

Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

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ABSTRACT. Let R be a prime ring with its Utumi ring of quotients U , $C = Z(U)$ the extended centroid of R , L a non-central Lie ideal of R and $0 \neq a \in R$. If R admits a generalized derivation F such that $a(F(u^2) \pm F(u)^2) = 0$ for all $u \in L$, then one of the following holds:

- (1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
- (2) $F(x) = \mp x$ for all $x \in R$;
- (3) $\text{char}(R) = 2$ and R satisfies s_4 ;
- (4) $\text{char}(R) \neq 2$, R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

We also study the situations (i) $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, and (ii) $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$ in prime and semiprime rings.

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1. INTRODUCTION

Let R be an associative prime ring with center $Z(R)$ and U the Utumi quotient ring of R . The center of U , denoted by C , is called the extended centroid of R (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of x, y is denoted by $[x, y] = xy - yx$. An additive mapping $d : R \rightarrow R$ is called a derivation, if it satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, d is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \rightarrow R$ is called generalized derivation, if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Let S be a nonempty subset of R and $F : R \rightarrow R$ be an additive mapping. Then we say that F acts as homomorphism or anti-homomorphism on S if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping F acts as a Jordan homomorphism on S if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Many results in literature indicate that global structure of a prime ring R is often tightly connected to the behavior of additive mappings defined on R . Asma, Rehman, Shakir in [1] proved that if d is a derivation of a 2-torsion free prime ring R which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of R such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation F acts as homomorphism or anti-homomorphism on some subsets of R and then determined the structure of ring R as well as associated map F (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let R be a prime ring of characteristic different from 2, F a generalized derivation of R associated to a derivation d , L a Lie ideal of R such that $u^2 \in L$ for all $u \in L$. If F acts as a homomorphism or anti-homomorphism on L , then either $d = 0$ or L is central in R . More recently in [9], Filippis studied the situation when generalized derivation F acts as a Jordan homomorphism on a non-central Lie ideal L of R .

Recently in [26], Rehman and Raza proved the following: Let R be a prime ring of char $(R) \neq 2$, Z the center of R , and L a nonzero Lie ideal of R . If R admits a generalized derivation F associated with a derivation d which acts as a homomorphism or as anti-homomorphism on L , then either $d = 0$ or $L \subseteq Z$.

In the above result, Rehman and Raza [26] did not give the complete structure of the map F .

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map F . The main results of this paper are as follows:

Theorem 1.1. *Let R be a prime ring with its Utumi ring of quotients U , $C = Z(U)$ the extended centroid of R , L a non-central Lie ideal of R and $0 \neq a \in R$. If R admits a generalized derivation F such that $a(F(u^2) \pm F(u)^2) = 0$ for all $u \in L$, then one of the following holds:*

- (1) *there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;*
- (2) *$F(x) = \mp x$ for all $x \in R$;*
- (3) *$\text{char}(R) = 2$ and R satisfies s_4 ;*
- (4) *$\text{char}(R) \neq 2$, R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.*

Theorem 1.2. *Let R be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients U , $C = Z(U)$ the extended centroid of R , F a generalized derivation on R and $0 \neq a \in R$.*

- (1) *If $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$ or $F(x) = \mp x$ for all $x \in R$.*
- (2) *If $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$, then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$.*

Theorem 1.3. *Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R , $C = Z(U)$ the extended centroid of R , $F(x) = bx + d(x)$ a generalized derivation on R associated to the derivation d and $0 \neq a \in R$. If any one of the following holds:*

- (1) *$a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$,*
- (2) *$a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$,*

then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1 U) = 0$, $e_2 a = 0$, and $e_3 U$ is commutative.

The following remarks are useful tools for the proof of main results.

Remark 1.4. Let R be a prime ring and L a noncentral Lie ideal of R . If $\text{char}(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If $\text{char}(R) = 2$ and $\dim_C RC > 4$, i.e., $\text{char}(R) = 2$ and R does not satisfy s_4 , then by [22, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either $\text{char}(R) \neq 2$ or R does not satisfy s_4 , then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

Remark 1.5. We denote by $\text{Der}(U)$ the set of all derivations on U . By a derivation word Δ of R we mean $\Delta = d_1 d_2 d_3 \dots d_m$ for some derivations $d_i \in \text{Der}(U)$.

Let D_{int} be the C -subspace of $\text{Der}(U)$ consisting of all inner derivations on U and let d be a non-zero derivation on R . By [21, Theorem 2] we have the following result:

If $\Phi(x_1, x_2, \dots, x_n, d(x_1), d(x_2) \cdots d(x_n))$ is a differential identity on R , then one of the following holds:

- (1) $d \in D_{int}$;
- (2) R satisfies the generalized polynomial identity $\Phi(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$.

Remark 1.6. In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \rightarrow U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense left ideal of R and d is a derivation from I into U . Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U , and thus all generalized derivations of R will be implicitly assumed to be defined on the whole of U . Lee obtained the following: every generalized derivation F on a dense left ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U .

2. PROOF OF THE MAIN RESULTS

Now we begin with the following Lemmas:

Lemma 2.1. *Let $R = M_2(C)$ be the ring of all 2×2 matrices over the field C of characteristic different from 2 and $b, c \in R$. Suppose that there exists $0 \neq a \in R$ such that*

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0,$$

for all $x, y \in R$. Then $c \in C \cdot I_2$.

Proof. If $c \in C \cdot I_2$, then nothing to prove. Let $c \notin C \cdot I_2$. In this case R is a dense ring of C -linear transformations over a vector space V . Assume that there exists $0 \neq v \in V$ such that $\{v, cv\}$ is linearly C -independent. By density, there exist $x, y \in R$ such that $xv = v, xcv = 0; yv = 0, ycv = v$. Then $[x, y]v = 0, [x, y]cv = v$ and hence $a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}v = av$.

Of course for any $u \in V$, $\{u, v\}$ linearly C -dependent implies $au = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $\{w, v\}$ are linearly C -independent. Also $a(w + v) = aw \neq 0$ and $a(w - v) = aw \neq 0$. By the above argument, it follows that w and cw are linearly C -dependent, as are $\{w + v, c(w + v)\}$ and $\{w - v, c(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).$$

In other words we have

$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v \tag{2.1}$$

and

$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{2.2}$$

By comparing (2.1) with (2.2) we get both

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \quad (2.3)$$

and

$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \quad (2.4)$$

By (2.3), and since $\{w, v\}$ are C -independent and $\text{char}(R) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (2.4) it follows $2cv = 2\alpha_w v$. This leads a contradiction with the fact that $\{v, cv\}$ is linear C -independent.

In light of this, we may assume that for any $v \in V$ there exists a suitable $\alpha_v \in C$ such that $cv = \alpha_v v$, and standard argument shows that there is $\alpha \in C$ such that $cv = \alpha v$ for all $v \in V$. Now let $r \in R, v \in V$. Since $cv = \alpha v$,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[c, r]v = 0$ for all $v \in V$ i.e., $[c, r]V = 0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space V , $[c, r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$, a contradiction. \square

Lemma 2.2. *Let $R = M_2(C)$ be the ring of all 2×2 matrices over the field C of characteristic different from 2 and $0 \neq p \in R$. Suppose that there exists $0 \neq a \in R$ such that*

$$a(px^m y^n - px^m py^n) = 0,$$

for all $x, y \in R$. Then either $ap = 0$ or $p = 1$.

Proof. Putting $x = y = I_2$, we get $ap = ap^2$. In this case R is a dense ring of C -linear transformations over a vector space V . Assume that there exists $0 \neq v \in V$ such that $\{v, pv\}$ is linearly C -independent. By density, there exist $x, y \in R$ such that $xv = v, xpv = 0; yv = v, ypv = 0$. Then we get $0 = a(px^m y^n - px^m py^n)v = apv$. Then by same argument as in Lemma 2.1, we get either $ap = 0$ or $p \in C \cdot I_2$. When $0 \neq p \in C \cdot I_2$, from $ap = ap^2$, we get $0 = a(p - 1)$. Since $a \neq 0$, we conclude $p = 1$. \square

Lemma 2.3. *Let $R = M_2(C)$ be the ring of all 2×2 matrices over the field C of characteristic different from 2 and $0 \neq p \in R$. Suppose that there exists $0 \neq a \in R$ such that*

$$a(px^m y^n - py^n px^m) = 0,$$

for all $x, y \in R$. Then $ap = 0$.

Proof. Putting $x = y = I_2$, we get $ap = ap^2$. Here R is a dense ring of C -linear transformations over a vector space V . Assume that there exists $0 \neq v \in V$ such that $\{v, pv\}$ is linearly C -independent. By density, there exist $x, y \in R$ such that $xv = v, xpv = 0; yv = 0, ypv = pv$. Then we have $0 = a(px^m y^n - py^n px^m)v = -ap^2 v = -apv$. Then by same argument as in Lemma 2.1, we get either $ap = 0$ or $p \in C \cdot I_2$. When $0 \neq p \in C \cdot I_2$, by hypothesis, we get $0 = a[x^m, y^n]$. Then for $x = e_{11}$ and $y = e_{11} + e_{12}$, we have

$0 = a[x^m, y^n] = a[e_{11}, e_{11} + e_{12}] = ae_{12}$. Again, for $x = e_{22}$ and $y = e_{22} + e_{21}$, we have $0 = a[x^m, y^n] = a[e_{22}, e_{22} + e_{21}] = ae_{21}$. These imply $a = 0$, a contradiction. \square

Lemma 2.4. *Let R be a noncommutative prime ring with extended centroid C and $b, c \in R$. Suppose that $0 \neq a \in R$ such that*

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0$$

for all $x, y \in R$. Then one of the following holds:

- (1) $c \in C$ and $a(b + c) = 0$;
- (2) $b, c \in C$ and $b + c = 1$;
- (3) $\text{char}(R) = 2$ and R satisfies s_4 ;
- (4) $\text{char}(R) \neq 2$, R satisfies s_4 and $c \in C$.

Proof. By assumption, R satisfies the generalized polynomial identity (GPI)

$$f(x, y) = a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}.$$

By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by U . Now we consider the following two cases:

Case-I. U does not satisfy any nontrivial GPI.

Let $T = U *_C C\{x, y\}$, the free product of U and $C\{x, y\}$, the free C -algebra in noncommuting indeterminates x and y . Thus

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}$$

is zero element in $T = U *_C C\{x, y\}$. Let $c \notin C$. Then $\{1, c\}$ is C -independent. Then from above

$$a\{[x, y]^2c - (b[x, y] + [x, y]c)[x, y]c\}$$

which is

$$a\{[x, y] - b[x, y] - [x, y]c\}[x, y]c,$$

is zero in T . Again, since $c \notin C$, we have that $a[x, y]c[x, y]c$ is zero element in T , implying $a = 0$ or $c = 0$, a contradiction. Thus we conclude that $c \in C$. Then the identity reduces to

$$a\{(b + c)[x, y] - (b + c)[x, y](b + c)\}[x, y],$$

is zero element in T . Again, if $b + c \notin C$, then $a(b + c)[x, y]^2$ becomes zero element in T , implying $a(b + c) = 0$. If $b + c \in C$, then $a(b + c)(b + c - 1)[x, y]^2$ becomes zero element in T , implying $b + c = 0$ or $b + c = 1$. When $b + c = 0$, then $a(b + c) = 0$, which is our conclusion (1). When $b + c = 1$, then $b = 1 - c \in C$, which is our conclusion (2).

Case-II. U satisfies a nontrivial GPI.

Thus we assume that

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0,$$

is a nontrivial GPI for U . In case C is infinite, we have $f(x, y) = 0$ for all $x, y \in U \otimes_C \overline{C}$, where \overline{C} is the algebraic closure of C . Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [17], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that R centrally closed over C which either finite or algebraically closed and $f(x, y) = 0$ for all $x, y \in R$. By Martindale's Theorem [25], R is then primitive ring having non-zero socle $\text{soc}(R)$ with C as the associated division ring. Hence by Jacobson's Theorem [20], R is isomorphic to a dense ring of linear transformations of a vector space V over C . Since R is noncommutative, $\dim_C V \geq 2$. If $\dim_C V = 2$, then $R \cong M_2(C)$. In this case by Lemma 2.1, either $c \in C$ or $\text{char}(R) = 2$. This gives conclusions (3) and (4).

Let $\dim_C V \geq 3$. Let for some $v \in V$, v and cv are linearly independent over C . By density there exist $x, y \in R$ such that

$$xv = v, \quad xcv = 0;$$

$$yv = 0, \quad ycv = v.$$

Then $[x, y]v = 0$, $[x, y]cv = v$ and hence $a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}v = av$.

This implies that if $av \neq 0$, then by contradiction we may conclude that v and cv are linearly C -dependent. Now choose $v \in V$ such that v and cv are linearly C -independent. Set $W = \text{Span}_C\{v, cv\}$. Then $av = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and then $a(v - w) = aw \neq 0$. By the previous argument we have that w, cw are linearly C -dependent and $(v - w), c(v - w)$ too. Thus there exist $\alpha, \beta \in C$ such that $cw = \alpha w$ and $c(v - w) = \beta(v - w)$. Then $cv = \beta(v - w) + cw = \beta(v - w) + \alpha w$ i.e., $(\alpha - \beta)w = cv - \beta v \in W$. Now $\alpha = \beta$ implies that $cv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$. Again, if $u \in V$ with $au = 0$ then $a(w + u) \neq 0$. So, $w + u \in W$ forcing $u \in W$. Thus it is observed that $w \in V$ with $aw \neq 0$ implies $w \in W$ and $u \in V$ with $au = 0$ implies $u \in W$. This implies that $V = W$ i.e., $\dim_C V = 2$, a contradiction.

Hence, in any case, v and cv are linearly C -dependent for all $v \in V$. Thus for each $v \in V$, $cv = \alpha_v v$ for some $\alpha_v \in C$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $cv = \alpha v$ for all $v \in V$ and $\alpha \in C$ fixed. Now let $r \in R, v \in V$. Since $cv = \alpha v$,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[c, r]v = 0$ for all $v \in V$ i.e., $[c, r]V = 0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space V , $[c, r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$.

Thus our identity reduces to

$$a\{(b'[x, y]^2) - (b'[x, y])^2\} = 0,$$

for all $x, y \in R$, where $b' = b + c$.

Let for some $v \in V$, v and $b'v$ are linearly independent over C . Since $\dim_C V \geq 3$, there exists $u \in V$ such that $v, b'v, u$ are linearly independent over C . By density there exist $x, y \in R$ such that

$$xv = v, \quad xb'v = 0, \quad xu = v;$$

$$yv = 0, \quad yb'v = u, \quad yu = v.$$

Then $[x, y]v = 0$, $[x, y]b'v = v$, $[x, y]u = v$ and hence $0 = a\{(b'[x, y]^2) - (b'[x, y])^2\}u = ab'v$. Then by same argument as before, we have either $ab' = 0$ or v and $b'v$ are linearly C -dependent for all $v \in V$. In the first case, $0 = ab' = a(b + c)$, which is conclusion (1). In the last case, again by standard argument, we have that $b' \in C$. If $b' = 0$, then also $ab' = a(b + c) = 0$ which gives conclusion (1). So assume that $0 \neq b' \in C$. Then our identity reduces to

$$ab'(b' - 1)[x, y]^2 = 0,$$

for all $x, y \in R$. This gives $0 = ab'(b' - 1) = a(b' - 1)$. Since $a \neq 0$, we get $b' = 1$. This gives conclusion (2). \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First we consider the case when

$$a(F(u^2) - F(u)^2) = 0,$$

for all $u \in L$. If $\text{char}(R) = 2$ and R satisfies s_4 , then we have our conclusion (3). So we assume that either $\text{char}(R) \neq 2$ or R does not satisfy s_4 . Since L is a noncentral by Remark 1.4, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Thus by assumption I satisfies the differential identity

$$a(F([x, y]^2) - F([x, y])^2) = 0.$$

Now since R is a prime ring and F is a generalized derivation of R , by Lee [23, Theorem 3], $F(x) = bx + d(x)$ for some $b \in U$ and derivation d on U . Since I, R and U satisfy the same differential identities [24], without loss of generality, U satisfies

$$a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0. \quad (2.5)$$

Here we divide the proof into two cases:

Case 1. Let d be inner derivation induced by element $c \in U$, that is, $d(x) = [c, x]$ for all $x \in U$. It follows that

$$a(b[x, y]^2 + [c, [x, y]^2] - (b[x, y] + [c, [x, y]])^2) = 0,$$

that is

$$a((b+c)[x, y]^2 - [x, y]^2 c - ((b+c)[x, y] - [x, y]c)^2) = 0,$$

for all $x, y \in U$. Now by Lemma 2.4, one of the following holds:

(1) $c \in C$ and $0 = a(b+c-c) = ab$. Thus $F(x) = bx$ for all $x \in R$, with $ab = 0$.

(2) $b+c, c \in C$ and $b+c-c = 1$. Thus $F(x) = x$ for all $x \in R$.

(3) $\text{char}(R) \neq 2$, R satisfies s_4 and $c \in C$. Thus $F(x) = bx$ for all $x \in R$.

Case 2. Assume that d is not inner derivation of U . We have from (2.5) that U satisfies

$$a(b[x, y]^2 + d([x, y])[x, y] + [x, y]d([x, y]) - (b[x, y] + d([x, y]))^2) = 0,$$

that is

$$a(b[x, y]^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y]([d(x), y] + [x, d(y)]) - (b[x, y] + [d(x), y] + [x, d(y)])^2) = 0.$$

Then by Kharchenko's Theorem [21], U satisfies

$$a(b[x, y]^2 + ([u, y] + [x, z])[x, y] + [x, y]([u, y] + [x, z]) - (b[x, y] + [u, y] + [x, z])^2) = 0. \quad (2.6)$$

Since R is noncommutative, we may choose $q \in U$ such that $q \notin C$. Then replacing u by $[q, x]$ and z by $[q, y]$ in (2.6), we get

$$a(b[x, y]^2 + ([q, x], y) + [x, [q, y]])[x, y] + [x, y]([q, x], y) + [x, [q, y]] - (b[x, y] + ([q, x], y) + [x, [q, y]])^2) = 0,$$

which is

$$a(b[x, y]^2 + [q, [x, y]^2]) - (b[x, y] + [q, [x, y]])^2 = 0.$$

Then by Lemma 2.4, we have $q \in C$, a contradiction.

Now replacing F with $-F$ in the above result, we obtain the conclusion for the situation $a(F(u^2) + F(u)^2) = 0$ for all $u \in L$.

Corollary 2.5. *Let R be a prime ring with extended centroid C , L a non-central Lie ideal of R and $0 \neq a \in R$. If R admits the generalized derivation F such that either $a(F(XY) \pm F(X)F(Y)) = 0$ for all $X, Y \in L$ or $a(F(XY) \pm F(Y)F(X)) = 0$ for all $X, Y \in L$, then one of the following holds:*

- (1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
- (2) $F(x) = \mp x$ for all $x \in R$;
- (3) $\text{char}(R) = 2$ and R satisfies s_4 ;
- (4) $\text{char}(R) \neq 2$, R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

Proof of Theorem 1.2. First consider the case when $a(F(x^m y^n) - F(x^m)F(y^n)) = 0$ for all $x, y \in R$. Let G_1 be the additive subgroup of R generated by the set $S_1 = \{x^m | x \in R\}$ and G_2 be the additive subgroup of R generated by the set $S_2 = \{x^n | x \in R\}$. Then by assumption

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in G_1, \quad \forall y \in G_2.$$

Then by [7], either $G_1 \subseteq Z(R)$ or $\text{char}(R) = 2$ and R satisfies s_4 , except when G_1 contains a noncentral Lie ideal L_1 of R . $G_1 \subseteq Z(R)$ implies that $x^m \in Z(R)$ for all $x \in R$. It is well known that in this case R must be commutative, which is a contradiction. Since $\text{char}(R) \neq 2$, we are to consider the case when G_1 contains a noncentral Lie ideal L_1 of R . In this case by [4, Lemma 1], there exists a nonzero ideal I_1 of R such that $[I_1, I_1] \subseteq L_1$.

Thus we have

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in G_2.$$

Analogously, we see that there exists a nonzero ideal I_2 of R such that

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in [I_2, I_2].$$

By Lee [23, Theorem 3], $F(x) = bx + d(x)$ for some $b \in U$ and derivations d on U . Since I_1, I_2, R and U satisfy the same differential identities [24], without loss of generality,

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x, y \in [R, R],$$

and in particular

$$a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].$$

Then by Theorem 1.1, we get

- (1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
- (2) $F(x) = x$ for all $x \in R$;
- (3) R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

In the last conclusion, R satisfies polynomial identity and hence $R \subseteq M_2(C)$ for some field C and $M_2(C)$ satisfies $a(bx^m y^n - bx^m b y^n) = 0$. By lemma 2.2, we get either $ab = 0$ or $b = 1$. If $ab = 0$, then $F(x) = bx$ for all $x \in R$, with $ab = 0$, which is our conclusion (1). If $b = 1$ then $F(x) = x$ for all $x \in R$, which is our conclusion (2).

Now replacing F with $-F$ in the hypothesis $a(F(x^m y^n) - F(x^m)F(y^n)) = 0$, we get $0 = a((-F)(x^m y^n) - (-F)(x^m)(-F)(y^n))$, that is $0 = a(F(x^m y^n) + F(x^m)F(y^n))$ for all $x, y \in R$ implies one of the following:

- (1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
- (2) $F(x) = -x$ for all $x \in R$;

Now consider the case when $a(F(x^m y^n) - F(y^n)F(x^m)) = 0$ for all $x, y \in R$. By similar argument as above we get

$$a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R, R],$$

and in particular

$$a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].$$

Then by Theorem 1.1, we get

- (1) there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $ab = 0$;
- (2) $F(x) = x$ for all $x \in R$;
- (3) R satisfies s_4 and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$.

In the conclusion (3), R satisfies polynomial identity and hence $R \subseteq M_2(C)$ for some field C and $M_2(C)$ satisfies $a(bx^m y^n - by^n bx^m) = 0$. Then by Lemma 2.3, we have $ab = 0$, which is our conclusion (1).

Now replacing F with $-F$ in the hypothesis $a(F(x^m y^n) - F(y^n)F(x^m)) = 0$, we get $0 = a((-F)(x^m y^n) - (-F)(y^n)(-F)(x^m))$. That is, $0 = a(F(x^m y^n) + F(y^n)F(x^m))$ for all $x, y \in R$. This implies that there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$ or $F(x) = -x$. This completes the proof.

In particular, we have the following corollary.

Corollary 2.6. *Let R be a prime ring of characteristic different from 2 and $0 \neq a \in R$. Suppose that R admits the generalized derivation F associated with a nonzero derivation d of R . If any one of the following conditions is satisfied:*

- (1) $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$ for all $x, y \in R$;
- (2) $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$ for all $x, y \in R$,

then R is commutative.

Proof of Theorem 1.3. First we consider the case $a(F(x^m y^n) + F(x^m)F(y^n)) = 0$ for all $x, y \in R$. Other cases are similar. We know the fact that any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U [24, Lemma 2]. Moreover R and U satisfy the same GPIs as well as same differential identities. Thus

$$a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0$$

for all $x, y \in U$. Let $M(C)$ be the set of all maximal ideals of C and $P \in M(C)$. Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have PU is a prime ideal of U invariant under all derivations of U . Moreover, $\bigcap \{PU \mid P \in M(C)\} = 0$. Set $\bar{U} = U/PU$. Then derivation d canonically induces a derivation \bar{d} on \bar{U} defined by $\bar{d}(\bar{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,

$$\bar{a}(b\bar{x}^m \bar{y}^n + d(\bar{x}^m \bar{y}^n) + (b\bar{x}^m + d(\bar{x}^m))(b\bar{y}^n + d(\bar{y}^n))) = 0$$

for all $\bar{x}, \bar{y} \in \bar{U}$. By the prime ring case of Corollary 2.6, we have either $\bar{d} = 0$ or $[\bar{U}, \bar{U}] = 0$ or $\bar{a} = 0$. In any case we have $ad(U)[U, U] \subseteq PU$ for all $P \in M(C)$. Since $\bigcap\{PU \mid P \in M(C)\} = 0$, $ad(U)[U, U] = 0$. In particular, $ad(R)[R, R] = 0$. This implies $0 = ad(R)[R^2, R] = ad(R)R[R, R] + ad(R)[R, R]R = ad(R)R[R, R]$. In particular, $ad(R)R[R, ad(R)] = 0$. Therefore, $[ad(R), R]R[ad(R), R] = 0$. Since R is semiprime, we obtain that $ad(R) \subseteq Z(R)$. By Theorem 3.2 in [10], there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $d(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative. Hence the theorem is proved.

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