Fractal Dimension of Graphs of Typical Continuous Functions on Manifolds

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Abstract. If $M$ is a compact Riemannian manifold and $C(M, R)$ is the set of all real valued continuous functions defined on $M$, then we show that for a typical element $f \in C(M, R)$, $\text{dim}_B(\text{graph}(f))$ is as big as possible and for a typical $f \in C(M, R)$, $\text{dim}_B(\text{graph}(f))$ is as small as possible.

Keywords: Manifold, Fractal, Box dimension.


1. Introduction

A subset $A$ of a topological space $X$ is called to be comeagre, if there is a countable collection $\{W_i\}$ of open and dense subsets of $X$ such that $\bigcap_i W_i \subset A$. Complement of a comeagre subset is called a meagre subset. A meagre subset can be considered as subset of a countable union of nowhere dense subsets and they are negligible in some sense. So, we say that some property holds for typical elements of $X$, if it holds on a comeagre subset. Study of properties of typical elements in $X$ is a classic and interesting problem. One can find many papers dealing with typical elements when $X$ is supposed to be the space $C(W, R)$ of all continuous functions defined on a compact topological space $W$, endowed with the metric topology defined by the metric $d(f, g) = \sup_{x \in W} |f(x) - g(x)|$. A well known theorem due to Banach [1], states that typical elements of $C([0, 1], R)$ are nowhere differentiable, so the image or graph of a typical $f$ in $C([0, 1], R)$ is a
fractal set. Calculating fractal dimensions (including box dimension, Hausdorff dimension, packing dimension, etc) of the image of $f$ or $\text{graph}(f)$ is a well known problem and one can find many results in the literature. It is proved in [6] that for a typical $g \in C([0, 1], R)$, $\dim_H(\text{graph}(g)) = 1$. It is proved in [3] that if $W \subset R$ is bounded with only finitely many isolated points and $X = \{f \in C(W, R) : f$ is uniformly continuous $\}$, then for a typical $f \in X$, $\dim_B(\text{graph}(f))$ is as big as possible and $\dim_B(\text{graph}(f))$ is as small as possible. In the previous paper [7] we generalized Banach’s theorem to the set $C(M, R)$, where $M$ is a compact Riemannian manifold. Now, we show in the present paper that the main results of [3] about upper and lower box dimensions are also true when $W$ is replaced by a compact Riemannian manifold $M$.

2. Preliminaries

In what follows, $M$ is a compact Riemannian manifold with the Riemannian metric $d$, and $C(M, R)$ will denote the collection of all continuous functions defined on $M$ endowed with the metric $d$ defined by $d(f, g) = \max_{x \in M} |f(x) - g(x)|$.

If $(X, d_1)$ and $(Y, d_2)$ are metric spaces then we will consider the usual product metric $d$ on $X \times Y$ defined by $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$.

If $E$ is a bounded subset of $M$ then the upper box dimension of $E$ is defined by

$$\dim_B(E) = \limsup_{\delta \to 0} \frac{N_\delta(E)}{-\log \delta}.$$ 

Where, $N_\delta(E)$ is the minimum number of balls of radius $\delta$ (or minimum number of sets of diameter at most $\delta$) covering $E$ (the lower box dimension $\dim_B(E)$ is defined in similar way). Another definition for dimension, which is widely used in fractal geometry is Hausdorff dimension (see [4]).

Now, we mention some facts which we need in the proofs of theorems.

Remark 2.1. If $E$ is a bounded subset of $R^m$ then $\dim_B(E \times I^n) = \dim_B(E) + n$. The similar result is true if we replace $\dim_B$ by $\dim_B$ or $\dim_H$.

Proof. We give the proof for $\dim_B(E \times I) = \dim_B(E) + 1$. The general case comes by induction. If $\delta > 0$ then the smallest number of intervals of length $\delta$ covering $I$ is equal to $[\frac{1}{\delta}]$ or $[\frac{1}{\delta}] + 1$. If $U_\delta (I_\delta)$ is a bounded subset of $R^m (I)$ with diameter $\delta$, then the diameter of $U_\delta \times I_\delta$ is equal to $\sqrt{2}\delta$. So,

$$N_{\sqrt{2}\delta}(E \times I) \leq ([\frac{1}{\delta}] + 1)N_\delta(E)$$
Then we have
\[ \overline{\dim}_B(E \times I) = \limsup_{\delta \to 0} \frac{\log(N_{\sqrt{2}\delta}(E \times I))}{-\log(\sqrt{2}\delta)} \]
\[ \leq \limsup_{\delta \to 0} \frac{\log([\frac{1}{2}] + 1)N_{\delta}(E)}{-\log(\sqrt{2}\delta)} \]
\[ = 1 + \limsup_{\delta \to 0} \frac{N_{\delta}(E)}{-\log\delta} = 1 + \overline{\dim}_B(E) \]
Also we know that \( \overline{\dim}_B(E \times I^n) \geq \overline{\dim}_B(E) + n \) (see [4]). So we get the equality.

**Remark 2.2.** If \( M \) is a compact metric space and \( f : M \to R \) is a locally lipschitz function, then \( f \) is globally lipschitz.

**Proof.** Since \( f \) is locally lipschitz and \( M \) is compact, then there is a finite collection of open cover of balls \( B_i, 1 \leq i \leq m \), and constants \( L_i \) such that
\[ d(f(x),f(y)) \leq L_id(x,y), \quad x,y \in B_i \]
Since \( M \) is compact then the function \( F : M \times M \to R \), defined by \( F(x,y) = d(f(x),f(y)) \) has a maximum which we denote it by \( N \). Let \( \delta \) be the lebesgue’s number related to the covering \( B_i \) of \( M \), and put \( L = \max\{\frac{N}{\delta}, L_i : i\} \). Then for given \( x,y \in M \), either there is a \( B_i \) such that \( x,y \in B_i \) or \( d(x,y) \geq \delta \). In the first case we have \( d(f(x),f(y)) \leq Ld(x,y) \). In the second case we have
\[ d(f(x),f(y)) \leq N \leq \frac{N}{\delta}d(x,y) \leq Ld(x,y) \]

If \( M \) and \( N \) are compact differentiable manifolds and \( f : M \to N \) is continuously differentiable, then \( f \) is a lipschitz function. So, we get the following remark easily.

**Remark 2.3.** If \( M \) and \( N \) are compact Riemannian manifolds and \( \phi : M \to N \) is a map such that \( \phi \) and its inverse are continuously differentiable, then the map \( \psi : M \times R \to N \times R \) defined by \( \psi(x,y) = (\phi(x), y) \) is bilipschitz.

**Remark 2.4.** If \( M \) is a compact Riemannian manifold, \( f : M \to R \) is continuously differentiable, \( g : M \to R \) is continuous and \( k = f + g \), then \( \overline{\dim}_B(\text{graph}(k)) = \overline{\dim}_B(\text{graph}(g)) \). The same result is true for \( \overline{\dim}_B \).

**Proof.** Consider the map \( \psi : \text{graph}(g) \to \text{graph}(k) \), defined by \( \psi(x,g(x)) = (x, k(x)) \). We show that \( \psi \) and \( \psi^{-1} \) are Lipschitz functions. We have
\[ d(\psi(x,g(x)), \psi(y,g(y))) = d((x,k(x)),(y,k(y))) = \sqrt{d^2(x,y) + (k(x) - k(y))^2} \]
Since $f$ is continuously differentiable, it is locally Lipschitz and by Remark 2.2, it must be Lipschitz. Then, there exist a positive number $N$ such that $|f(x) - f(y)| \leq Nd(x, y)$, $x, y \in M$. Thus

$$(k(x) - k(y))^2 = (f(x) - f(y) + g(x) - g(y))^2 \leq (Nd(x, y) + |g(x) - g(y)|)^2$$

$$= N^2d^2(x, y) + 2N|g(x) - g(y)| + |g(x) - g(y)|^2$$

$$\leq N^2d^2(x, y) + N^2d^2(x, y) + |g(x) - g(y)|^2 + |g(x) - g(y)|^2$$

$$= 2N^2d^2(x, y) + 2|g(x) - g(y)|^2$$

Then

$$d(\psi(x, g(x)), \psi(y, g(y))) \leq \sqrt{d^2(x, y) + 2N^2d^2(x, y) + 2|g(x) - g(y)|^2}$$

$$\leq \sqrt{2(N^2 + 1)} \sqrt{d^2(x, y) + (g(x) - g(y))^2} \leq \sqrt{2(N^2 + 1)} d((x, g(x)), (y, g(y))).$$

Therefore, $\psi$ is Lipschitz. In a similar way we can show that $\psi^{-1}$ is Lipschitz.

Remark 2.5. (generalized Stone-Weierstrass Theorem). Suppose $X$ is a compact Hausdorff space and $A$ is a subalgebra of $C(X, R)$ which contains a non-zero constant function. Then $A$ is dense in $C(X, R)$ if and only if it separates points.

3. Results

Lemma 3.1. If $f : M \to R$ is continuously differentiable and $\epsilon > 0$, then there exists $g \in C(M, R)$ such that $d(f, g) < \epsilon$ and $\overline{\dim}(\text{graph}(g)) = n + 1$, $n = \dim M$.

Proof. Let $N$ be a compact Riemannian manifold. Consider a function $g_1 \in C(I, R^+)$ such that $\overline{\dim}(\text{graph}(g_1)) = 2$ and put

$$g_2 : I^n = I \times I^{n-1} \to R^+, \quad g_2(t_1, t_2) = g_1(t_1).$$

Then

$$\text{graph}(g_2) = \{(t_1, t_2), g_1(t_1)) \times I^{n-1} \} \simeq$$

$$\{(t_1, g_1(t_1)), t_2 \in I \times I^{n-1} = \text{graph}(g_1) \times I^{n-1}.\}

So, by Remark 2.1

$$\overline{\dim}(\text{graph}(g_2)) = 2 + n - 1 = n + 1.$$
n + 1. By compactness condition we can choose \( \delta \) small enough such that for all \( y = (x, f(x)) \in N, g_5(y) < \epsilon \).

Now, consider the function \( g_6 : M \to R \), defined by \( g_6(x) = g_5(x, f(x)) \) and put \( \psi : M \times R \to N \times R, \psi(x, y) = ((x, f(x)), y) \). We have

\[ \psi \circ g = g \]

By Remark 2.3, \( \psi \) is bilipshitz, so

\[ \overline{\dim}_B(\text{graph}(g_6)) = \overline{\dim}_B(\text{graph}(g_5)) = n + 1 \]

Put \( g : M \to R, g(x) = f(x) + g_6(x) \). Since \( f \) is differentiable, then by Remark 2.4, \( \overline{\dim}_B(\text{graph}(g)) = \overline{\dim}_B(\text{graph}(g_6)) = n + 1 \). Also, we have \( d(f, g) = \max_{x \in M} |g(x) - f(x)| = \max_{x \in M} |g_6(x)| = \max_{x \in M} g_5(x, f(x)) < \epsilon \). \( \square \)

**Theorem 3.2.** Let \( M \) be a compact Riemannian manifold, \( \dim(M) = n \), and \( C(M, R) \) be the set of all continuous functions defined on \( M \). Then for typical members \( f \) in \( C(M, R) \), \( \overline{\dim}_B(\text{graph}(f)) = n \).

**Proof.** Put

\[ A = \{ f \in C(M, R) : \dim_B(\text{graph}(f)) = n \}. \]

Let \( f \in A \) and consider a positive number \( \epsilon > 0 \) and \( g \in C(M, R) \) such that \( d(f, g) < \epsilon \). If a collection of balls of radius \( \delta \) in \( M \times R \) covers \( \text{graph}(f) \) and \( \epsilon < \delta \), then the same number of balls with radius \( 2\delta \) covers \( \text{graph}(g) \). Since each ball of radius \( 2\delta \) can be covered by \( 4^{n+1} \) balls of radius \( \delta \), then

\[ N_{\delta}(\text{graph}(g)) \leq 4^{n+1}N_{\delta}(\text{graph}(f)) \]

If \( \delta < 1 \) then

\[ \frac{\log N_{\delta}(\text{graph}(g))}{-\log(\delta)} \leq (n + 1) \frac{\log 4}{-\log(\delta)} + \frac{\log N_{\delta}(\text{graph}(f))}{-\log(\delta)} \]

Since \( \dim_B(\text{graph}(f)) = n \) and \( \lim_{\delta \to 0} \frac{\log 4}{-\log(\delta)} = 0 \), then for each \( k \in N \) there exists \( \delta = \delta(f, k) > 0 \) such that

\[ \frac{\log N_{\delta}(\text{graph}(g))}{-\log(\delta)} \leq (n + 1) \frac{\log 4}{-\log(\delta)} + \frac{\log N_{\delta}(\text{graph}(f))}{-\log(\delta)} < n + \frac{1}{k} \]

Put

\[ U_{f,k} = \{ g \in C(M, R) : d(f, g) < \delta(f, k) \} \]

and

\[ W_k = \bigcup_{(f \in A)} U_{f,k} \]

\( W_{f,k} \) is an open set in \( C(M, R) \) such that for each \( g \in W_k \),

\[ \dim_B(\text{graph}(g)) < n + \frac{1}{k} \]

Clearly \( A \subset \bigcap_k W_k \). If \( g \in \bigcap_k W_k \) then \( \dim_B(\text{graph}(g)) \leq n \), and since for all \( g \in C(M, R) \), \( n \leq \dim_B(\text{graph}(g)) \) then \( \dim_B(\text{graph}(g)) = n \). Thus
∩_k W_k = A. Now, we show that W_k is dense for all k, then the proof will be complete. Given g ∈ C(M, R) and ϵ > 0. By Remark 2.5, collection of differentiable functions is dense, so there exists a differentiable function f : M → R such that d(f, g) < ϵ. But for a differentiable function f, \( \dim_B(\text{graph}(f)) = \dim_B(\text{graph}(f)) = n \). So f ∈ A ⊂ W_k.

**Lemma 3.3.** If g ∈ C(M, R) and ϵ > 0, then there exists k ∈ C(M, R) such that d(g, k) < ϵ and \( \dim_B(\text{graph}(k)) = n + 1 \).

**Proof.** By Remark 2.5, for a given δ > 0 there exists a differentiable function f ∈ C(M, R) such that d(f, g) < δ. Consider a function f_1 ∈ C(M, R) such that \( \overline{\dim}_B(\text{graph}(f_1)) = n + 1 \). Since M is compact, for a given number δ_2 > 0 there is a positive number δ_1 such that |δ_1f_1(x)| < δ_2 for all x ∈ M. Now, put k = f + δ_1f_1. By Remark 2.4, we have

\[
\overline{\dim}_B(\text{graph}(k)) = \overline{\dim}_B(\text{graph}(δ_1f_1)) = \overline{\dim}_B(\text{graph}(f_1)) = n + 1.
\]

If we choose δ and δ_2 smaller than \( \frac{\epsilon}{2} \), then

\[
d(g, k) ≤ d(g, f) + d(f, k) ≤ δ + δ_1||f_1|| ≤ δ + δ_2 < ϵ.
\]

□

**Theorem 3.4.** Let M be a compact Riemannian manifold, \( \dim(M) = n \), and C(M, R) be the set of all continuous functions defined on M. Then for typical members f in C(M, R), \( \overline{\dim}_B(\text{graph}(f)) = n + 1 \).

**Proof.** Clearly for all f ∈ C(M, R), \( \overline{\dim}_B(\text{graph}(f)) ≤ n + 1 \). Put

\[
A = \{ f ∈ C(M, R) : \overline{\dim}_B(\text{graph}(f)) = n + 1 \}.
\]

Consider f ∈ A, a positive number ϵ > 0 and g ∈ C(M, R) such that d(f, g) < ϵ.

If a collection of balls of radius δ in M × R covers graph(g) and ϵ < δ, then the same number of balls with radius 2δ covers graph(f). Since each ball of radius 2δ can be covered by \( 4^{n+1} \) balls of radius δ, then

\[
N_δ(\text{graph}(f)) < 4^{n+1}N_δ(\text{graph}(g))
\]

So, if δ < 1 then

\[
\frac{\log N_δ(\text{graph}(f))}{-\log(δ)} < (n + 1)\frac{\log N_δ(\text{graph}(g))}{-\log(δ)} + \frac{\log N_δ(\text{graph}(g))}{-\log(δ)}
\]

Since \( \overline{\dim}_B(\text{graph}(f)) = n + 1 \), then for each k ∈ N there is δ(k) = δ(f, k) > 0 such that

\[
n + 1 - \frac{1}{k} < \frac{\log N_δ(k)(\text{graph}(f))}{-\log(δ(k))} - (n + 1)\frac{\log 4}{-\log(δ(k))} < \frac{\log N_δ(k)(\text{graph}(g))}{-\log(δ(k))}
\]

Put

\[
U_{f,k} = \{ g ∈ C(M, R) : d(f, g) < δ(f, k) \}
\]
Fractal dimension of graphs of typical continuous functions on manifolds

and

\[ W_k = \bigcup_{f \in A} U_{f,k} \]

\( W_k \) is an open set in \( C(M, R) \) such that for each \( g \in W_k \),

\[ \overline{\dim_B(\text{graph}(g))} > n + 1 - \frac{1}{k} \]

Clearly

\[ \bigcap_k W_k = A \]

Now it remains to show that \( W_k \) is dense for all \( k \). Let \( h \in C(M, R) \) and \( \varepsilon > 0 \) we show that there exists \( g \in W_k \) such that \( d(h, g) < \varepsilon \). Since by Remark 2.5, the collection of all differentiable functions is dense in \( C(M, R) \) then there exists a differentiable function \( g_1 \in C(M, R) \) such that \( d(h, g_1) < \frac{\varepsilon}{2} \). Consider a function \( f \in A \subset W_k \). Since \( f \) is continuous and \( M \) is compact then there exists \( \delta > 0 \) such that \( |\delta f(x)| < \frac{\varepsilon}{2} \) for all \( x \in M \). Now, put \( g = g_1 + \delta f \). Since \( g_1 \) is differentiable then \( \overline{\dim_B(\text{graph}(g))} = \overline{\dim_B(\text{graph}\delta f)} = \overline{\dim_B(\text{graph}(f))} = n + 1 \). So, \( g \in A \subset W_k \) and we have

\[ d(h, g) \leq d(h, g_1) + d(g_1, g) \leq \frac{\varepsilon}{2} + \max_{x \in M} |\delta f| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

\[ \square \]

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