Fractal Dimension of Graphs of Typical Continuous Functions on Manifolds

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Abstract. If $M$ is a compact Riemannian manifold and $C(M, R)$ is the set of all real valued continuous functions defined on $M$, then we show that for a typical element $f \in C(M, R)$, $\dim_B(\text{graph}(f))$ is as big as possible and for a typical $f \in C(M, R)$, $\dim_B(\text{graph}(f))$ is as small as possible.

Keywords: Manifold, Fractal, Box dimension.


1. Introduction

A subset $A$ of a topological space $X$ is called to be comeagre, if there is a countable collection $\{W_i\}$ of open and dense subsets of $X$ such that $\bigcap_i W_i \subset A$. Complement of a comeagre subset is called a meagre subset. A meagre subset can be considered as subset of a countable union of nowhere dense subsets and they are negligible in some sense. So, we say that some property holds for typical elements of $X$, if it holds on a comeagre subset. Study of properties of typical elements in $X$ is a classic and interesting problem. One can find many papers dealing with typical elements when $X$ is supposed to be the space $C(W, R)$ of all continuous functions defined on a compact topological space $W$, endowed with the metric topology defined by the metric $d(f, g) = \sup_{x \in W} |f(x) - g(x)|$. A well known theorem due to Banach [1], states that typical elements of $C([0, 1], R)$ are nowhere differentiable, so the image or graph of a typical $f$ in $C([0, 1], R)$ is a
fractal set. Calculating fractal dimensions (including box dimension, Hausdorff dimension, packing dimension, etc) of the image of \( f \) or \( \text{graph}(f) \) is a well known problem and one can find many results in the literature. It is proved in [6] that for a typical \( g \in C([0,1], R) \), \( \text{dim}_H(\text{graph}(g)) = 1 \). It is proved in [3] that if \( W \subset R \) is bounded with only finitely many isolated points and \( X = \{ f \in C(W, R) : f \text{ is uniformly continuous} \} \), then for a typical \( f \in X \), \( \text{dim}_B(\text{graph}(f)) \) is as big as possible and \( \text{dim}_B(\text{graph}(f)) \) is as small as possible. In the previous paper [7] we generalized Banach’s theorem to the set \( C(M, R) \), where \( M \) is a compact Riemannian manifold. Now, we show in the present paper that the main results of [3] about upper and lower box dimensions are also true when \( W \) is replaced by a compact Riemannian manifold \( M \).

2. Preliminaries

In what follows, \( M \) is a compact Riemannian manifold with the Riemannian metric \( d \), and \( C(M, R) \) will denote the collection of all continuous functions defined on \( M \) endowed with the metric \( d \) defined by \( d(f, g) = \max_{x \in M} |f(x) - g(x)| \).

If \( (X, d_1) \) and \( (Y, d_2) \) are metric spaces then we will consider the usual product metric \( d \) on \( X \times Y \) defined by \( d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)} \).

If \( E \) is a bounded subset of \( M \) then the upper box dimension of \( E \) is defined by

\[
\text{dim}_B(E) = \limsup_{\delta \to 0} \frac{N_\delta(E)}{-\log \delta}.
\]

Where, \( N_\delta(E) \) is the minimum number of balls of radius \( \delta \) (or minimum number of sets of diameter at most \( \delta \)) covering \( E \) (The lower box dimension \( \text{dim}_B(E) \) is defined in similar way). Another definition for dimension, which is widely used in fractal geometry is Hausdorff dimension (see [4]).

Now, we mention some facts which we need in the proofs of theorems.

**Remark 2.1.** If \( E \) is a bounded subset of \( R^m \) then \( \text{dim}_B(E \times I^n) = \text{dim}_B(E) + n \). The similar result is true if we replace \( \text{dim}_B \) by \( \text{dim}_B \) or \( \text{dim}_H \).

**Proof.** We give the proof for \( \text{dim}_B(E \times I) = \text{dim}_B(E) + 1 \). The general case comes by induction. If \( \delta > 0 \) then the smallest number of intervals of length \( \delta \) covering \( I \) is equal to \( \lfloor \frac{1}{\delta} \rfloor \) or \( \lceil \frac{1}{\delta} \rceil + 1 \). If \( U_\delta(I_\delta) \) is a bounded subset of \( R^m \) (\( I \)) with diameter \( \delta \), then the diameter of \( U_\delta \times I_\delta \) is equal to \( \sqrt{2} \delta \). So,

\[
N_{\sqrt{2}\delta}(E \times I) \leq (\lfloor \frac{1}{\delta} \rfloor + 1)N_\delta(E)
\]
Then we have
\[
\overline{\dim}_B(E \times I) = \limsup_{\delta \to 0} \frac{\log(N_{\sqrt{\delta}}(E \times I))}{-\log(\sqrt{2\delta})} \\
\leq \limsup_{\delta \to 0} \frac{\log(\frac{1}{2} \cdot 1 + 1)N_{\delta}(E)}{-\log(\sqrt{2\delta})} \\
= 1 + \limsup_{\delta \to 0} \frac{N_{\delta}(E)}{-\log\delta} = 1 + \overline{\dim}_B(E)
\]
Also we know that \( \overline{\dim}_B(E \times I^n) \geq \overline{\dim}_B(E) + n \) (see [4]). So we get the equality. \(\square\)

Remark 2.2. If \( M \) is a compact metric space and \( f : M \to R \) is a locally lipschitz function, then \( f \) is globally lipschitz.

Proof. Since \( f \) is locally lischitz and \( M \) is compact, then there is a finite collection of open cover of balls \( B_i, 1 \leq i \leq m \), and constants \( L_i \) such that
\[
d(f(x), f(y)) \leq L_id(x, y), \quad x, y \in B_i
\]
Since \( M \) is compact then the function \( F : M \times M \to R \), defined by \( F(x, y) = d(f(x), f(y)) \) has a maximum which we denote it by \( N \). Let \( \delta \) be the lebesgue’s number related to the covering \( B_i \) of \( M \), and put \( L = \max\{\frac{N}{\delta}, L_i : i\} \). Then for given \( x, y \in M \), either there is a \( B_i \) such that \( x, y \in B_i \) or \( d(x, y) \geq \delta \). In the first case we have \( d(f(x), f(y)) \leq Ld(x, y) \). In the second case we have
\[
d(f(x), f(y)) \leq N \leq \frac{N}{\delta}d(x, y) \leq Ld(x, y)
\]
\(\square\)

If \( M \) and \( N \) are compact differentiable manifolds and \( f : M \to N \) is continuously differentiable, then \( f \) is a lipschitz function. So, we get the following remark easily.

Remark 2.3. If \( M \) and \( N \) are compact Riemannian manifolds and \( \phi : M \to N \) is a map such that \( \phi \) and its inverse are continuously differentiable, then the map \( \psi : M \times R \to N \times R \) defined by \( \psi(x, y) = (\phi(x), y) \) is bilipschitz.

Remark 2.4. If \( M \) is a compact Riemannian manifold, \( f : M \to R \) is continuously differentiable, \( g : M \to R \) is continuous and \( k = f + g \), then \( \overline{\dim}_B(graph(k)) = \overline{\dim}_B(graph(g)) \). The same result is true for \( \overline{\dim}_B \).

Proof. Consider the map \( \psi : graph(g) \to graph(k) \), defined by \( \psi(x, g(x)) = (x, k(x)) \). We show that \( \psi \) and \( \psi^{-1} \) are Lipschitz functions. We have
\[
d(\psi(x, g(x)), \psi(y, g(y))) = d((x, k(x)), (y, k(y))) = \sqrt{d^2(x, y) + (k(x) - k(y))^2}
\]
Since \( f \) is continuously differentiable, it is locally Lipschitz and by Remark 2.2, it must be Lipschitz. Then, there exist a positive number \( N \) such that 
\[
|f(x) - f(y)| \leq Nd(x, y), \quad x, y \in M.
\]
Thus
\[
(k(x) - k(y))^2 = (f(x) - f(y) + g(x) - g(y))^2 \leq (Nd(x, y) + |g(x) - g(y)|)^2
\]
\[
= N^2d^2(x, y) + 2Nd(x, y)|g(x) - g(y)| + |g(x) - g(y)|^2
\]
\[
\leq N^2d^2(x, y) + N^2d^2(x, y) + |g(x) - g(y)|^2 + |g(x) - g(y)|^2
\]
\[
= 2N^2d^2(x, y) + 2|g(x) - g(y)|^2
\]
Then
\[
d(\psi(x, g(x)), \psi(y, g(y))) \leq \sqrt{d^2(x, y) + 2N^2d^2(x, y) + 2|g(x) - g(y)|^2}
\]
\[
\leq \sqrt{2(N^2 + 1)d^2(x, y) + (g(x) - g(y))^2} = \sqrt{2(N^2 + 1)d((x, g(x)), (y, g(y)))}.
\]
Therefore, \( \psi \) is Lipschitz. In a similar way we can show that \( \psi^{-1} \) is Lipschitz.

**Remark 2.5.** (generalized Stone-Weierstrass Theorem) Suppose \( X \) is a compact Hausdorff space and \( A \) is a subalgebra of \( C(X, \mathbb{R}) \) which contains a nonzero constant function. Then \( A \) is dense in \( C(X, \mathbb{R}) \) if and only if it separates points.

### 3. Results

**Lemma 3.1.** If \( f : M \to \mathbb{R} \) is continuously differentiable and \( \epsilon > 0 \), then there exists \( g \in C(M, \mathbb{R}) \) such that \( d(f, g) < \epsilon \) and \( \overline{\text{dim}}_{B}(\text{graph}(g)) = n + 1 \), \( n = \text{dim}M \).

**Proof.** Let \( N \) be a compact Riemannian manifold. Consider a function \( g_1 \in C(I, \mathbb{R}^+) \) such that \( \overline{\text{dim}}_{B}(\text{graph}(g_1)) = 2 \) and put
\[
g_2 : I^n = I \times I^{n-1} \to \mathbb{R}^+, \quad g_2(t_1, t_2) = g_1(t_1).
\]
Then
\[
\text{graph}(g_2) = \{(t_1, t_2, g_1(t_1)), (t_1, t_2) \in I \times I^{n-1}\} \approx
\]
\[
\{(t_1, g_1(t_1), t_2), (t_1, t_2) \in I \times I^{n-1}\} = \text{graph}(g_1) \times I^{n-1}.
\]
So, by Remark 2.1
\[
\overline{\text{dim}}_{B}(\text{graph}(g_2)) = 2 + n - 1 = n + 1.
\]
Consider a chart \( (U, \phi) \) on \( N \) such that \( I^n \subset \phi(U) \) and put \( W = \phi^{-1}(I^n) \). Now, put \( g_3 = g_2 \circ \phi : W \to \mathbb{R} \). By Remark 2.3, the function \( \psi : W \times \mathbb{R} \to I^n \times \mathbb{R} \), defined by \( \psi(x, y) = (\phi(x), y) \) is bilipschitz. Since \( \psi(\text{graph}(g_3)) = \text{graph}(g_2) \), then \( \overline{\text{dim}}_{B}(\text{graph}(g_3)) = n + 1 \). Extend the function \( g_3 \) to a continuous function \( g_4 : N \to \mathbb{R} \). Since \( \text{graph}(g_3) \subset \text{graph}(g_4) \) then \( \overline{\text{dim}}_{B}(\text{graph}(g_4)) = n + 1 \). Now put \( N = \text{graph}(f) \). We know that \( N \) is a submanifold of \( M \times \mathbb{R} \), which with the induced metric is a riemannian manifold. Given \( \delta > 0 \), the function \( g_5 = \delta g_4 : N \to \mathbb{R} \) is a positive function such that \( \overline{\text{dim}}(\text{graph}(g_5)) = \overline{\text{dim}}(\text{graph}(g_4)) = n + 1 \).
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By compactness condition we can choose \( \delta \) small enough such that for all \( y = (x, f(x)) \in N, g_5(y) < \epsilon \).

Now, consider the function \( g_6 : M \to R \), defined by \( g_6(x) = g_5(x, f(x)) \) and put \( \psi : M \times R \to N \times R, \psi(x, y) = ((x, f(x)), y) \). We have

\[
\psi : \text{graph}(g_6) = \text{graph}(g_5)
\]

By Remark 2.3, \( \psi \) is bilipshitz, so

\[
\overline{\dim}_B(\text{graph}(g_6)) = \overline{\dim}_B(\text{graph}(g_5)) = n + 1
\]

Put \( g : M \to R, g(x) = f(x) + g_6(x) \). Since \( f \) is differentiable, then by Remark 2.4, \( \overline{\dim}_B(\text{graph}(g)) = \overline{\dim}_B(\text{graph}(g_6)) = n + 1 \). Also, we have \( d(f, g) = \max_{x \in M} |g(x) - f(x)| = \max_{x \in M} |g_6(x)| = \max_{x \in M} g_5(x, f(x)) < \epsilon \). \( \square \)

**Theorem 3.2.** Let \( M \) be a compact Riemannian manifold, \( \dim(M) = n \), and \( C(M, R) \) be the set of all continuous functions defined on \( M \). Then for typical members \( f \) in \( C(M, R) \), \( \overline{\dim}_B(\text{graph}(f)) = n \).

**Proof.** Put

\[
A = \{ f \in C(M, R) : \dim_B(\text{graph}(f)) = n \}.
\]

Let \( f \in A \) and consider a positive number \( \epsilon > 0 \) and \( g \in C(M, R) \) such that \( d(f, g) < \epsilon \). If a collection of balls of radius \( \delta \) in \( M \times R \) covers \( \text{graph}(f) \) and \( \epsilon < \delta \), then the same number of balls with radius \( 2\delta \) covers \( \text{graph}(g) \). Since each ball of radius \( 2\delta \) can be covered by \( 4^{n+1} \) balls of radius \( \delta \), then

\[
N_\delta(\text{graph}(g)) \leq 4^{n+1}N_\delta(\text{graph}(f))
\]

If \( \delta < 1 \) then

\[
\frac{\log N_\delta(\text{graph}(g))}{-\log(\delta)} \leq (n + 1) \frac{\log 4}{-\log(\delta)} + \frac{\log N_\delta(\text{graph}(f))}{-\log(\delta)}
\]

Since \( \dim_B(\text{graph}(f)) = n \) and \( \lim_{\delta \to 0} \frac{\log 4}{-\log(\delta)} = 0 \), then for each \( k \in \mathbb{N} \) there exists \( \delta = \delta(f, k) > 0 \) such that

\[
\frac{\log N_\delta(\text{graph}(g))}{-\log(\delta)} \leq (n + 1) \frac{\log 4}{-\log(\delta)} + \frac{\log N_\delta(\text{graph}(f))}{-\log(\delta)} < n + \frac{1}{k}
\]

Put

\[
U_{f,k} = \{ g \in C(M, R) : d(f, g) < \delta(f, k) \}
\]

and

\[
W_k = \bigcup_{(f \in A)} U_{f,k}
\]

\( W_{f,k} \) is an open set in \( C(M, R) \) such that for each \( g \in W_k \),

\[
\dim_B(\text{graph}(g)) < n + \frac{1}{k}
\]

Clearly \( A \subset \bigcap_k W_k \). If \( g \in \bigcap_k W_k \) then \( \dim_B(\text{graph}(g)) \leq n \), and since for all \( g \in C(M, R) \), \( n \leq \dim_B(\text{graph}(g)) \) then \( \dim_B(\text{graph}(g)) = n \). Thus
Now, we show that $W_k$ is dense for all $k$, then the proof will be complete. Given $g \in C(M, R)$ and $\epsilon > 0$. By Remark 2.5, collection of differentiable functions is dense, so there exists a differentiable function $f : M \to R$ such that $d(f, g) < \epsilon$. But for a differentiable function $f$, $\dim_B(\text{graph}(f)) = \overline{\dim}_B(\text{graph}(f)) = n$. So $f \in A \subset W_k$.

Lemma 3.3. If $g \in C(M, R)$ and $\epsilon > 0$, then there exists $k \in C(M, R)$ such that $d(g, k) < \epsilon$ and $\dim_B( \text{graph}(k)) = n + 1$.

Proof. By Remark 2.5, for a given $\delta > 0$ there exists a differentiable function $f \in C(M, R)$ such that $d(f, g) < \delta$. Consider a function $f_1 \in C(M, R)$ such that $\overline{\dim}_B(\text{graph}(f_1)) = n + 1$. Since $M$ is compact, for a given number $\delta_2 > 0$ there is a positive number $\delta_1$ such that $|\delta_1 f_1(x)| < \delta_2$ for all $x \in M$. Now, put $k = f + \delta_1 f_1$. By Remark 2.4, we have

$$\overline{\dim}_B(\text{graph}(k)) = \overline{\dim}_B(\text{graph}(\delta_1 f_1)) = \overline{\dim}_B(\text{graph}(f_1)) = n + 1.$$

If we choose $\delta$ and $\delta_2$ smaller than $\frac{\delta_1}{2}$, then

$$d(g, k) \leq d(g, f) + d(f, k) \leq \delta + \delta_1 |f_1| \leq \delta + \delta_2 < \epsilon.$$

\[\square\]

Theorem 3.4. Let $M$ be a compact Riemannian manifold, $\dim(M) = n$, and $C(M, R)$ be the set of all continuous functions defined on $M$. Then for typical members $f$ in $C(M, R)$, $\overline{\dim}_B(\text{graph}(f)) = n + 1$.

Proof. Clearly for all $f \in C(M, R)$, $\overline{\dim}_B(\text{graph}(f)) \leq n + 1$. Put

$$A = \{f \in C(M, R) : \overline{\dim}_B(\text{graph}(f)) = n + 1\}.$$

Consider $f \in A$, a positive number $\epsilon > 0$ and $g \in C(M, R)$ such that $d(f, g) < \epsilon$. If a collection of balls of radius $\delta$ in $M \times R$ covers $\text{graph}(g)$ and $\epsilon < \delta$, then the same number of balls with radius $2 \delta$ covers $\text{graph}(f)$. Since each ball of radius $2 \delta$ can be covered by $4^{n+1}$ balls of radius $\delta$, then

$$N_{\delta}(\text{graph}(f)) < 4^{n+1} N_{\delta}(\text{graph}(g))$$

So, if $\delta < 1$ then

$$\frac{\log N_{\delta}(\text{graph}(f))}{-\log(\delta)} < (n + 1) \frac{\log 4}{-\log \delta} + \frac{\log N_{\delta}(\text{graph}(g))}{-\log(\delta)}$$

Since $\overline{\dim}_B(\text{graph}(f)) = n + 1$, then for each $k \in N$ there is $\delta(k) = \delta(f, k) > 0$ such that

$$n + 1 - \frac{1}{k} < \frac{\log N_{\delta(k)}(\text{graph}(f))}{-\log(\delta(k))} - (n + 1) \frac{\log 4}{-\log \delta(k)} < \frac{\log N_{\delta(k)}(\text{graph}(g))}{-\log(\delta(k))}$$

Put

$$U_{f, k} = \{g \in C(M, R) : d(f, g) < \delta(f, k)\}$$
and
\[ W_k = \bigcup_{f \in A} U_{f,k} \]

\( W_k \) is an open set in \( C(M,R) \) such that for each \( g \in W_k \),
\[ \overline{\dim}_B(\text{graph}(g)) > n + 1 - \frac{1}{k} \]

Clearly
\[ \bigcap_k W_k = A \]

Now it remains to show that \( W_k \) is dense for all \( k \). Let \( h \in C(M,R) \) and \( \epsilon > 0 \) we show that there exists \( g \in W_k \) such that \( d(h,g) < \epsilon \). Since by Remark 2.5, the collection of all differentiable functions is dense in \( C(M,R) \) then there exists a differentiable function \( g_1 \in C(M,R) \) such that \( d(h,g_1) < \frac{\epsilon}{2} \). Consider a function \( f \in A \subset W_k \). Since \( f \) is continuous and \( M \) is compact then there exists \( \delta > 0 \) such that \( |\delta f(x)| < \frac{\epsilon}{2} \) for all \( x \in M \). Now, put \( g = g_1 + \delta f \). Since \( g_1 \) is differentiable then \( \overline{\dim}_B(\text{graph}(g)) = \overline{\dim}_B(\text{graph}\delta f) = \overline{\dim}_B(\text{graph}(f)) = n + 1 \). So, \( g \in A \subset W_k \) and we have
\[ d(h,g) \leq d(h,g_1) + d(g_1,g) < \frac{\epsilon}{2} + \max_{x \in M} |\delta f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

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