Fractal Dimension of Graphs of Typical Continuous Functions on Manifolds

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Abstract. If $M$ is a compact Riemannian manifold and $C(M, R)$ is the set of all real valued continuous functions defined on $M$, then we show that for a typical element $f \in C(M, R)$, $\dim_B(\text{graph}(f))$ is as big as possible and for a typical $f \in C(M, R)$, $\dim_H(\text{graph}(f))$ is as small as possible.

Keywords: Manifold, Fractal, Box dimension.


1. Introduction

A subset $A$ of a topological space $X$ is called to be comeagre, if there is a countable collection $\{W_i\}$ of open and dense subsets of $X$ such that $\bigcap_i W_i \subset A$. Complement of a comeagre subset is called a meagre subset. A meagre subset can be considered as subset of a countable union of nowhere dense subsets and they are negligible in some sense. So, we say that some property holds for typical elements of $X$, if it holds on a comeagre subset. Study of properties of typical elements in $X$ is a classic and interesting problem. One can find many papers dealing with typical elements when $X$ is supposed to be the space $C(W, R)$ of all continuous functions defined on a compact topological space $W$, endowed with the metric topology defined by the metric $d(f, g) = \sup_{x \in W} |f(x) - g(x)|$. A well known theorem due to Banach [1], states that typical elements of $C([0, 1], R)$ are nowhere differentiable, so the image or graph of a typical $f$ in $C([0, 1], R)$ is a
fractal set. Calculating fractal dimensions (including box dimension, Hausdorff dimension, packing dimension, etc) of the image of \( f \) or \( \text{graph}(f) \) is a well-known problem and one can find many results in the literature. It is proved in [6] that for a typical \( g \in C([0,1],R) \), \( \text{dim}_H(\text{graph}(g)) = 1 \). It is proved in [3] that if \( W \subset R \) is bounded with only finitely many isolated points and \( X = \{ f \in C(W,R) : f \text{ is uniformly continuous} \} \), then for a typical \( f \in X \), \( \text{dim}_B(\text{graph}(f)) \) is as big as possible and \( \text{dim}_B(\text{graph}(f)) \) is as small as possible. In the previous paper [7] we generalized Banach’s theorem to the set \( C(M,R) \), where \( M \) is a compact Riemannian manifold. Now, we show in the present paper that the main results of [3] about upper and lower box dimensions are also true when \( W \) is replaced by a compact Riemannian manifold \( M \).

2. Preliminaries

In what follows, \( M \) is a compact Riemannian manifold with the Riemannian metric \( d \), and \( C(M,R) \) will denote the collection of all continuous functions defined on \( M \) endowed with the metric \( d \) defined by \( d(f,g) = \max_{x \in M} |f(x) - g(x)| \).

If \((X,d_1)\) and \((Y,d_2)\) are metric spaces then we will consider the usual product metric \( d \) on \( X \times Y \) defined by \( d((x_1,y_1),(x_2,y_2)) = \sqrt{d_1^2(x_1,x_2) + d_2^2(y_1,y_2)} \).

If \( E \) is a bounded subset of \( M \) the upper box dimension of \( E \) is defined by

\[
\overline{\text{dim}}_B(E) = \limsup_{\delta \to 0} \frac{N_\delta(E)}{-\log \delta}.
\]

Where, \( N_\delta(E) \) is the minimum number of balls of radius \( \delta \) (or minimum number of sets of diameter at most \( \delta \)) covering \( E \) (The lower box dimension \( \underline{\text{dim}}_B(E) \) is defined in similar way). Another definition for dimension, which is widely used in fractal geometry is Hausdorff dimension (see [4]).

Now, we mention some facts which we need in the proofs of theorems.

Remark 2.1. If \( E \) is a bounded subset of \( R^m \) then \( \overline{\text{dim}}_B(E \times I^n) = \overline{\text{dim}}_B(E) + n \). The similar result is true if we replace \( \overline{\text{dim}}_B \) by \( \underline{\text{dim}}_B \) or \( \text{dim}_H \).

Proof. We give the proof for \( \overline{\text{dim}}_B(E \times I) = \overline{\text{dim}}_B(E) + 1 \). The general case comes by induction. If \( \delta > 0 \) then the smallest number of intervals of length \( \delta \) covering \( I \) is equal to \( \left[ \frac{1}{\delta} \right] \) or \( \left[ \frac{1}{\delta} \right] + 1 \). If \( U_\delta \) \((I_\delta)\) is a bounded subset of \( R^m \) \((I)\) with diameter \( \delta \), then the diameter of \( U_\delta \times I_\delta \) is equal to \( \sqrt{2}\delta \). So,

\[
N_{\sqrt{2}\delta}(E \times I) \leq \left( \left[ \frac{1}{\delta} \right] + 1 \right)N_\delta(E)
\]
Then we have
\[
\dim B(E \times I) = \limsup_{\delta \to 0} \frac{\log(N_{\sqrt{2}\delta}(E \times I))}{-\log(\sqrt{2}\delta)}
\]
\[
\leq \limsup_{\delta \to 0} \frac{\log\left(\frac{1}{\delta} + 1\right)N_{\delta}(E)}{-\log(\sqrt{2}\delta)}
\]
\[
= 1 + \limsup_{\delta \to 0} \frac{N_{\delta}(E)}{-\log(\delta)} = 1 + \overline{\dim}_B(E)
\]
Also we know that \(\overline{\dim}_B(E \times I^n) \geq \overline{\dim}_B(E) + n\) (see [4]). So we get the equality.

\[\square\]

Remark 2.2. If \(M\) is a compact metric space and \(f : M \to R\) is a locally lipschitz function, then \(f\) is globally lipschitz.

**Proof.** Since \(f\) is locally lipschitz and \(M\) is compact, then there is a finite collection of open cover of balls \(B_i, 1 \leq i \leq m\), and constants \(L_i\) such that
\[
d(f(x), f(y)) \leq L_i d(x, y), \quad x, y \in B_i
\]
Since \(M\) is compact then the function \(F : M \times M \to R\), defined by \(F(x, y) = d(f(x), f(y))\) has a maximum which we denote it by \(N\). Let \(\delta\) be the lebesgue’s number related to the covering \(B_i\) of \(M\), and put \(L = \max\{\frac{N}{\delta}, L_i : i\}\). Then for given \(x, y \in M\), either there is a \(B_i\) such that \(x, y \in B_i\) or \(d(x, y) \geq \delta\). In the first case we have \(d(f(x), f(y)) \leq L d(x, y)\). In the second case we have
\[
d(f(x), f(y)) \leq N \leq \frac{N}{\delta} d(x, y) \leq L d(x, y)
\]

\[\square\]

If \(M\) and \(N\) are compact differentiable manifolds and \(f : M \to N\) is continuously differentiable, then \(f\) is a lipschitz function. So, we get the following remark easily.

**Remark 2.3.** If \(M\) and \(N\) are compact Riemannian manifolds and \(\phi : M \to N\) is a map such that \(\phi\) and its inverse are continuously differentiable, then the map \(\psi : M \times R \to N \times R\) defined by \(\psi(x, y) = (\phi(x), y)\) is bilipschitz.

**Remark 2.4.** If \(M\) is a compact Riemannian manifold, \(f : M \to R\) is continuously differentiable, \(g : M \to R\) is continuous and \(k = f + g\), then \(\overline{\dim}_B(graph(k)) = \overline{\dim}_B(graph(g))\). The same result is true for \(\overline{\dim}_B\).

**Proof.** Consider the map \(\psi : graph(g) \to graph(k)\), defined by \(\psi(x, g(x)) = (x, k(x))\). We show that \(\psi\) and \(\psi^{-1}\) are Lipschitz functions. We have
\[
d(\psi(x, g(x)), \psi(y, g(y))) = d((x, k(x)), (y, k(y))) = \sqrt{d^2(x, y) + (k(x) - k(y))^2}
\]
Since \( f \) is continuously differentiable, it is locally Lipschitz and by Remark 2.2, it must be Lipschitz. Then, there exist a positive number \( N \) such that 
\[
|f(x) - f(y)| \leq N d(x, y), \quad x, y \in M.
\]
Thus 
\[
(k(x) - k(y))^2 = (f(x) - f(y) + g(x) - g(y))^2 \leq (N d(x, y) + |g(x) - g(y)|)^2 \\
= N^2 d^2(x, y) + 2 N d(x, y) |g(x) - g(y)| + |g(x) - g(y)|^2 \\
\leq N^2 d^2(x, y) + N^2 d^2(x, y) + |g(x) - g(y)|^2 + |g(x) - g(y)|^2 \\
= 2 N^2 d^2(x, y) + 2 |g(x) - g(y)|^2
\]
Then 
\[
d(\psi(x, g(x)), \psi(y, g(y))) \leq \sqrt{d^2(x, y) + 2 N^2 d^2(x, y) + 2 |g(x) - g(y)|^2} \\
\leq \sqrt{2 (N^2 + 1 N^2 d^2(x, y) + (g(x) - g(y))^2} = \sqrt{2 (N^2 + 1) d((x, g(x)), (y, g(y)))}.
\]
Therefore, \( \psi \) is Lipschitz. In a similar way we can show that \( \psi^{-1} \) is Lipschitz. 

\[\square\]

Remark 2.5. (generalized Stone-Weierstrass Theorem) Suppose \( X \) is a compact Hausdorff space and \( A \) is a subalgebra of \( C(X, R) \) which contains a nonzero constant function. Then \( A \) is dense in \( C(X, R) \) if and only if it separates points.

3. Results

Lemma 3.1. If \( f : M \to R \) is continuously differentiable and \( \epsilon > 0 \), then there exists \( g \in C(M, R) \) such that \( d(f, g) < \epsilon \) and \( \overline{\dim}_B(\text{graph}(g)) = n + 1 \), \( n = \dim M \).

Proof. Let \( N \) be a compact Riemannian manifold. Consider a function \( g_1 \in C(I, R^+) \) such that \( \overline{\dim}_B(\text{graph}(g_1)) = 2 \) and put 
\[
g_2 : I^n = I \times I^{n-1} \to R^+, \quad g_2(t_1, t_2) = g_1(t_1).
\]
Then 
\[
\text{graph}(g_2) = \{(t_1, t_2), g_1(t_1)) \mid (t_1, t_2) \in I \times I^{n-1} \} \simeq \\
\{(t_1, g_1(t_1), t_2) \mid (t_1, t_2) \in I \times I^{n-1} \} = \text{graph}(g_1) \times I^{n-1}.
\]
So, by Remark 2.1 
\[
\overline{\dim}_B(\text{graph}(g_2)) = 2 + n - 1 = n + 1.
\]
Consider a chart \((U, \phi)\) on \( N \) such that \( I^n \subset \phi(U) \) and put \( W = \phi^{-1}(I^n) \). Now, put \( g_3 = g_2 \circ \phi : W \to R \). By Remark 2.3, the function \( \psi : W \times R \to I^n \times R \), defined by \( \psi(x, y) = (\phi(x), y) \) is bilipschitz. Since \( \psi(\text{graph}(g_3)) = \text{graph}(g_2) \), then \( \overline{\dim}_B(\text{graph}(g_3)) = n + 1 \). Extend the function \( g_3 \) to a continuous function \( g_4 : N \to R \). Since \( \text{graph}(g_3) \subset \text{graph}(g_4) \) then \( \overline{\dim}_B(\text{graph}(g_4)) = n + 1 \). Now put \( N = \text{graph}(f) \). We know that \( N \) is a submanifold of \( M \times R \), which with the induced metric is a Riemannian manifold. Given \( \delta > 0 \), the function \( g_5 = \delta g_4 : N \to R \) is a positive function such that \( \overline{\dim}(\text{graph}(g_5)) = \overline{\dim}(\text{graph}(g_4)) = ...
n + 1. By compactness condition we can choose δ small enough such that for all y = (x, f(x)) ∈ N, g_5(y) < ε.

Now, consider the function g_6 : M → R, defined by g_6(x) = g_5(x, f(x)) and put ψ : M × R → N × R, ψ(x, y) = ((x, f(x)), y). We have

\[ \psi : \text{graph}(g_6) = \text{graph}(g_5) \]

By Remark 2.3, ψ is bilipschitz, so

\[ \overline{\dim}_B(\text{graph}(g_6)) = \overline{\dim}_B(\text{graph}(g_5)) = n + 1 \]

Put g : M → R, g(x) = f(x) + g_6(x). Since f is differentiable, then by Remark 2.4, \( \overline{\dim}_B(\text{graph}(g)) = \overline{\dim}_B(\text{graph}(g_6)) = n + 1 \). Also, we have \( d(f, g) = \max_{x ∈ M} |g(x) - f(x)| = \max_{x ∈ M} |g_6(x)| = \max_{x ∈ M} g_5(x, f(x)) < ε \).

**Theorem 3.2.** Let M be a compact Riemannian manifold, \( \dim(M) = n \), and C(M, R) be the set of all continuous functions defined on M. Then for typical members f in C(M, R), \( \dim_B(\text{graph}(f)) = n \).

**Proof.**

Put

\[ A = \{ f ∈ C(M, R) : \dim_B(\text{graph}(f)) = n \}. \]

Let f ∈ A and consider a positive number ε > 0 and g ∈ C(M, R) such that \( d(f, g) < ε \). If a collection of balls of radius δ in M × R covers \( \text{graph}(f) \) and \( ε < δ \), then the same number of balls with radius 2δ covers \( \text{graph}(g) \). Since each ball of radius 2δ can be covered by \( 4^{n+1} \) balls of radius δ, then

\[ N_δ(\text{graph}(g)) \leq 4^{n+1} N_δ(\text{graph}(f)) \]

If δ < 1 then

\[ \frac{\log N_δ(\text{graph}(g))}{-\log(δ)} \leq (n + 1) \frac{\log 4}{-\log(δ)} + \frac{\log N_δ(\text{graph}(f))}{-\log(δ)} \]

Since \( \dim_B(\text{graph}(f)) = n \) and \( \lim_{δ → 0} \frac{\log 4}{-\log(δ)} = 0 \), then for each k ∈ N there exists \( δ = δ(f, k) > 0 \) such that

\[ \frac{\log N_δ(\text{graph}(g))}{-\log(δ)} \leq (n + 1) \frac{\log 4}{-\log(δ)} + \frac{\log N_δ(\text{graph}(f))}{-\log(δ)} < n + \frac{1}{k} \]

Put

\[ U_{f,k} = \{ g ∈ C(M, R) : d(f, g) < δ(f, k) \} \]

and

\[ W_k = \bigcup_{(f ∈ A)} U_{f,k} \]

\( W_{f,k} \) is an open set in C(M, R) such that for each \( g ∈ W_k \),

\[ \dim_B(\text{graph}(g)) < n + \frac{1}{k} \]

Clearly A ⊂ ∩_k W_k. If \( g ∈ \bigcap_k W_k \) then \( \dim_B(\text{graph}(g)) \leq n \), and since for all \( g ∈ C(M, R) \), \( n ≤ \dim_B(\text{graph}(g)) \) then \( \dim_B(\text{graph}(g)) = n \). Thus
Now, we show that $W_k$ is dense for all $k$, then the proof will be complete. Given $g \in C(M, R)$ and $\epsilon > 0$. By Remark 2.5, collection of differentiable functions is dense, so there exists a differentiable function $f : M \to R$ such that $d(f, g) < \epsilon$. But for a differentiable function $f$, $\text{dim}_B(\text{graph}(f)) = \overline{\text{dim}}_B(\text{graph}(f)) = n$. So $f \in A \subset W_k$. \hfill \Box

**Lemma 3.3.** If $g \in C(M, R)$ and $\epsilon > 0$, then there exists $k \in C(M, R)$ such that $d(g, k) < \epsilon$ and $\text{dim}_B(\text{graph}(k)) = n + 1$.

**Proof.** By Remark 2.5, for a given $\delta > 0$ there exists a differentiable function $f \in C(M, R)$ such that $d(f, g) < \delta$. Consider a function $f_1 \in C(M, R)$ such that $\overline{\text{dim}}_B(\text{graph}(f_1)) = n + 1$. Since $M$ is compact, for a given number $\delta_2 > 0$ there is a positive number $\delta_1$ such that $|\delta_1 f_1(x)| < \delta_2$ for all $x \in M$. Now, put $k = f + \delta_1 f_1$. By Remark 2.4, we have

$$\overline{\text{dim}}_B(\text{graph}(k)) = \overline{\text{dim}}_B(\text{graph}(\delta_1 f_1)) = \overline{\text{dim}}_B(\text{graph}(f_1)) = n + 1.$$ 

If we choose $\delta$ and $\delta_2$ smaller than $\frac{\epsilon}{2}$, then

$$d(g, k) \leq d(g, f) + d(f, k) \leq \delta + \delta_1 ||f_1|| \leq \delta + \delta_2 < \epsilon.$$ 

\hfill \Box

**Theorem 3.4.** Let $M$ be a compact Riemannian manifold, $\text{dim}(M) = n$, and $C(M, R)$ be the set of all continuous functions defined on $M$. Then for typical members $f$ in $C(M, R)$, $\overline{\text{dim}}_B(\text{graph}(f)) = n + 1$.

**Proof.** Clearly for all $f \in C(M, R)$, $\overline{\text{dim}}_B(\text{graph}(f)) \leq n + 1$. Put

$$A = \{ f \in C(M, R) : \overline{\text{dim}}_B(\text{graph}(f)) = n + 1 \}.$$ 

Consider $f \in A$, a positive number $\epsilon > 0$ and $g \in C(M, R)$ such that $d(f, g) < \epsilon$. If a collection of balls of radius $\delta$ in $M \times R$ covers $\text{graph}(g)$ and $\epsilon < \delta$, then the same number of balls with radius $2\delta$ covers $\text{graph}(f)$. Since each ball of radius $2\delta$ can be covered by $4^{n+1}$ balls of radius $\delta$, then

$$N_\delta(\text{graph}(f)) < 4^{n+1}N_\delta(\text{graph}(g))$$

So, if $\delta < 1$ then

$$\frac{\log N_\delta(\text{graph}(f))}{-\log(\delta)} < (n + 1)\frac{\log 4}{-\log(\delta)} + \frac{\log N_\delta(\text{graph}(g))}{-\log(\delta)}$$

Since $\overline{\text{dim}}_B(\text{graph}(f)) = n + 1$, then for each $k \in N$ there is $\delta(k) = \delta(f, k) > 0$ such that

$$n + 1 - \frac{1}{k} < \frac{\log N_{\delta(k)}(\text{graph}(f))}{-\log(\delta(k))} - (n + 1)\frac{\log 4}{-\log(\delta(k))} < \frac{\log N_{\delta(k)}(\text{graph}(g))}{-\log(\delta(k))}$$

Put

$$U_{f,k} = \{ g \in C(M, R) : d(f, g) < \delta(f, k) \}$$
and
\[ W_k = \bigcup_{(f \in A)} U_{f,k} \]

\( W_k \) is an open set in \( C(M, R) \) such that for each \( g \in W_k \),
\[ \dim_B(\text{graph}(g)) > n + 1 - \frac{1}{k} \]

Clearly
\[ \bigcap_k W_k = A \]

Now it remains to show that \( W_k \) is dense for all \( k \). Let \( h \in C(M, R) \) and \( \epsilon > 0 \) we show that there exists \( g \in W_k \) such that \( d(h, g) < \epsilon \). Since by Remark 2.5, the collection of all differentiable functions is dense in \( C(M, R) \) then there exists a differentiable function \( g_1 \in C(M, R) \) such that \( d(h, g_1) < \frac{\epsilon}{2} \). Consider a function \( f \in A \subset W_k \). Since \( f \) is continuous and \( M \) is compact then there exists \( \delta > 0 \) such that \( |\delta f(x)| < \frac{\epsilon}{2} \) for all \( x \in M \). Now, put \( g = g_1 + \delta f \). Since \( g_1 \) is differentiable then \( \dim_B(\text{graph}(g)) = \dim_B(\text{graph}\delta f) = \dim_B(\text{graph}(f)) = n + 1 \). So, \( g \in A \subset W_k \) and we have
\[ d(h, g) \leq d(h, g_1) + d(g_1, g) \leq \frac{\epsilon}{2} + \max_{x \in M} |\delta f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

\[ \square \]

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