Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

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Abstract. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a (F(u^2) \pm F(u^2)) = 0$ for all $u \in L$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. $\text{char } (R) = 2$ and $R$ satisfies $s_4$;
4. $\text{char } (R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

We also study the situations (i) $a (F(x^m y^n) \pm F(x^m) F(y^n)) = 0$ for all $x, y \in R$, and (ii) $a (F(x^m y^n) \pm F(y^n) F(x^m)) = 0$ for all $x, y \in R$ in prime and semiprime rings.

Keywords: Prime ring, Generalized derivation, Utumi quotient ring.

1. Introduction

Let $R$ be an associative prime ring with center $Z(R)$ and $U$ the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \to R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, $d$ is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let $S$ be a nonempty subset of $R$ and $F : R \to R$ be an additive mapping. Then we say that $F$ acts as homomorphism or anti-homomorphism on $S$ if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping $F$ acts as a Jordan homomorphism on $S$ if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. Asma, Rehman, Shakir in [1] proved that if $d$ is a derivation of a 2-torsion free prime ring $R$ which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of $R$ such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation $F$ acts as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring $R$ as well as associated map $F$ (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let $R$ be a prime ring of characteristic different from 2, $F$ a generalized derivation of $R$ associated to a derivation $d$, $L$ a Lie ideal of $R$ such that $u^2 \in L$ for all $u \in L$. If $F$ acts as a homomorphism or anti-homomorphism on $L$, then either $d = 0$ or $L$ is central in $R$. More recently in [9], Filippis studied the situation when generalized derivation $F$ acts as a Jordan homomorphism on a non-central Lie ideal $L$ of $R$.

Recently in [26], Rehman and Raza proved the following: Let $R$ be a prime ring of char $(R) \neq 2$, $Z$ the center of $R$, and $L$ a nonzero Lie ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ which acts as a homomorphism or as anti-homomorphism on $L$, then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map $F$.

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map $F$.

The main results of this paper are as follows:
Theorem 1.1. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$ and $0 \neq a \in R$. If $R$ admits a generalized derivation $F$ such that $a(F(a^2) \pm F(a)^2) = 0$ for all $a \in R$, then one of the following holds:

1. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
2. $F(x) = \mp x$ for all $x \in R$;
3. $\text{char}(R) = 2$ and $R$ satisfies $s_4$;
4. $\text{char}(R) \neq 2$, $R$ satisfies $s_4$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

Theorem 1.2. Let $R$ be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients $U$, $C = Z(U)$ the extended centroid of $R$, $F$ a generalized derivation on $R$ and $0 \neq a \in R$.

1. If $a(F(x^my^n) \pm F(x^n)F(y^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$ or $F(x) = \mp x$ for all $x \in R$.
2. If $a(F(x^my^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$.

Theorem 1.3. Let $R$ be a noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $F(x) = bx + d(x)$ a generalized derivation on $R$ associated to the derivation $d$ and $0 \neq a \in R$. If any one of the following holds:

1. $a(F(x^my^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$,
2. $a(F(x^my^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$,

then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and $e_3U$ is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let $R$ be a prime ring and $L$ a noncentral Lie ideal of $R$. If $\text{char}(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. If $\text{char}(R) = 2$ and $\dim_RC > 4$, i.e., $\text{char}(R) = 2$ and $R$ does not satisfy $s_4$, then by [22, Theorem 13] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. Thus if either $\text{char}(R) \neq 2$ or $R$ does not satisfy $s_4$, then we may conclude that there exists a nonzero ideal $I$ of $R$ such that $[I, I] \subseteq L$.

Remark 1.5. We denote by $\text{Der}(U)$ the set of all derivations on $U$. By a derivation word $\Delta$ of $R$ we mean $\Delta = d_1d_2d_3 \ldots d_m$ for some derivations $d_i \in \text{Der}(U)$.

Let $D_{in}$ be the $C$-subspace of $\text{Der}(U)$ consisting of all inner derivations on $U$ and let $d$ be a non-zero derivation on $R$. By [21, Theorem 2] we have the following result:
If \( \Phi(x_1, x_2, \cdots, x_n, d(x_1), d(x_2) \cdots d(x_n)) \) is a differential identity on \( R \), then one of the following holds:

1. \( d \in D_{int} \);
2. \( R \) satisfies the generalized polynomial identity \( \Phi(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \).

**Remark 1.6.** In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping \( F : I \to U \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in I \), where \( I \) is a dense left ideal of \( R \) and \( d \) is a derivation from \( I \) into \( U \). Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of \( U \), and thus all generalized derivations of \( R \) will be implicitly assumed to be defined on the whole of \( U \). Lee obtained the following: every generalized derivation \( F \) on a dense left ideal of \( R \) can be uniquely extended to \( U \) and assumes the form \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \).

**2. Proof of the Main Results**

Now we begin with the following Lemmas:

**Lemma 2.1.** Let \( R = M_2(C) \) be the ring of all \( 2 \times 2 \) matrices over the field \( C \) of characteristic different from 2 and \( b, c \in R \). Suppose that there exists \( 0 \neq a \in R \) such that

\[
\alpha \{ (b[x,y]^2 + [x,y]^2)c - (b[x,y] + [x,y]c)^2 \} = 0,
\]

for all \( x, y \in R \). Then \( c \in C \cdot I_2 \).

**Proof.** If \( c \in C \cdot I_2 \), then nothing to prove. Let \( c \notin C \cdot I_2 \). In this case \( R \) is a dense ring of \( C \)-linear transformations over a vector space \( V \). Assume that there exists \( 0 \neq v \in V \) such that \( \{ v, cv \} \) is linearly \( C \)-independent. By density, there exist \( x, y \in R \) such that \( xv = v, xcv = 0, yv = 0, ycv = v \). Then \( [x,y]v = 0, [x,y]cv = v \) and hence \( \alpha \{ (b[x,y]^2 + [x,y]^2)c - (b[x,y] + [x,y]c)^2 \} = av \).

Of course for any \( u \in V \), \( \{ u, v \} \) linearly \( C \)-dependent implies \( au = 0 \). Since \( a \neq 0 \), there exists \( w \in V \) such that \( aw \neq 0 \) and so \( \{ w, v \} \) are linearly \( C \)-independent. Also \( a(w+v) = aw \neq 0 \) and \( a(w-v) = aw \neq 0 \). By the above argument, it follows that \( w \) and \( cw \) are linearly \( C \)-dependent, as are \( \{ w+v, c(w+v) \} \) and \( \{ w-v, c(w-v) \} \). Therefore there exist \( \alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C \) such that

\[
cw = \alpha_w w, \quad c(w+v) = \alpha_{w+v} (w+v), \quad c(w-v) = \alpha_{w-v} (w-v).
\]

In other words we have

\[
\alpha_w w + cw = \alpha_{w+v} w + \alpha_{w+v} v \tag{2.1}
\]

and

\[
\alpha_w w - cw = \alpha_{w-v} w - \alpha_{w-v} v \tag{2.2}
\]
By comparing (2.1) with (2.2) we get both
\[(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0\] (2.3)
and
\[2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.\] (2.4)
By (2.3), and since \(w, v\) are \(C\)-independent and \(\text{char } (R) \neq 2\), we have \(\alpha_w = \alpha_{w+v} = \alpha_{w-v}\). Thus by (2.4) it follows \(2cv = 2\alpha_wv\). This leads to a contradiction with the fact that \(\{v, cv\}\) is linear \(C\)-independent.

In light of this, we may assume that for any \(v \in V\) there exists a suitable \(\alpha_v \in C\) such that \(cv = \alpha_vv\), and standard argument shows that there is \(\alpha \in C\) such that \(cv = \alpha v\) for all \(v \in V\). Now let \(r \in R\), \(v \in V\). Since \(cv = \alpha v\),
\[ [c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.\]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a \(C\)-linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\), a contradiction. \(\Box\)

**Lemma 2.2.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from \(2\) and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^m y^n - px^m py^n) = 0,\]
for all \(x, y \in R\). Then either \(ap = 0\) or \(p = 1\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). In this case \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = v, ypv = 0\). Then we get
\[0 = a(px^m y^n - px^m py^n)v = apv.\]
Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), from \(ap = ap^2\), we get \(0 = a(p - 1)\). Since \(a \neq 0\), we conclude \(p = 1\). \(\Box\)

**Lemma 2.3.** Let \(R = M_2(C)\) be the ring of all \(2 \times 2\) matrices over the field \(C\) of characteristic different from \(2\) and \(0 \neq p \in R\). Suppose that there exists \(0 \neq a \in R\) such that
\[a(px^m y^n - py^n px^m) = 0,\]
for all \(x, y \in R\). Then \(ap = 0\).

**Proof.** Putting \(x = y = I_2\), we get \(ap = ap^2\). Here \(R\) is a dense ring of \(C\)-linear transformations over a vector space \(V\). Assume that there exists \(0 \neq v \in V\) such that \(\{v, pv\}\) is linearly \(C\)-independent. By density, there exist \(x, y \in R\) such that \(xv = v, xpv = 0; yv = 0, ypv = pv\). Then we have
\[0 = a(px^m y^n - py^n px^m)v = -ap^2v = -apv.\]
Then by same argument as in Lemma 2.1, we get either \(ap = 0\) or \(p \in C \cdot I_2\). When \(0 \neq p \in C \cdot I_2\), by hypothesis, we get \(0 = a[x^m, y^n]\). Then for \(x = e_{11}\) and \(y = e_{11} + e_{12}\), we have
Lemma 2.4. Let $R$ be a noncommutative prime ring with extended centroid $C$ and $b, c \in R$. Suppose that $0 \neq a \in R$ such that
\[
a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0
\]
for all $x, y \in R$. Then one of the following holds:

1. $c \in C$ and $a(b + c) = 0$;
2. $b, c \in C$ and $b + c = 1$;
3. char $(R) = 2$ and $R$ satisfies $s_4$;
4. char $(R) \neq 2$, $R$ satisfies $s_4$ and $c \in C$.

Proof. By assumption, $R$ satisfies the generalized polynomial identity (GPI)
\[
f(x, y) = a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}.
\]
By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by $U$. Now we consider the following two cases:

Case-I. $U$ does not satisfy any nontrivial GPI.

Let $T = U \ast_C C[x, y]$, the free product of $U$ and $C[x, y]$, the free $C$-algebra in noncommuting indeterminates $x$ and $y$. Thus
\[
a\{[x, y]^2c - (b[x, y] + [x, y]c)[x, y]c, \}
\]
is zero element in $T = U \ast_C C[x, y]$. Let $c \notin C$. Then $\{1, c\}$ is $C$-independent. Then from above
\[
a\{[x, y]^2c - (b[x, y] + [x, y]c)[x, y]c, \}
\]
which is
\[
a\{[x, y] - b[x, y] - [x, y]c\}[x, y]c,
\]
is zero in $T$. Again, since $c \notin C$, we have that $a[x, y]c[x, y]c$ is zero element in $T$, implying $a = 0$ or $c = 0$, a contradiction. Thus we conclude that $c \in C$. Then the identity reduces to
\[
a\{(b + c)[x, y] - (b + c)[x, y](b + c)\}[x, y],
\]
is zero element in $T$. Again, if $b + c \notin C$, then $ab + c)\[x, y]^2$ becomes zero element in $T$, implying $a(b + c) = 0$. If $b + c \in C$, then $a(b + c)(b + c)\[x, y]^2$ becomes zero element in $T$, implying $b + c = 0$ or $b + c = 1$. When $b + c = 0$, then $a(b + c) = 0$, which is our conclusion (1). When $b + c = 1$, then $b = 1 - c \in C$, which is our conclusion (2).

Case-II. $U$ satisfies a nontrivial GPI.
Thus we assume that
\[ a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0, \]
is a nontrivial GPI for \(U\). In case \(C\) is infinite, we have \(f(x, y) = 0\) for all \(x, y \in U \otimes_C \bar{C}\), where \(\bar{C}\) is the algebraic closure of \(C\). Since both \(U\) and \(U \otimes_C \bar{C}\) are prime and centrally closed [17], we may replace \(R\) by \(U\) or \(U \otimes_C \bar{C}\) according to \(C\) finite or infinite. Thus we may assume that \(R\) centrally closed over \(C\) which either finite or algebraically closed and \(f(x, y) = 0\) for all \(x, y \in R\).

By Martindale’s Theorem [25], \(R\) is then primitive ring having non-zero socle \(soc(R)\) with \(C\) as the associated division ring. Hence by Jacobson’s Theorem [20], \(R\) is isomorphic to a dense ring of linear transformations of a vector space \(V\) over \(C\). Since \(R\) is noncommutative, \(dim_C V \geq 2\). If \(dim_C V = 2\), then \(R \cong M_2(C)\). In this case by Lemma 2.1, either \(c \in C\) or char \((R) = 2\). This gives conclusions (3) and (4).

Let \(dim_C V \geq 3\). Let for some \(v \in V\), \(v\) and \(cv\) are linearly independent over \(C\). By density there exist \(x, y \in R\) such that

\[ xv = v, \quad xcv = 0; \]
\[ yv = 0, \quad ycv = v. \]

Then \([x, y]v = 0, \ [x, y]cv = v\) and hence \(a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}v = av\).

This implies that if \(av \neq 0\), then by contradiction we may conclude that \(v\) and \(cv\) are linearly \(C\)-dependent. Now choose \(v \in V\) such that \(v\) and \(cv\) are linearly \(C\)-independent. Set \(W = Span_C\{v, cv\}\). Then \(av = 0\). Since \(a \neq 0\), there exists \(w \in V\) such that \(aw \neq 0\) and then \(a(v - w) = aw \neq 0\). By the previous argument we have that \(w, cw\) are linearly \(C\)-dependent and \((v - w), c(v - w)\) too. Thus there exist \(\alpha, \beta \in C\) such that \(cw = \alpha w\) and \(c(v - w) = \beta(v - w)\). Then \(cv = \beta(v - w) + cw = \beta(v - w) + \alpha w\) i.e., \((\alpha - \beta)w = cv - \beta v \in W\). Now \(\alpha = \beta\) implies that \(cv = \beta v\), a contradiction. Hence \(\alpha \neq \beta\) and so \(w \in W\). Again, if \(u \in V\) with \(au = 0\) then \(a(u + w) \neq 0\).

So, \(w + u \in W\) forcing \(u \in W\). Thus it is observed that \(w \in V\) with \(aw \neq 0\) implies \(w \in W\) and \(u \in V\) with \(au = 0\) implies \(u \in W\). This implies that \(V = W\) i.e., \(dim_C V = 2\), a contradiction.

Hence, in any case, \(v\) and \(cv\) are linearly \(C\)-dependent for all \(v \in V\). Thus for each \(v \in V\), \(cv = \alpha_v v\) for some \(\alpha_v \in C\). It is very easy to prove that \(\alpha_v\) is independent of the choice of \(v \in V\). Thus we can write \(cv = \alpha v\) for all \(v \in V\) and \(\alpha \in C\) fixed. Now let \(r \in R, v \in V\). Since \(cv = \alpha v\),

\[ [e, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0. \]
Thus \([c, r]v = 0\) for all \(v \in V\) i.e., \([c, r]V = 0\). Since \([c, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([c, r] = 0\) for all \(r \in R\). Therefore, \(c \in Z(R)\).

Thus our identity reduces to
\[
a\{ (b'[x, y]^2) - (b'[x, y])^2 \} = 0,
\]
for all \(x, y \in R\), where \(b' = b + c\).

Let for some \(v \in V\), \(v\) and \(b'v\) are linearly independent over \(C\). Since \(\dim_C V \geq 3\), there exists \(u \in V\) such that \(v, b'v, u\) are linearly independent over \(C\). By density there exist \(x, y \in R\) such that
\[
xv = v, \quad xb'v = 0, \quad xu = v;
\]
\[
yv = 0, \quad yb'v = u, \quad yu = v.
\]

Then \([x, y]v = 0, [x, y]b'v = v, [x, y]u = v\) and hence \(0 = a\{ (b'[x, y]^2) - (b'[x, y])^2 \}u = ab'v\). Then by same argument as before, we have either \(ab' = 0\) or \(b'v\) are linearly \(C\)-dependent for all \(v \in V\). In the first case, \(0 = ab' = a(b + c)\), which is conclusion (1). In the last case, again by standard argument, we have that \(b' \in C\). If \(b' = 0\), then also \(ab' = a(b + c) = 0\) which gives conclusion (1). So assume that \(0 \neq b' \in C\). Then our identity reduces to
\[
ab'(b' - 1)[x, y]^2 = 0,
\]
for all \(x, y \in R\). This gives \(0 = ab'(b' - 1) = a(b' - 1)\). Since \(a \neq 0\), we get \(b' = 1\). This gives conclusion (2).

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when
\[
a(F(u^2) - F(u)^2) = 0,
\]
for all \(u \in L\). If \(\text{char}(R) = 2\) and \(R\) satisfies \(s_4\), then we have our conclusion (3). So we assume that either \(\text{char}(R) \neq 2\) or \(R\) does not satisfy \(s_4\). Since \(L\) is a noncentral by Remark 1.4, there exists a nonzero ideal \(I\) of \(R\) such that \([I, I] \subseteq L\). Thus by assumption \(I\) satisfies the differential identity
\[
a(F([x, y]^2) - F([x, y])^2) = 0.
\]
Now since \(R\) is a prime ring and \(F\) is a generalized derivation of \(R\), by Lee [23, Theorem 3], \(F(x) = bx + d(x)\) for some \(b \in U\) and derivation \(d\) on \(U\). Since \(I, R\) and \(U\) satisfy the same differential identities [24], without loss of generality, \(U\) satisfies
\[
a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0. \tag{2.5}
\]
Here we divide the proof into two cases:
Case 1. Let \( d \) be inner derivation induced by element \( c \in U \), that is, 
\[ d(x) = [c, x] \] 
for all \( x \in U \). It follows that
\[ a(b[x, y]^2 + [c, [x, y]^2] - (b[x, y] + [c, [x, y]])^2) = 0, \]
that is
\[ a((b + c)[x, y]^2 - [x, y]^2c - ((b + c)[x, y] - [x, y]c)^2) = 0, \]
for all \( x, y \in U \). Now by Lemma 2.4, one of the following holds:
(1) \( c \in C \) and \( 0 = a(b + c - c) = ab \). Thus \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \).
(2) \( b + c, c \in C \) and \( b + c - c = 1 \). Thus \( F(x) = x \) for all \( x \in R \).
(3) \( \text{char } (R) \neq 2 \), \( R \) satisfies \( s_4 \) and \( c \in C \). Thus \( F(x) = bx \) for all \( x \in R \).

Case 2. Assume that \( d \) is not inner derivation of \( U \). We have from (2.5) that \( U \) satisfies
\[ a(b[x, y]^2 + d([x, y])\left[ x, y + [x, y][d([x, y]) - (b[x, y] + d([x, y]))^2 \right] = 0, \]
that is
\[ a(b[x, y]^2 + ([d(x), y] + [x, d(y)])\left[ x, y + [x, y]([d(x), y] + [x, d(y)]) - (b[x, y] + [d(x), y] + [x, d(y)])^2 \right] = 0. \]
Then by Kharchenko’s Theorem [21], \( U \) satisfies
\[ a(b[x, y]^2 + ([u, y] + [x, z])\left[ x, y + [x, y]([u, y] + [x, z]) - (b[x, y] + [u, y] + [x, z])^2 \right] = 0. \]

Since \( R \) is noncommutative, we may choose \( q \in U \) such that \( q \notin C \). Then replacing \( u \) by \([q, x] \) and \( z \) by \([y, y] \) in (2.6), we get
\[ a(b[x, y]^2 + ([[q, x], y] + [x, [q, y]])[x, y] + [x, y][[[q, x], y] + [x, [q, y]]] - (b[x, y] + [[[q, x], y] + [x, [q, y]]])^2) = 0, \]
which is
\[ a(b[x, y]^2 + [q, [x, y]]^2) - (b[x, y] + [q, [x, y]])^2) = 0. \]

Then by Lemma 2.4, we have \( q \in C \), a contradiction.

Now replacing \( F \) with \( -F \) in the above result, we obtain the conclusion for the situation \( a(F(a^2)) + F(a^2) = 0 \) for all \( a \in U \).

**Corollary 2.5.** Let \( R \) be a prime ring with extended centroid \( C \), \( L \) a noncentral Lie ideal of \( R \) and \( 0 \neq a \in R \). If \( R \) admits the generalized derivation \( F \) such that either \( aF(XY) \pm F(X)F(Y) \) or \( aF(XY) \pm F(Y)F(X) \) or \( 0 \) for all \( X, Y \in L \) or \( aF(XY) \pm F(Y)F(X) \) or \( 0 \) for all \( X, Y \in L \), then one of the following holds:
(1) \( \text{there exists } b \in U \text{ such that } F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
(2) \( F(x) = \mp x \) for all \( x \in R \);
(3) \( \text{char } (R) = 2 \) and \( R \) satisfies \( s_4 \);
(4) \( \text{char } (R) \neq 2 \), \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).
Proof of Theorem 1.2. First consider the case when \(a(F(x^m y^n) - F(x^m) F(y^n)) = 0\) for all \(x, y \in R\). Let \(G_1\) be the additive subgroup of \(R\) generated by the set \(S_1 = \{x^m | x \in R\}\) and \(G_2\) be the additive subgroup of \(R\) generated by the set \(S_2 = \{x^n | x \in R\}\). Then by assumption
\[
a(F(xy) - F(x) F(y)) = 0 \quad \forall x \in G_1, \forall y \in G_2.
\]

Then by [7], either \(G_1 \subseteq Z(R)\) or \(\text{char}(R) = 2\) and \(R\) satisfies \(s_4\), except when \(G_1\) contains a noncentral Lie ideal \(L_1\) of \(R\). \(G_1 \subseteq Z(R)\) implies that \(x^m \in Z(R)\) for all \(x \in R\). It is well known that in this case \(R\) must be commutative, which is a contradiction. Since \(\text{char}(R) \neq 2\), we are to consider the case when \(G_1\) contains a noncentral Lie ideal \(L_1\) of \(R\). In this case by [4, Lemma 1], there exists a nonzero ideal \(I_1\) of \(R\) such that \([I_1, I_1] \subseteq L_1\).

Thus we have
\[
a(F(xy) - F(x) F(y)) = 0 \quad \forall x \in [I_1, I_1], \forall y \in G_2.
\]

Analogously, we see that there exists a nonzero ideal \(I_2\) of \(R\) such that
\[
a(F(xy) - F(x) F(y)) = 0 \quad \forall x \in [I_1, I_1], \forall y \in [I_2, I_2].
\]

By Lee [23, Theorem 3], \(F(x) = bx + d(x)\) for some \(b \in U\) and derivations \(d\) on \(U\). Since \(I_1, I_2, R\) and \(U\) satisfy the same differential identities [24], without loss of generality,
\[
a(F(xy) - F(x) F(y)) = 0 \quad \forall x, y \in [R, R],
\]
and in particular
\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].
\]

Then by Theorem 1.1, we get
\begin{enumerate}
  \item there exists \(b \in U\) such that \(F(x) = bx\) for all \(x \in R\), with \(ab = 0\);
  \item \(F(x) = x\) for all \(x \in R\);
  \item \(R\) satisfies \(s_4\) and there exists \(b \in U\) such that \(F(x) = bx\) for all \(x \in R\).
\end{enumerate}

In the last conclusion, \(R\) satisfies polynomial identity and hence \(R \subseteq M_2(C)\) for some field \(C\) and \(M_2(C)\) satisfies \(a(bx^m y^n - bx^m by^n) = 0\). By lemma 2.2, we get either \(ab = 0\) or \(b = 1\). If \(ab = 0\), then \(F(x) = bx\) for all \(x \in R\), with \(ab = 0\), which is our conclusion (1). If \(b = 1\) then \(F(x) = x\) for all \(x \in R\), which is our conclusion (2).

Now replacing \(F\) with \(-F\) in the hypothesis \(a(F(x^m y^n) - F(x^m) F(y^n)) = 0\), we get 0 = \(a((-F)(x^m y^n) - (-F)(x^m) (-F)(y^n))\), that is 0 = \(a(F(x^m y^n) + F(x^m) F(y^n))\) for all \(x, y \in R\) implies one of the following:
\begin{enumerate}
  \item there exists \(b \in U\) such that \(F(x) = bx\) for all \(x \in R\), with \(ab = 0\);
  \item \(F(x) = -x\) for all \(x \in R\);
\end{enumerate}
Now consider the case when \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \).

By similar argument as above we get
\[
a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R,R],
\]
and in particular
\[
a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R,R].
\]

Then by Theorem 1.1, we get

1. there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \), with \( ab = 0 \);
2. \( F(x) = x \) for all \( x \in R \);
3. \( R \) satisfies \( s_4 \) and there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \).

In the conclusion (3), \( R \) satisfies polynomial identity and hence \( R \subseteq M_2(C) \) for some field \( C \) and \( M_2(C) \) satisfies \( a(bx^my^n - by^nbx^m) = 0 \). Then by Lemma 2.3, we have \( ab = 0 \), which is our conclusion (1).

Now replacing \( F \) with \(-F\) in the hypothesis \( a(F(x^m y^n) - F(y^n)F(x^m)) = 0 \), we get
\[
0 = a(-F)(x^m y^n) - (-F)(y^n)(-F)(x^m).
\]
That is, \( 0 = a(F(x^m y^n) + F(y^n)F(x^m)) \) for all \( x, y \in R \). This implies that there exists \( b \in U \) such that \( F(x) = bx \) for all \( x \in R \) with \( ab = 0 \) or \( F(x) = -x \). This completes the proof.

In particular, we have the following corollary.

**Corollary 2.6.** Let \( R \) be a prime ring of characteristic different from 2 and \( 0 \neq a \in R \). Suppose that \( R \) admits the generalized derivation \( F \) associated with a nonzero derivation \( d \) of \( R \). If any one of the following conditions is satisfied:

1. \( a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \);
2. \( a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0 \) for all \( x, y \in R \),

then \( R \) is commutative.

**Proof of Theorem 1.3.** First we consider the case \( a(F(x^m y^n) + F(x^m)F(y^n)) = 0 \) for all \( x, y \in R \). Other cases are similar. We know the fact that any derivation of a semiprime ring \( R \) can be uniquely extended to a derivation of its left Utumi quotient ring \( U \) and so any derivation of \( R \) can be defined on the whole of \( U \) [24, Lemma 2]. Moreover \( R \) and \( U \) satisfy the same GPIs as well as same differential identities. Thus
\[
a(bx^my^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0
\]
for all \( x, y \in U \). Let \( M(C) \) be the set of all maximal ideals of \( C \) and \( P \in M(C) \).

Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have \( PU \) is a prime ideal of \( U \) invariant under all derivations of \( U \). Moreover, \( \bigcap \{ PU \mid P \in M(C) \} = 0 \). Set \( \overline{U} = U/PU \). Then derivation \( d \) canonically induces a derivation \( \overline{d} \) on \( \overline{U} \) defined by \( \overline{d}(x) = \overline{d}(x) \) for all \( x \in U \). Therefore,
\[
\overline{a}(bx^my^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0
\]
for all \( \pi, \bar{\pi} \in U \). By the prime ring case of Corollary 2.6, we have either 
\[ d = 0 \text{ or } [U, U] = 0 \text{ or } \pi = 0. \]
In any case we have \( \text{ad}(U)[U, U] \subseteq PU \) for all \( P \in M(C) \). Since 
\[ \bigcap\{PU \mid P \in M(C)\} = 0, \text{ad}(U)[U, U] = 0. \]
In particular, \( \text{ad}(R)[R, R] = 0 \). This implies \( 0 = \text{ad}(R)[R^2, R] = \text{ad}(R)[R, R] + \text{ad}(R)[R, R]R + \text{ad}(R)[R, R][R, R] = 0. \)
Therefore, \( \text{ad}(R)[R, R][\text{ad}(R), R] = 0. \) Since \( R \) is semiprime, we obtain that \( \text{ad}(R) \subseteq Z(R) \). By Theorem 3.2 in [10], there exist orthogonal central idempotents \( e_1, e_2, e_3 \in U \) with
\[ e_1 + e_2 + e_3 = 1 \]
such that \( d(e_1U) = 0, e_2a = 0, \) and \( e_3U \) is commutative. Hence the theorem is proved.

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**References**