

## Fractal Dimension of Graphs of Typical Continuous Functions on Manifolds

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ABSTRACT. If  $M$  is a compact Riemannian manifold and  $C(M, R)$  is the set of all real valued continuous functions defined on  $M$ , then we show that for a typical element  $f \in C(M, R)$ ,  $\overline{\dim}_B(\text{graph}(f))$  is as big as possible and for a typical  $f \in C(M, R)$ ,  $\underline{\dim}_B(\text{graph}(f))$  is as small as possible.

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### 1. INTRODUCTION

A subset  $A$  of a topological space  $X$  is called to be *comeagre*, if there is a countable collection  $\{W_i\}$  of open and dense subsets of  $X$  such that  $\bigcap_i W_i \subset A$ . Complement of a comeagre subset is called a meagre subset. A meagre subset can be considered as subset of a countable union of nowhere dense subsets and they are negligible in some sense. So, we say that some property holds for *typical* elements of  $X$ , if it holds on a comeagre subset. Study of properties of typical elements in  $X$  is a classic and interesting problem. One can find many papers dealing with typical elements when  $X$  is supposed to be the space  $C(W, R)$  of all continuous functions defined on a compact topological space  $W$ , endowed with the metric topology defined by the metric  $d(f, g) = \sup_{x \in W} |f(x) - g(x)|$ . A well known theorem due to Banach [1], states that typical elements of  $C([0, 1], R)$  are nowhere differentiable, so the image or graph of a typical  $f$  in  $C([0, 1], R)$  is a

fractal set. Calculating fractal dimensions (including box dimension, Hausdorff dimension, packing dimension, etc) of the image of  $f$  or  $graph(f)$  is a well known problem and one can find many results in the literature. It is proved in [6] that for a typical  $g \in C([0, 1], R)$ ,  $dim_H(graph(g)) = 1$ . It is proved in [3] that if  $W \subset R$  is bounded with only finitely many isolated points and  $X = \{f \in C(W, R) : f \text{ is uniformly continuous}\}$ , then for a typical  $f \in X$ ,  $\overline{dim}_B(graph(f))$  is as big as possible and  $\underline{dim}_B(graph(f))$  is as small as possible. In the previous paper [7] we generalized Banach's theorem to the set  $C(M, R)$ , where  $M$  is a compact Riemannian manifold. Now, we show in the present paper that the main results of [3] about upper and lower box dimensions are also true when  $W$  is replaced by a compact Riemannian manifold  $M$ .

## 2. PRELIMINARIES

In what follows,  $M$  is a compact Riemannian manifold with the Riemannian metric  $d$ , and  $C(M, R)$  will denote the collection of all continuous functions defined on  $M$  endowed with the metric  $d$  defined by  $d(f, g) = \max_{x \in M} |f(x) - g(x)|$ .

If  $(X, d_1)$  and  $(Y, d_2)$  are metric spaces then we will consider the usual product metric  $d$  on  $X \times Y$  defined by  $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$ .

If  $E$  is a bounded subset of  $M$  then the upper box dimension of  $E$  is defined by

$$\overline{dim}_B(E) = \limsup_{\delta \rightarrow 0} \frac{N_\delta(E)}{-\log \delta}.$$

Where,  $N_\delta(E)$  is the minimum number of balls of radius  $\delta$  (or minimum number of sets of diameter at most  $\delta$ ) covering  $E$  (The lower box dimension  $\underline{dim}_B(E)$  is defined in similar way). Another definition for dimension, which is widely used in fractal geometry is Hausdorff dimension (see [4]).

Now, we mention some facts which we need in the proofs of theorems.

*Remark 2.1.* If  $E$  is a bounded subset of  $R^m$  then  $\overline{dim}_B(E \times I^n) = \overline{dim}_B(E) + n$ . The similar result is true if we replace  $\overline{dim}_B$  by  $\underline{dim}_B$  or  $dim_H$ .

*Proof.* We give the proof for  $\overline{dim}_B(E \times I) = \overline{dim}_B(E) + 1$ . The general case comes by induction. If  $\delta > 0$  then the smallest number of intervals of length  $\delta$  covering  $I$  is equal to  $\lceil \frac{1}{\delta} \rceil$  or  $\lceil \frac{1}{\delta} \rceil + 1$ . If  $U_\delta(I_\delta)$  is a bounded subset of  $R^m$  ( $I$ ) with diameter  $\delta$ , then the diameter of  $U_\delta \times I_\delta$  is equal to  $\sqrt{2}\delta$ . So,

$$N_{\sqrt{2}\delta}(E \times I) \leq (\lceil \frac{1}{\delta} \rceil + 1)N_\delta(E)$$

Then we have

$$\begin{aligned}\overline{\dim}_B(E \times I) &= \limsup_{\delta \rightarrow 0} \frac{\log(N_{\sqrt{2}\delta}(E \times I))}{-\log(\sqrt{2}\delta)} \\ &\leq \limsup_{\delta \rightarrow 0} \frac{\log([\frac{1}{\delta}] + 1)N_\delta(E)}{-\log(\sqrt{2}\delta)} \\ &= 1 + \limsup_{\delta \rightarrow 0} \frac{N_\delta(E)}{-\log\delta} = 1 + \overline{\dim}_B(E)\end{aligned}$$

Also we know that  $\overline{\dim}_B(E \times I^n) \geq \overline{\dim}_B(E) + n$  (see [4]). So we get the equality.  $\square$

*Remark 2.2.* If  $M$  is a compact metric space and  $f : M \rightarrow R$  is a locally lipschitz function, then  $f$  is globally lipschitz.

*Proof.* Since  $f$  is locally lipschitz and  $M$  is compact, then there is a finite collection of open cover of balls  $B_i, 1 \leq i \leq m$ , and constants  $L_i$  such that

$$d(f(x), f(y)) \leq L_i d(x, y), \quad x, y \in B_i$$

Since  $M$  is compact then the function  $F : M \times M \rightarrow R$ , defined by  $F(x, y) = d(f(x), f(y))$  has a maximum which we denote it by  $N$ . Let  $\delta$  be the lebesgue's number related to the covering  $B_i$  of  $M$ , and put  $L = \max\{\frac{N}{\delta}, L_i : i\}$ . Then for given  $x, y \in M$ , either there is a  $B_i$  such that  $x, y \in B_i$  or  $d(x, y) \geq \delta$ . In the first case we have  $d(f(x), f(y)) \leq Ld(x, y)$ . In the second case we have

$$d(f(x), f(y)) \leq N \leq \frac{N}{\delta} d(x, y) \leq Ld(x, y)$$

$\square$

If  $M$  and  $N$  are compact differentiable manifolds and  $f : M \rightarrow N$  is continuously differentiable, then  $f$  is a lipschitz function. So, we get the following remark easily.

*Remark 2.3.* If  $M$  and  $N$  are compact Riemannian manifolds and  $\phi : M \rightarrow N$  is a map such that  $\phi$  and its inverse are continuously differentiable, then the map  $\psi : M \times R \rightarrow N \times R$  defined by  $\psi(x, y) = (\phi(x), y)$  is bilipschitz.

*Remark 2.4.* If  $M$  is a compact Riemannian manifold,  $f : M \rightarrow R$  is continuously differentiable,  $g : M \rightarrow R$  is continuous and  $k = f + g$ , then  $\overline{\dim}_B(\text{graph}(k)) = \overline{\dim}_B(\text{graph}(g))$ . The same result is true for  $\underline{\dim}_B$ .

*Proof.* Consider the map  $\psi : \text{graph}(g) \rightarrow \text{graph}(k)$ , defined by  $\psi(x, g(x)) = (x, k(x))$ . We show that  $\psi$  and  $\psi^{-1}$  are Lipschitz functions. We have

$$d(\psi(x, g(x)), \psi(y, g(y))) = d((x, k(x)), (y, k(y))) = \sqrt{d^2(x, y) + (k(x) - k(y))^2}$$

Since  $f$  is continuously differentiable, it is locally Lipschitz and by Remark 2.2, it must be Lipschitz. Then, there exist a positive number  $N$  such that  $|f(x) - f(y)| \leq Nd(x, y)$ ,  $x, y \in M$ . Thus

$$\begin{aligned} (k(x) - k(y))^2 &= (f(x) - f(y) + g(x) - g(y))^2 \leq (Nd(x, y) + |g(x) - g(y)|)^2 \\ &= N^2d^2(x, y) + 2Nd(x, y)|g(x) - g(y)| + |g(x) - g(y)|^2 \\ &\leq N^2d^2(x, y) + N^2d^2(x, y) + |g(x) - g(y)|^2 + |g(x) - g(y)|^2 \\ &= 2N^2d^2(x, y) + 2|g(x) - g(y)|^2 \end{aligned}$$

Then

$$\begin{aligned} d(\psi(x, g(x)), \psi(y, g(y))) &\leq \sqrt{d^2(x, y) + 2N^2d^2(x, y) + 2|g(x) - g(y)|^2} \\ &\leq \sqrt{2(N^2 + 1)d^2(x, y) + 2|g(x) - g(y)|^2} = \sqrt{2(N^2 + 1)}d((x, g(x)), (y, g(y))). \end{aligned}$$

Therefore,  $\psi$  is Lipschitz. In a similar way we can show that  $\psi^{-1}$  is Lipschitz.  $\square$

*Remark 2.5.* (generalized StoneWeierstrass Theorem) . Suppose  $X$  is a compact Hausdorff space and  $A$  is a subalgebra of  $C(X, \mathbb{R})$  which contains a non-zero constant function. Then  $A$  is dense in  $C(X, \mathbb{R})$  if and only if it separates points.

### 3. RESULTS

**Lemma 3.1.** *If  $f : M \rightarrow \mathbb{R}$  is continuously differentiable and  $\epsilon > 0$ , then there exists  $g \in C(M, \mathbb{R})$  such that  $d(f, g) < \epsilon$  and  $\overline{\dim}_B(\text{graph}(g)) = n + 1$ ,  $n = \dim M$ .*

*Proof.* Let  $N$  be a compact Riemannian manifold. Consider a function  $g_1 \in C(I, \mathbb{R}^+)$  such that  $\overline{\dim}_B(\text{graph}(g_1)) = 2$  and put

$$g_2 : I^n = I \times I^{n-1} \rightarrow \mathbb{R}^+, \quad g_2(t_1, t_2) = g_1(t_1).$$

Then

$$\begin{aligned} \text{graph}(g_2) &= \{(t_1, t_2), g_1(t_1)\}, (t_1, t_2) \in I \times I^{n-1}\} \simeq \\ &\{(t_1, g_1(t_1)), t_2\}, (t_1, t_2) \in I \times I^{n-1}\} = \text{graph}(g_1) \times I^{n-1}. \end{aligned}$$

So, by Remark 2.1

$$\overline{\dim}_B(\text{graph}(g_2)) = 2 + n - 1 = n + 1.$$

Consider a chart  $(U, \phi)$  on  $N$  such that  $I^n \subset \phi(U)$  and put  $W = \phi^{-1}(I^n)$ . Now, put  $g_3 = g_2 \circ \phi : W \rightarrow \mathbb{R}$ . By Remark 2.3, the function  $\psi : W \times \mathbb{R} \rightarrow I^n \times \mathbb{R}$ , defined by  $\psi(x, y) = (\phi(x), y)$  is bilipschitz. Since  $\psi(\text{graph}(g_3)) = \text{graph}(g_2)$ , then  $\overline{\dim}_B(\text{graph}(g_3)) = n + 1$ . Extend the function  $g_3$  to a continuous function  $g_4 : N \rightarrow \mathbb{R}$ . Since  $\text{graph}(g_3) \subset \text{graph}(g_4)$  then  $\overline{\dim}_B(\text{graph}(g_4)) = n + 1$ . Now put  $N = \text{graph}(f)$ . We know that  $N$  is a submanifold of  $M \times \mathbb{R}$ , which with the induced metric is a Riemannian manifold. Given  $\delta > 0$ , the function  $g_5 = \delta g_4 : N \rightarrow \mathbb{R}$  is a positive function such that  $\overline{\dim}(\text{graph}(g_5)) = \overline{\dim}(\text{graph}(g_4)) =$

$n + 1$ . By compactness condition we can choose  $\delta$  small enough such that for all  $y = (x, f(x)) \in N$ ,  $g_5(y) < \epsilon$ .

Now, consider the function  $g_6 : M \rightarrow R$ , defined by  $g_6(x) = g_5(x, f(x))$  and put  $\psi : M \times R \rightarrow N \times R$ ,  $\psi(x, y) = ((x, f(x)), y)$ . We have

$$\psi : \text{graph}(g_6) = \text{graph}(g_5)$$

By Remark 2.3,  $\psi$  is bilipshitz, so

$$\overline{\dim}_B(\text{graph}(g_6)) = \overline{\dim}_B(\text{graph}(g_5)) = n + 1$$

Put  $g : M \rightarrow R$ ,  $g(x) = f(x) + g_6(x)$ . Since  $f$  is differentiable, then by Remark 2.4,  $\overline{\dim}_B(\text{graph}(g)) = \overline{\dim}_B(\text{graph}(g_6)) = n + 1$ . Also, we have  $d(f, g) = \max_{x \in M} |g(x) - f(x)| = \max_{x \in M} |g_6(x)| = \max_{x \in M} g_5(x, f(x)) < \epsilon$ .  $\square$

**Theorem 3.2.** *Let  $M$  be a compact Riemannian manifold,  $\dim(M) = n$ , and  $C(M, R)$  be the set of all continuous functions defined on  $M$ . Then for typical members  $f$  in  $C(M, R)$ ,  $\underline{\dim}_B(\text{graph}(f)) = n$ .*

*Proof.* Put

$$A = \{f \in C(M, R) : \underline{\dim}_B(\text{graph}(f)) = n\}.$$

Let  $f \in A$  and consider a positive number  $\epsilon > 0$  and  $g \in C(M, R)$  such that  $d(f, g) < \epsilon$ . If a collection of balls of radius  $\delta$  in  $M \times R$  covers  $\text{graph}(f)$  and  $\epsilon < \delta$ , then the same number of balls with radius  $2\delta$  covers  $\text{graph}(g)$ . Since each ball of radius  $2\delta$  can be covered by  $4^{n+1}$  balls of radius  $\delta$ , then

$$N_\delta(\text{graph}(g)) \leq 4^{n+1} N_\delta(\text{graph}(f))$$

If  $\delta < 1$  then

$$\frac{\log N_\delta(\text{graph}(g))}{-\log(\delta)} \leq (n+1) \frac{\log 4}{-\log \delta} + \frac{\log N_\delta(\text{graph}(f))}{-\log \delta}$$

Since  $\underline{\dim}_B(\text{graph}(f)) = n$  and  $\lim_{\delta \rightarrow 0} \frac{\log 4}{-\log \delta} = 0$ , then for each  $k \in \mathbb{N}$  there exists  $\delta = \delta(f, k) > 0$  such that

$$\frac{\log N_\delta(\text{graph}(g))}{-\log(\delta)} \leq (n+1) \frac{\log 4}{-\log \delta} + \frac{\log N_\delta(\text{graph}(f))}{-\log \delta} < n + \frac{1}{k}$$

Put

$$U_{f,k} = \{g \in C(M, R) : d(f, g) < \delta(f, k)\}$$

and

$$W_k = \bigcup_{(f \in A)} U_{f,k}$$

$W_{f,k}$  is an open set in  $C(M, R)$  such that for each  $g \in W_k$ ,

$$\underline{\dim}_B(\text{graph}(g)) < n + \frac{1}{k}.$$

Clearly  $A \subset \bigcap_k W_k$ . If  $g \in \bigcap_k W_k$  then  $\underline{\dim}_B(\text{graph}(g)) \leq n$ , and since for all  $g \in C(M, R)$ ,  $n \leq \underline{\dim}_B(\text{graph}(g))$  then  $\underline{\dim}_B(\text{graph}(g)) = n$ . Thus

$\bigcap_k W_k = A$ . Now, we show that  $W_k$  is dense for all  $k$ , then the proof will be complete. Given  $g \in C(M, R)$  and  $\epsilon > 0$ . By Remark 2.5, collection of differentiable functions is dense, so there exists a differentiable function  $f : M \rightarrow R$  such that  $d(f, g) < \epsilon$ . But for a differentiable function  $f$ ,  $\underline{\dim}_B(\text{graph}(f)) = \overline{\dim}_B(\text{graph}(f)) = n$ . So  $f \in A \subset W_k$ .  $\square$

**Lemma 3.3.** *If  $g \in C(M, R)$  and  $\epsilon > 0$ , then there exists  $k \in C(M, R)$  such that  $d(g, k) < \epsilon$  and  $\overline{\dim}_B(\text{graph}(k)) = n + 1$ .*

*Proof.* By Remark 2.5, for a given  $\delta > 0$  there exists a differentiable function  $f \in C(M, R)$  such that  $d(f, g) < \delta$ . Consider a function  $f_1 \in C(M, R)$  such that  $\overline{\dim}_B(\text{graph}(f_1)) = n + 1$ . Since  $M$  is compact, for a given number  $\delta_2 > 0$  there is a positive number  $\delta_1$  such that  $|\delta_1 f_1(x)| < \delta_2$  for all  $x \in M$ . Now, put  $k = f + \delta_1 f_1$ . By Remark 2.4, we have

$$\overline{\dim}_B(\text{graph}(k)) = \overline{\dim}_B(\text{graph}(\delta_1 f_1)) = \overline{\dim}_B(\text{graph}(f_1)) = n + 1.$$

If we choose  $\delta$  and  $\delta_2$  smaller than  $\frac{\epsilon}{2}$ , then

$$d(g, k) \leq d(g, f) + d(f, k) \leq \delta + \delta_1 \|f_1\| \leq \delta + \delta_2 < \epsilon.$$

$\square$

**Theorem 3.4.** *Let  $M$  be a compact Riemannian manifold,  $\dim(M) = n$ , and  $C(M, R)$  be the set of all continuous functions defined on  $M$ . Then for typical members  $f$  in  $C(M, R)$ ,  $\overline{\dim}_B(\text{graph}(f)) = n + 1$ .*

*Proof.* Clearly for all  $f \in C(M, R)$ ,  $\overline{\dim}_B(\text{graph}(f)) \leq n + 1$ . Put

$$A = \{f \in C(M, R) : \overline{\dim}_B(\text{graph}(f)) = n + 1\}.$$

Consider  $f \in A$ , a positive number  $\epsilon > 0$  and  $g \in C(M, R)$  such that  $d(f, g) < \epsilon$ . If a collection of balls of radius  $\delta$  in  $M \times R$  covers  $\text{graph}(g)$  and  $\epsilon < \delta$ , then the same number of balls with radius  $2\delta$  covers  $\text{graph}(f)$ . Since each ball of radius  $2\delta$  can be covered by  $4^{n+1}$  balls of radius  $\delta$ , then

$$N_\delta(\text{graph}(f)) < 4^{n+1} N_\delta(\text{graph}(g))$$

So, if  $\delta < 1$  then

$$\frac{\log N_\delta(\text{graph}(f))}{-\log(\delta)} < (n+1) \frac{\log 4}{-\log \delta} + \frac{\log N_\delta(\text{graph}(g))}{-\log \delta}$$

Since  $\overline{\dim}_B(\text{graph}(f)) = n + 1$ , then for each  $k \in N$  there is  $\delta(k) = \delta(f, k) > 0$  such that

$$n + 1 - \frac{1}{k} < \frac{\log N_{\delta(k)}(\text{graph}(f))}{-\log(\delta(k))} - (n+1) \frac{\log 4}{-\log \delta(k)} < \frac{\log N_{\delta(k)}(\text{graph}(g))}{-\log \delta(k)}$$

Put

$$U_{f,k} = \{g \in C(M, R) : d(f, g) < \delta(f, k)\}$$

and

$$W_k = \bigcup_{(f \in A)} U_{f,k}$$

$W_k$  is an open set in  $C(M, R)$  such that for each  $g \in W_k$ ,

$$\overline{\dim}_B(\text{graph}(g)) > n + 1 - \frac{1}{k}$$

Clearly

$$\bigcap_k W_k = A$$

Now it remains to show that  $W_k$  is dense for all  $k$ . Let  $h \in C(M, R)$  and  $\epsilon > 0$  we show that there exists  $g \in W_k$  such that  $d(h, g) < \epsilon$ . Since by Remark 2.5, the collection of all differentiable functions is dense in  $C(M, R)$  then there exists a differentiable function  $g_1 \in C(M, R)$  such that  $d(h, g_1) < \frac{\epsilon}{2}$ . Consider a function  $f \in A \subset W_k$ . Since  $f$  is continuous and  $M$  is compact then there exists  $\delta > 0$  such that  $|\delta f(x)| < \frac{\epsilon}{2}$  for all  $x \in M$ . Now, put  $g = g_1 + \delta f$ . Since  $g_1$  is differentiable then  $\overline{\dim}_B(\text{graph}(g)) = \overline{\dim}_B(\text{graph}(\delta f)) = \overline{\dim}_B(\text{graph}(f)) = n + 1$ . So,  $g \in A \subset W_k$  and we have

$$d(h, g) \leq d(h, g_1) + d(g_1, g) \leq \frac{\epsilon}{2} + \max_{x \in M} |\delta f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

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