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Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

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Abstract. Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R and $0 \neq a \in R$. If R admits a generalized derivation F such that $a(F(u^2) \pm F(u)^2) = 0$ for all $u \in L$, then one of the following holds:

(1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
(2) $F(x) = \mp x$ for all $x \in R$;
(3) char $(R) = 2$ and R satisfies s_4;
(4) char $(R) \neq 2$, R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

We also study the situations (i) $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, and (ii) $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$ in prime and semiprime rings.

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1. Introduction

Let R be an associative prime ring with center $Z(R)$ and U the Utumi quotient ring of R. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of x, y is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \to R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, d is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let S be a nonempty subset of R and $F : R \to R$ be an additive mapping. Then we say that F acts as homomorphism or anti-homomorphism on S if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping F acts as a Jordan homomorphism on S if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring R is often tightly connected to the behavior of additive mappings defined on R. Asma, Rehman, Shakir in [1] proved that if d is a derivation of a 2-torsion free prime ring R which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of R such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation F acts as homomorphism or anti-homomorphism on some subsets of R and then determined the structure of ring R as well as associated map F (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let R be a prime ring of characteristic different from 2, F a generalized derivation of R associated to a derivation d, L a Lie ideal of R such that $u^2 \in L$ for all $u \in L$. If F acts as a homomorphism or anti-homomorphism on L, then either $d = 0$ or L is central in R. More recently in [9], Filippis studied the situation when generalized derivation F acts as a Jordan homomorphism on a non-central Lie ideal L of R.

Recently in [26], Rehman and Raza proved the following: Let R be a prime ring of char $(R) \neq 2$, Z the center of R, and L a nonzero Lie ideal of R. If R admits a generalized derivation F associated with a derivation d which acts as a homomorphism or as anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map F.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map F.
The main results of this paper are as follows:

Theorem 1.1. *Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R and $0 \neq a \in R$. If R admits a generalized derivation F such that $a(F(u^2) \pm F(u)^2) = 0$ for all $u \in L$, then one of the following holds:*

(1) *there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;*
(2) *$F(x) = \mp x$ for all $x \in R$;*
(3) *char $(R) = 2$ and R satisfies s_4;*
(4) *char $(R) \neq 2$, R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.*

Theorem 1.2. *Let R be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients U, $C = Z(U)$ the extended centroid of R, F a generalized derivation on R and $0 \neq a \in R$.*

(1) *If $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$ or $F(x) = \mp x$ for all $x \in R$.*
(2) *If $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$.*

Theorem 1.3. *Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, $F(x) = bx + d(x)$ a generalized derivation on R associated to the derivation d and $0 \neq a \in R$. If any one of the following holds:*

(1) *$a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$,*
(2) *$a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$,*

then there exist orthogonal central idempotents e_1, e_2, $e_3 \in U$ with $e_1+e_2+e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let R be a prime ring and L a noncentral Lie ideal of R. If char$(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If char$(R) = 2$ and $\dim_C RC > 4$, i.e., char$(R) = 2$ and R does not satisfy s_4, then by [22, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either char$(R) \neq 2$ or R does not satisfy s_4, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

Remark 1.5. We denote by Der(U) the set of all derivations on U. By a derivation word Δ of R we mean $\Delta = d_1 d_2 d_3 \dots d_m$ for some derivations $d_i \in$ Der(U).

Let D_{int} be the C-subspace of Der(U) consisting of all inner derivations on U and let d be a non-zero derivation on R. By [21, Theorem 2] we have the following result:

If $\Phi(x_1, x_2, \cdots, x_n, d(x_1), d(x_2) \cdots d(x_n))$ is a differential identity on R, then one of the following holds:

(1) $d \in D_{int}$;
(2) R satisfies the generalized polynomial identity $\Phi(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$.

Remark 1.6. In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense left ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U, and thus all generalized derivations of R will be implicitly assumed to be defined on the whole of U. Lee obtained the following: every generalized derivation F on a dense left ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U.

2. Proof of the Main Results

Now we begin with the following Lemmas:

Lemma 2.1. *Let $R = M_2(C)$ be the ring of all 2×2 matrices over the field C of characteristic different from 2 and $b, c \in R$. Suppose that there exists $0 \neq a \in R$ such that*

$$a\{(b[x, y]^2 + [x, y]^2 c) - (b[x, y] + [x, y]c)^2\} = 0,$$

for all $x, y \in R$. Then $c \in C \cdot I_2$.

Proof. If $c \in C \cdot I_2$, then nothing to prove. Let $c \notin C \cdot I_2$. In this case R is a dense ring of C-linear transformations over a vector space V. Assume that there exists $0 \neq v \in V$ such that $\{v, cv\}$ is linearly C-independent. By density, there exist $x, y \in R$ such that $xv = v, xcv = 0; yv = 0, ycv = v$. Then $[x, y]v = 0$, $[x, y]cv = v$ and hence $a\{(b[x, y]^2 + [x, y]^2 c) - (b[x, y] + [x, y]c)^2\}v = av$.

Of course for any $u \in V$, $\{u, v\}$ linearly C-dependent implies $au = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $\{w, v\}$ are linearly C-independent. Also $a(w + v) = aw \neq 0$ and $a(w - v) = aw \neq 0$. By the above argument, it follows that w and cw are linearly C-dependent, as are $\{w+v, c(w+v)\}$ and $\{w-v, c(w-v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).$$

In other words we have

$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v \tag{2.1}$$

and

$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{2.2}$$

By comparing (2.1) with (2.2) we get both

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \tag{2.3}$$

and

$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \tag{2.4}$$

By (2.3), and since $\{w, v\}$ are C-independent and char $(R) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (2.4) it follows $2cv = 2\alpha_w v$. This leads a contradiction with the fact that $\{v, cv\}$ is linear C-independent.

In light of this, we may assume that for any $v \in V$ there exists a suitable $\alpha_v \in C$ such that $cv = \alpha_v v$, and standard argument shows that there is $\alpha \in C$ such that $cv = \alpha v$ for all $v \in V$. Now let $r \in R$, $v \in V$. Since $cv = \alpha v$,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[c, r]v = 0$ for all $v \in V$ i.e., $[c, r]V = 0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space V, $[c, r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$, a contradiction. $\square$

Lemma 2.2. *Let $R = M_2(C)$ be the ring of all 2×2 matrices over the field C of characteristic different from 2 and $0 \neq p \in R$. Suppose that there exists $0 \neq a \in R$ such that*

$$a(px^m y^n - px^m py^n) = 0,$$

for all $x, y \in R$. Then either $ap = 0$ or $p = 1$.

Proof. Putting $x = y = I_2$, we get $ap = ap^2$. In this case R is a dense ring of C-linear transformations over a vector space V. Assume that there exists $0 \neq v \in V$ such that $\{v, pv\}$ is linearly C-independent. By density, there exist $x, y \in R$ such that $xv = v, xpv = 0; yv = v, ypv = 0$. Then we get $0 = a(px^m y^n - px^m py^n)v = apv$. Then by same argument as in Lemma 2.1, we get either $ap = 0$ or $p \in C \cdot I_2$. When $0 \neq p \in C \cdot I_2$, from $ap = ap^2$, we get $0 = a(p - 1)$. Since $a \neq 0$, we conclude $p = 1$. $\square$

Lemma 2.3. *Let $R = M_2(C)$ be the ring of all 2×2 matrices over the field C of characteristic different from 2 and $0 \neq p \in R$. Suppose that there exists $0 \neq a \in R$ such that*

$$a(px^m y^n - py^n px^m) = 0,$$

for all $x, y \in R$. Then $ap = 0$.

Proof. Putting $x = y = I_2$, we get $ap = ap^2$. Here R is a dense ring of C-linear transformations over a vector space V. Assume that there exists $0 \neq v \in V$ such that $\{v, pv\}$ is linearly C-independent. By density, there exist $x, y \in R$ such that $xv = v, xpv = 0; yv = 0, ypv = pv$. Then we have $0 = a(px^m y^n - py^n px^m)v = -ap^2 v = -apv$. Then by same argument as in Lemma 2.1, we get either $ap = 0$ or $p \in C \cdot I_2$. When $0 \neq p \in C \cdot I_2$, by hypothesis, we get $0 = a[x^m, y^n]$. Then for $x = e_{11}$ and $y = e_{11} + e_{12}$, we have

$0 = a[x^m, y^n] = a[e_{11}, e_{11} + e_{12}] = ae_{12}$. Again, for $x = e_{22}$ and $y = e_{22} + e_{21}$, we have $0 = a[x^m, y^n] = a[e_{22}, e_{22} + e_{21}] = ae_{21}$. These imply $a = 0$, a contradiction. □

Lemma 2.4. *Let R be a noncommutative prime ring with extended centroid C and $b, c \in R$. Suppose that $0 \neq a \in R$ such that*

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0$$

for all $x, y \in R$. Then one of the following holds:

(1) $c \in C$ and $a(b + c) = 0$;

(2) $b, c \in C$ and $b + c = 1$;

(3) char $(R) = 2$ and R satisfies s_4;

(4) char $(R) \neq 2$, R satisfies s_4 and $c \in C$.

Proof. By assumption, R satisfies the generalized polynomial identity (GPI)

$$f(x, y) = a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}.$$

By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by U. Now we consider the following two cases:

Case-I. U does not satisfy any nontrivial GPI.

Let $T = U *_C C\{x, y\}$, the free product of U and $C\{x, y\}$, the free C-algebra in noncommuting indeterminates x and y. Thus

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2, \}$$

is zero element in $T = U *_C C\{x, y\}$. Let $c \notin C$. Then $\{1, c\}$ is C-independent. Then from above

$$a\{[x, y]^2c - (b[x, y] + [x, y]c)[x, y]c, \}$$

which is

$$a\{[x, y] - b[x, y] - [x, y]c)\}[x, y]c,$$

is zero in T. Again, since $c \notin C$, we have that $a[x, y]c[x, y]c$ is zero element in T, implying $a = 0$ or $c = 0$, a contradiction. Thus we conclude that $c \in C$. Then the identity reduces to

$$a\{(b + c)[x, y] - (b + c)[x, y](b + c)\}[x, y],$$

is zero element in T. Again, if $b + c \notin C$, then $a(b + c)[x, y]^2$ becomes zero element in T, implying $a(b + c) = 0$. If $b + c \in C$, then $a(b + c)(b + c - 1)[x, y]^2$ becomes zero element in T, implying $b+c = 0$ or $b+c = 1$. When $b+c = 0$, then $a(b + c) = 0$, which is our conclusion (1). When $b + c = 1$, then $b = 1 - c \in C$, which is our conclusion (2).

Case-II. U satisfies a nontrivial GPI.

Thus we assume that

$$a\{(b[x,y]^2+[x,y]^2c)-(b[x,y]+[x,y]c)^2\}=0,$$

is a nontrivial GPI for U. In case C is infinite, we have $f(x,y)=0$ for all $x,y\in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U\otimes_C\overline{C}$ are prime and centrally closed [17], we may replace R by U or $U\otimes_C\overline{C}$ according to C finite or infinite. Thus we may assume that R centrally closed over C which either finite or algebraically closed and $f(x,y)=0$ for all $x,y\in R$. By Martindale's Theorem [25], R is then primitive ring having non-zero socle $soc(R)$ with C as the associated division ring. Hence by Jacobson's Theorem [20], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Since R is noncommutative, $\dim_C V\geq 2$. If $\dim_C V=2$, then $R\cong M_2(C)$. In this case by Lemma 2.1, either $c\in C$ or char $(R)=2$. This gives conclusions (3) and (4).

Let $\dim_C V\geq 3$. Let for some $v\in V$, v and cv are linearly independent over C. By density there exist $x,y\in R$ such that

$$xv=v,\quad xcv=0;$$

$$yv=0,\quad ycv=v.$$

Then $[x,y]v=0$, $[x,y]cv=v$ and hence $a\{(b[x,y]^2+[x,y]^2c)-(b[x,y]+[x,y]c)^2\}v=av$.

This implies that if $av\neq 0$, then by contradiction we may conclude that v and cv are linearly C-dependent. Now choose $v\in V$ such that v and cv are linearly C-independent. Set $W=Span_C\{v,cv\}$. Then $av=0$. Since $a\neq 0$, there exists $w\in V$ such that $aw\neq 0$ and then $a(v-w)=aw\neq 0$. By the previous argument we have that w,cw are linearly C-dependent and $(v-w),c(v-w)$ too. Thus there exist $\alpha,\beta\in C$ such that $cw=\alpha w$ and $c(v-w)=\beta(v-w)$. Then $cv=\beta(v-w)+cw=\beta(v-w)+\alpha w$ i.e., $(\alpha-\beta)w=cv-\beta v\in W$. Now $\alpha=\beta$ implies that $cv=\beta v$, a contradiction. Hence $\alpha\neq\beta$ and so $w\in W$. Again, if $u\in V$ with $au=0$ then $a(w+u)\neq 0$. So, $w+u\in W$ forcing $u\in W$. Thus it is observed that $w\in V$ with $aw\neq 0$ implies $w\in W$ and $u\in V$ with $au=0$ implies $u\in W$. This implies that $V=W$ i.e., $\dim_C V=2$, a contradiction.

Hence, in any case, v and cv are linearly C-dependent for all $v\in V$. Thus for each $v\in V$, $cv=\alpha_v v$ for some $\alpha_v\in C$. It is very easy to prove that α_v is independent of the choice of $v\in V$. Thus we can write $cv=\alpha v$ for all $v\in V$ and $\alpha\in C$ fixed. Now let $r\in R$, $v\in V$. Since $cv=\alpha v$,

$$[c,r]v=(cr)v-(rc)v=c(rv)-r(cv)=\alpha(rv)-r(\alpha v)=0.$$

Thus $[c, r]v = 0$ for all $v \in V$ i.e., $[c, r]V = 0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space V, $[c, r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$.

Thus our identity reduces to

$$a\{(b'[x, y]^2) - (b'[x, y])^2\} = 0,$$

for all $x, y \in R$, where $b' = b + c$.

Let for some $v \in V$, v and $b'v$ are linearly independent over C. Since $\dim_C V \geq 3$, there exists $u \in V$ such that $v, b'v, u$ are linearly independent over C. By density there exist $x, y \in R$ such that

$$xv = v, \quad xb'v = 0, \quad xu = v;$$

$$yv = 0, \quad yb'v = u, \quad yu = v.$$

Then $[x, y]v = 0$, $[x, y]b'v = v$, $[x, y]u = v$ and hence $0 = a\{(b'[x, y]^2) - (b'[x, y])^2\}u = ab'v$. Then by same argument as before, we have either $ab' = 0$ or v and $b'v$ are linearly C-dependent for all $v \in V$. In the first case, $0 = ab' = a(b + c)$, which is conclusion (1). In the last case, again by standard argument, we have that $b' \in C$. If $b' = 0$, then also $ab' = a(b + c) = 0$ which gives conclusion (1). So assume that $0 \neq b' \in C$. Then our identity reduces to

$$ab'(b' - 1)[x, y]^2 = 0,$$

for all $x, y \in R$. This gives $0 = ab'(b' - 1) = a(b' - 1)$. Since $a \neq 0$, we get $b' = 1$. This gives conclusion (2). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First we consider the case when

$$a(F(u^2) - F(u)^2) = 0,$$

for all $u \in L$. If char $(R) = 2$ and R satisfies s_4, then we have our conclusion (3). So we assume that either char $(R) \neq 2$ or R does not satisfy s_4. Since L is a noncentral by Remark 1.4, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Thus by assumption I satisfies the differential identity

$$a(F([x, y]^2) - F([x, y])^2) = 0.$$

Now since R is a prime ring and F is a generalized derivation of R, by Lee [23, Theorem 3], $F(x) = bx + d(x)$ for some $b \in U$ and derivation d on U. Since I, R and U satisfy the same differential identities [24], without loss of generality, U satisfies

$$a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0. \tag{2.5}$$

Here we divide the proof into two cases:

Case 1. Let d be inner derivation induced by element $c \in U$, that is, $d(x) = [c, x]$ for all $x \in U$. It follows that

$$a(b[x, y]^2 + [c, [x, y]^2] - (b[x, y] + [c, [x, y]])^2) = 0,$$

that is

$$a((b + c)[x, y]^2 - [x, y]^2c - ((b + c)[x, y] - [x, y]c)^2) = 0,$$

for all $x, y \in U$. Now by Lemma 2.4, one of the following holds:

(1) $c \in C$ and $0 = a(b + c - c) = ab$. Thus $F(x) = bx$ for all $x \in R$, with $ab = 0$.

(2) $b + c, c \in C$ and $b + c - c = 1$. Thus $F(x) = x$ for all $x \in R$.

(3) char $(R) \neq 2$, R satisfies s_4 and $c \in C$. Thus $F(x) = bx$ for all $x \in R$.

Case 2. Assume that d is not inner derivation of U. We have from (2.5) that U satisfies

$$a(b[x, y]^2 + d([x, y])[x, y] + [x, y]d([x, y]) - (b[x, y] + d([x, y]))^2) = 0,$$

that is

$$\begin{gathered} a(b[x, y]^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y]([d(x), y] + [x, d(y)]) \\ -(b[x, y] + [d(x), y] + [x, d(y)])^2) = 0. \end{gathered}$$

Then by Kharchenko's Theorem [21], U satisfies

$$\begin{gathered} a(b[x, y]^2 + ([u, y] + [x, z])[x, y] + [x, y]([u, y] + [x, z]) \\ -(b[x, y] + [u, y] + [x, z])^2) = 0. \end{gathered} \tag{2.6}$$

Since R is noncommutative, we may choose $q \in U$ such that $q \notin C$. Then replacing u by $[q, x]$ and z by $[q, y]$ in (2.6), we get

$$\begin{gathered} a(b[x, y]^2 + ([[q, x], y] + [x, [q, y]])[x, y] + [x, y]([[q, x], y] + [x, [q, y]]) \\ -(b[x, y] + ([[q, x], y] + [x, [q, y]]))^2) = 0, \end{gathered}$$

which is

$$a(b[x, y]^2 + [q, [x, y]^2]) - (b[x, y] + [q, [x, y]])^2) = 0.$$

Then by Lemma 2.4, we have $q \in C$, a contradiction.

Now replacing F with $-F$ in the above result, we obtain the conclusion for the situation $a(F(u^2) + F(u)^2) = 0$ for all $u \in L$.

Corollary 2.5. *Let R be a prime ring with extended centroid C, L a non-central Lie ideal of R and $0 \neq a \in R$. If R admits the generalized derivation F such that either $a(F(XY) \pm F(X)F(Y)) = 0$ for all $X, Y \in L$ or $a(F(XY) \pm F(Y)F(X)) = 0$ for all $X, Y \in L$, then one of the following holds:*

(1) *there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;*

(2) *$F(x) = \mp x$ for all $x \in R$;*

(3) *char $(R) = 2$ and R satisfies s_4;*

(4) *char $(R) \neq 2$, R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.*

Proof of Theorem 1.2. First consider the case when $a(F(x^my^n)-F(x^m)F(y^n)) = 0$ for all $x, y \in R$. Let G_1 be the additive subgroup of R generated by the set $S_1 = \{x^m | x \in R\}$ and G_2 be the additive subgroup of R generated by the set $S_2 = \{x^n | x \in R\}$. Then by assumption

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in G_1, \quad \forall y \in G_2.$$

Then by [7], either $G_1 \subseteq Z(R)$ or char $(R) = 2$ and R satisfies s_4, except when G_1 contains a noncentral Lie ideal L_1 of R. $G_1 \subseteq Z(R)$ implies that $x^m \in Z(R)$ for all $x \in R$. It is well known that in this case R must be commutative, which is a contradiction. Since char $(R) \neq 2$, we are to consider the case when G_1 contains a noncentral Lie ideal L_1 of R. In this case by [4, Lemma 1], there exists a nonzero ideal I_1 of R such that $[I_1, I_1] \subseteq L_1$.

Thus we have

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in G_2.$$

Analogously, we see that there exists a nonzero ideal I_2 of R such that

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in [I_2, I_2].$$

By Lee [23, Theorem 3], $F(x) = bx + d(x)$ for some $b \in U$ and derivations d on U. Since I_1, I_2, R and U satisfy the same differential identities [24], without loss of generality,

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x, y \in [R, R],$$

and in particular

$$a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].$$

Then by Theorem 1.1, we get

(1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
(2) $F(x) = x$ for all $x \in R$;
(3) R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

In the last conclusion, R satisfies polynomial identity and hence $R \subseteq M_2(C)$ for some field C and $M_2(C)$ satisfies $a(bx^my^n - bx^mby^n) = 0$. By lemma 2.2, we get either $ab = 0$ or $b = 1$. If $ab = 0$, then $F(x) = bx$ for all $x \in R$, with $ab = 0$, which is our conclusion (1). If $b = 1$ then $F(x) = x$ for all $x \in R$, which is our conclusion (2).

Now replacing F with $-F$ in the hypothesis $a(F(x^my^n)-F(x^m)F(y^n)) = 0$, we get $0 = a((-F)(x^my^n) - (-F)(x^m)(-F)(y^n))$, that is $0 = a(F(x^my^n) + F(x^m)F(y^n))$ for all $x, y \in R$ implies one of the following:

(1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
(2) $F(x) = -x$ for all $x \in R$;

Now consider the case when $a(F(x^my^n)-F(y^n)F(x^m))=0$ for all $x,y\in R$. By similar argument as above we get

$$a(F(xy)-F(y)F(x))=0 \quad \forall x,y\in[R,R],$$

and in particular

$$a(F(x^2)-F(x)^2)=0 \quad \forall x\in[R,R].$$

Then by Theorem 1.1, we get

(1) there exists $b\in U$ such that $F(x)=bx$ for all $x\in R$, with $ab=0$;
(2) $F(x)=x$ for all $x\in R$;
(3) R satisfies s_4 and there exists $b\in U$ such that $F(x)=bx$ for all $x\in R$.

In the conclusion (3), R satisfies polynomial identity and hence $R\subseteq M_2(C)$ for some field C and $M_2(C)$ satisfies $a(bx^my^n-by^nbx^m)=0$. Then by Lemma 2.3, we have $ab=0$, which is our conclusion (1).

Now replacing F with $-F$ in the hypothesis $a(F(x^my^n)-F(y^n)F(x^m))=0$, we get $0=a((-F)(x^my^n)-(-F)(y^n)(-F)(x^m))$. That is, $0=a(F(x^my^n)+F(y^n)F(x^m))$ for all $x,y\in R$. This implies that there exists $b\in U$ such that $F(x)=bx$ for all $x\in R$ with $ab=0$ or $F(x)=-x$. This completes the proof.

In particular, we have the following corollary.

Corollary 2.6. *Let R be a prime ring of characteristic different from 2 and $0\neq a\in R$. Suppose that R admits the generalized derivation F associated with a nonzero derivation d of R. If any one of the following conditions is satisfied:*
(1) $a(F(x^my^n)\pm F(x^m)F(y^n))=0$ for all $x,y\in R$;
(2) $a(F(x^my^n)\pm F(y^n)F(x^m))=0$ for all $x,y\in R$,
then R is commutative.

Proof of Theorem 1.3. First we consider the case $a(F(x^my^n)+F(x^m)F(y^n))=0$ for all $x,y\in R$. Other cases are similar. We know the fact that any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U [24, Lemma 2]. Moreover R and U satisfy the same GPIs as well as same differential identities. Thus

$$a(bx^my^n+d(x^my^n)+(bx^m+d(x^m))(by^n+d(y^n)))=0$$

for all $x,y\in U$. Let $M(C)$ be the set of all maximal ideals of C and $P\in M(C)$. Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P\in M(C)\}=0$. Set $\overline{U}=U/PU$. Then derivation d canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x})=\overline{d(x)}$ for all $x\in U$. Therefore,

$$\overline{a}(b\overline{x}^m\overline{y}^n+d(\overline{x}^m\overline{y}^n)+(b\overline{x}^m+d(\overline{x}^m))(b\overline{y}^n+d(\overline{y}^n)))=0$$

for all $\overline{x}, \overline{y} \in \overline{U}$. By the prime ring case of Corollary 2.6, we have either $\overline{d} = 0$ or $[\overline{U}, \overline{U}] = 0$ or $\overline{a} = 0$. In any case we have $ad(U)[U,U] \subseteq PU$ for all $P \in M(C)$. Since $\bigcap\{PU \mid P \in M(C)\} = 0$, $ad(U)[U,U] = 0$. In particular, $ad(R)[R,R] = 0$. This implies $0 = ad(R)[R^2,R] = ad(R)R[R,R] + ad(R)[R,R]R = ad(R)R[R,R]$. In particular, $ad(R)R[R,ad(R)] = 0$. Therefore, $[ad(R),R]R[ad(R),R] = 0$. Since R is semiprime, we obtain that $ad(R) \subseteq Z(R)$. By Theorem 3.2 in [10], there exist orthogonal central idempotents e_1, e_2, $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative. Hence the theorem is proved.

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