid P \in M(C)\} = 0$, $ad(U)[U, U] = 0$. In particular, $ad(R)[R, R] = 0$. This implies $0 = ad(R)[R^2, R] = ad(R)R[R, R] + ad(R)[R, R]R = ad(R)[R, R]$. In particular, $ad(R)R[R, ad(R)] = 0$. Therefore, $ad(R)R[ad(R), R] = 0$. Since $R$ is semiprime, we obtain that $ad(R) \subseteq Z(R)$. By Theorem 3.2 in [10], there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1 U) = 0$, $e_2 a = 0$, and $e_3 U$ is commutative. Hence the theorem is proved.

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