Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

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Abstract. In this paper, we define the almost uniform convergence and the almost everywhere convergence for cone-valued functions with respect to an operator valued measure. We prove the Egoroff theorem for \( \mathcal{P} \)-valued functions and operator valued measure \( \theta : \mathcal{R} \to \mathcal{L}(\mathcal{P}, \mathcal{Q}) \), where \( \mathcal{R} \) is a \( \sigma \)-ring of subsets of \( X \neq \emptyset \), \( (\mathcal{P}, V) \) is a quasi-full locally convex cone and \( (\mathcal{Q}, W) \) is a locally convex complete lattice cone.

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1. Introduction

The theory of locally convex cones as developed in [7] and [9] uses an order theoretical concept or convex quasi-uniform structure to introduce a topological structure on a cone. For recent researches see [1, 2, 3, 4, 8].

A cone is a set \( \mathcal{P} \) endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element \( 0 \in \mathcal{P} \). For the scalar multiplication the usual associative and distributive properties hold, that is \( \alpha(\beta a) = (\alpha\beta)a \),

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\[(\alpha + \beta)a = \alpha a + \beta a,\ \alpha(a + b) = \alpha a + \alpha b,\ 1a = a \text{ and } 0a = 0 \text{ for all } a, b \in \mathcal{P}\]
and \(\alpha, \beta \geq 0\).

An ordered cone \(\mathcal{P}\) carries a reflexive transitive relation \(\leq\) such that \(a \leq b\) implies \(a + c \leq b + c\) and \(\alpha a \leq \alpha b\) for all \(a, b, c \in \mathcal{P}\) and \(\alpha \geq 0\). The extended real numbers \(\mathbb{R} = \mathbb{R} \cup \{+\infty\}\) is a natural example of an ordered cone with the usual order and algebraic operations in \(\mathbb{R}\), in particular \(0 \cdot (+\infty) = 0\).

A subset \(\mathcal{V}\) of the ordered cone \(\mathcal{P}\) is called an abstract neighborhood system, if the following properties hold:

1. \(0 < v\) for all \(v \in \mathcal{V}\);
2. for all \(u, v \in \mathcal{V}\) there is a \(w \in \mathcal{V}\) with \(w \leq u\) and \(w \leq v\);
3. \(u + v \in \mathcal{V}\) and \(\alpha v \in \mathcal{V}\) whenever \(u, v \in \mathcal{V}\) and \(\alpha > 0\).

For every \(a \in \mathcal{P}\) and \(v \in \mathcal{V}\) we define

\[v(a) = \{b \in \mathcal{P} | b \leq a + v\} \quad \text{resp.} \quad (a)v = \{b \in \mathcal{P} | a \leq b + v\},\]
to be a neighborhood of \(a\) in the upper, resp. lower topologies on \(\mathcal{P}\). Their common refinement is called the symmetric topology generated by the neighborhoods \(v^\circ(a) = v(a) \cap (a)v\). If we suppose that all elements of \(\mathcal{P}\) are bounded below, that is for every \(a \in \mathcal{P}\) and \(v \in \mathcal{V}\) we have \(0 \leq a + \lambda v\) for some \(\lambda > 0\), then the pair \((\mathcal{P}, \mathcal{V})\) is called a full locally convex cone. A locally convex cone \((\mathcal{P}, \mathcal{V})\) is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system \(\mathcal{V}\). For example, the extended real number system \(\mathbb{R} = \mathbb{R} \cup \{+\infty\}\) endowed with the usual order and algebraic operations and the neighborhood system \(\mathcal{V} = \{\varepsilon \in \mathbb{R} | \varepsilon > 0\}\) is a full locally convex cone.

A subset \(B\) of the locally convex cone \((\mathcal{P}, \mathcal{V})\) is called bounded below whenever for every \(v \in \mathcal{V}\) there is \(\lambda > 0\), such that \(0 \leq b + \lambda v\) for all \(b \in B\).

For cones \(\mathcal{P}\) and \(\mathcal{Q}\) a mapping \(T : \mathcal{P} \to \mathcal{Q}\) is called a linear operator if \(T(a + b) = T(a) + T(b)\) and \(T(\alpha a) = \alpha T(a)\) hold for all \(a, b \in \mathcal{P}\) and \(\alpha \geq 0\). If both \(\mathcal{P}\) and \(\mathcal{Q}\) are ordered, then \(T\) is called monotone, if \(a \leq b\) implies \(T(a) \leq T(b)\). If both \((\mathcal{P}, \mathcal{V})\) and \((\mathcal{Q}, \mathcal{W})\) are locally convex cones, the operator \(T\) is called (uniformly) continuous if for every \(w \in \mathcal{W}\) one can find \(v \in \mathcal{V}\) such that \(T(a) \leq T(b) + w\) whenever \(a \leq b + v\) for all \(a, b \in \mathcal{P}\).

A linear functional on \(\mathcal{P}\) is a linear operator \(\mu : \mathcal{P} \to \mathbb{R} = \mathbb{R} \cup \{+\infty\}\). The dual cone \(\mathcal{P}^*\) of a locally convex cone \((\mathcal{P}, \mathcal{V})\) consists of all continuous linear functionals on \(\mathcal{P}\) and is the union of all polars \(v^\circ\) of neighborhoods \(v \in \mathcal{V}\), where \(\mu \in v^\circ\) means that \(\mu(a) \leq \mu(b) + 1\), whenever \(a \leq b + v\) for all \(a, b \in \mathcal{P}\).

In addition to the given order \(\leq\) on the locally convex cone \((\mathcal{P}, \mathcal{V})\), the weak preorder \(\preceq\) is defined for \(a, b \in \mathcal{P}\) by

\[a \preceq b \quad \text{if} \quad a \leq \gamma b + \varepsilon v\]

for all \(v \in \mathcal{V}\) and \(\varepsilon > 0\) with some \(1 \leq \gamma \leq 1 + \varepsilon\) (for details, see [9], I.3). It is obviously coarser than the given order, that is \(a \leq b\) implies \(a \preceq b\) for all \(a, b \in \mathcal{P}\).
Given a neighborhood \( v \in \mathcal{V} \) and \( \varepsilon > 0 \), the corresponding upper and lower relative neighborhoods \( v_+(a) \) and \( (a)v_-(a) \) for an element \( a \in \mathcal{P} \) are defined by

\[
v_+(a) = \{ b \in \mathcal{P} | b \leq \gamma a + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon \},
\]

\[
(a)v_-(a) = \{ b \in \mathcal{P} | a \leq \gamma b + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon \}.
\]

Their intersection \( v_+(a) = v_-(a) \cap (a)v_-(a) \) is the corresponding symmetric relative neighborhood. Suppose \( v \in \mathcal{V} \). If we consider the abstract neighborhood system \( \mathcal{V}_v = \{ \alpha v : \alpha > 0 \} \) on \( \mathcal{P} \), then the corresponding upper (lower or symmetric) relative topology on \( \mathcal{P} \) is called upper (lower or symmetric) relative \( v \)-topology.

We shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is a \textit{locally convex} \( \vee \)-\textit{semilattice cone} if its order is antisymmetric and if for any two elements \( a, b \in \mathcal{P} \) their supremum \( a \vee b \) exists in \( \mathcal{P} \) and if

\[
(\forall 1) \quad (a + c) \vee (b + c) = a \vee b + c \text{ holds for all } a, b, c \in \mathcal{P},
\]

\[
(\forall 2) \quad a \leq c + v \text{ and } b \leq c + w \text{ for } a, b, c \in \mathcal{P} \text{ and } v, w \in \mathcal{V} \text{ imply that } a \vee b \leq c + (v + w).
\]

Likewise, \((\mathcal{P}, \mathcal{V})\) is a \textit{locally convex} \( \wedge \)-\textit{semilattice cone} if its order is antisymmetric and if for any two elements \( a, b \in \mathcal{P} \) their infimum \( a \wedge b \) exists in \( \mathcal{P} \) and if

\[
(\wedge 1) \quad (a + c) \wedge (b + c) = a \wedge b + c \text{ holds for all } a, b, c \in \mathcal{P},
\]

\[
(\wedge 2) \quad c \leq a + v \text{ and } c \leq b + w \text{ for } a, b, c \in \mathcal{P} \text{ and } v, w \in \mathcal{V} \text{ imply that } c \leq a \wedge b + (v + w).
\]

If both sets of the above conditions hold, then \((\mathcal{P}, \mathcal{V})\) is called a \textit{locally convex lattice cone} (cf. [9]).

We shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is a locally convex \( \vee^c \)-\textit{semilattice cone} if \( \mathcal{P} \) carries the weak preorder (that is the given order coincides with the weak preorder for the elements and the neighborhoods in \( \mathcal{P} \)), this order is antisymmetric and if

\[
(\forall 1) \quad \text{every non-empty subset } A \subseteq \mathcal{P} \text{ has a supremum } \sup A \in \mathcal{P} \text{ and } \sup(A + b) = \sup A + b \text{ holds for all } b \in \mathcal{P},
\]

\[
(\forall 2) \quad \text{let } \emptyset \neq A \subseteq \mathcal{P}, b \in \mathcal{P} \text{ and } v \in \mathcal{V}. \text{ If } a \leq b + v \text{ for all } a \in A, \text{ then } \sup A \leq b + v.
\]

Likewise, \((\mathcal{P}, \mathcal{V})\) is said to be a locally convex \( \wedge^c \)-\textit{semilattice cone} if \( \mathcal{P} \) carries the weak preorder, this order is antisymmetric and if

\[
(\wedge 1) \quad \text{every bounded below subset } A \subseteq \mathcal{P} \text{ has an infimum } \inf A \in \mathcal{P} \text{ and } \inf(A + b) = \inf A + b \text{ holds for all } b \in \mathcal{P},
\]

\[
(\wedge 2) \quad \text{let } A \subseteq \mathcal{P} \text{ be bounded below, } b \in \mathcal{P} \text{ and } v \in \mathcal{V}. \text{ If } b \leq a + v \text{ for all } a \in A, \text{ then } b \leq \inf A + v.
\]

Combining both of the above notions, we shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is a \textit{locally convex complete lattice cone} if \( \mathcal{P} \) is both a \( \vee^c \)-semilattice cone and a \( \wedge^c \)-semilattice cone.
As a simple example, the locally convex cone \((\mathbb{R}, \mathcal{V})\), where \(\mathbb{R} = \mathbb{R} \cup \{\infty\}\) and \(\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}\), is a locally convex lattice cone and a locally convex complete lattice cone.

Suppose \((\mathcal{P}, \mathcal{V})\) is a locally convex complete lattice cone. A net \((a_i)_{i \in I}\) in \(\mathcal{P}\) is called bounded below if there is \(i_0 \in I\) such that the set \(\{a_i \mid i \geq i_0\}\) is bounded below. We define the superior and the inferior limits of a bounded below net \((a_i)_{i \in I}\) in \(\mathcal{P}\) by
\[
\liminf_{i \in I} a_i = \sup \{\inf_{i \geq i_0} a_k \mid i_0 \in I\} \quad \text{and} \quad \limsup_{i \in I} a_i = \inf \{\sup_{i \geq i_0} a_k \mid i_0 \in I\}.
\]
If \(\liminf_{i \in I} a_i\) and \(\limsup_{i \in I} a_i\) coincide, then we denote their common value by \(\lim_{i \in I} a_i\) and say that the net \((a_i)_{i \in I}\) is order convergent. A series \(\sum_{i=1}^{\infty} a_i\) in \((\mathcal{P}, \mathcal{V})\) is said to be order convergent to \(s \in \mathcal{P}\) if the sequence \(s_n = \sum_{i=1}^{n} a_i\) is order convergent to \(s\).

2. Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

The classical Egoroff theorem states that almost everywhere convergent sequences of measurable functions on a finite measure space converge almost uniformly. In this paper, we prove the Egoroff theorem for operator-valued measures in locally convex cones.

We shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is quasi-full if
\[(QF1)\ a \leq b + v \ \text{for} \ a, b \in \mathcal{P} \ \text{and} \ v \in \mathcal{V} \ \text{if and only if} \ a \leq b + s \ \text{for some} \ s \in \mathcal{P} \ \text{such that} \ s \leq v, \]
\[(QF2)\ a \leq u + v \ \text{for} \ a \in \mathcal{P} \ \text{and} \ u, v \in \mathcal{V} \ \text{if and only if} \ a \leq s + t \ \text{for some} \ s, t \in \mathcal{P} \ \text{such that} \ s \leq u \ \text{and} \ t \leq v.\]

The collection \(\mathcal{R}\) of subsets of a set \(X\) is called a (weak) \(\sigma\)-ring whenever:
\[(R1)\ \emptyset \in \mathcal{R}, \quad (R2)\ \text{If} \ E_1, E_2 \in \mathcal{R}, \ \text{then} \ E_1 \cup E_2 \in \mathcal{R} \ \text{and} \ E_1 \setminus E_2 \in \mathcal{R}, \quad (R3)\ \text{If} \ E_n \in \mathcal{R} \ \text{for} \ n \in \mathbb{N} \ \text{and} \ E_n \subseteq E \ \text{for some} \ E \in \mathcal{R}, \ \text{then} \ \bigcup_{n \in \mathbb{N}} E_n \in \mathcal{R} \ \text{(see [9]).}\]

Any \(\sigma\)-algebra is a \(\sigma\)-ring and a \(\sigma\)-ring \(\mathcal{R}\) is a \(\sigma\)-algebra if and only if \(X \in \mathcal{R}\). However, we can associate with \(\mathcal{R}\) in a canonical way the \(\sigma\)-algebra
\[\mathcal{U}_\mathcal{R} = \{A \subset X : A \cap E \in \mathcal{R} \ \text{for all} \ E \in \mathcal{R}\}.\]

A subset \(A\) of \(X\) is said to be measurable whenever \(A \in \mathcal{U}_\mathcal{R}\).

We consider the symmetric relative topology on \(\mathcal{P}\). The function \(f : X \to \mathcal{P}\) is measurable with respect to the \(\sigma\)-ring \(\mathcal{R}\) if for every \(v \in \mathcal{V}\),
\[(M_1)\ f^{-1}(O) \cap E \in \mathcal{R} \ \text{for every open subset} \ O \ \text{of} \ \mathcal{P} \ \text{and every} \ E \in \mathcal{R}, \quad (M_2)\ f(E) \ \text{is separable in} \ \mathcal{P} \ \text{for every} \ E \in \mathcal{R}.\]

The operator–valued measures in locally convex cones have been defined in [9]. Let \((\mathcal{P}, \mathcal{V})\) be a quasi-full locally convex cone and let \((\mathcal{Q}, \mathcal{W})\) be a locally convex complete lattice cone. Let \(\mathcal{L}(\mathcal{P}, \mathcal{Q})\) denote the cone of all (uniformly)
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continuous linear operators from $\mathcal{P}$ to $\mathcal{Q}$. Recall from Section 3 in Chapter I from [9] that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because $\mathcal{Q}$ carries its weak preorder, this implies monotonicity with respect to the given orders of $\mathcal{P}$ and $\mathcal{Q}$ as well. Let $X$ be a set and $\mathfrak{A}$ a $\sigma$-ring of subsets of $X$. An $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measure $\theta$ on $\mathfrak{A}$ is a set function $E \mapsto \theta_E : \mathfrak{A} \rightarrow \mathcal{L}(\mathcal{P}, \mathcal{Q})$

such that $\theta_0 = 0$ and $\theta(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \theta_{E_n}$

holds whenever the sets $E_n \in \mathfrak{A}$ are disjoint and $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{A}$. Convergence for the series on the right-hand side is meant in the following way: For every $a \in \mathcal{P}$ the series $\sum_{n \in \mathbb{N}} \theta_{E_n}(a)$ is order convergent in $\mathcal{Q}$. We note that the order convergence is implied by convergence in the symmetric relative topology.

Let $(\mathcal{P}, \mathcal{V})$ be a quasi-full locally convex cone and let $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone. Suppose $\theta$ is a fixed $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measure on $\mathfrak{A}$. For a neighborhood $v \in \mathcal{V}$ and a set $E \in \mathfrak{A}$, semivariation of $\theta$ is defined as follows:

$|\theta|(E, v) = \sup \left\{ \sum_{i \in \mathbb{N}} \theta_{E_i}(s_i) : s_i \in \mathcal{P}, s_i \leq v, E_i \in \mathfrak{A} \text{ disjoint subsets of } E \right\}.$

It is proved in Lemma 3.3 chapter II from [9], that if $v \in \mathcal{P}$, then $|\theta|(E, v) = \theta_E(v)$.

**Proposition 2.1.** Let $(\mathcal{P}, \mathcal{V})$ be a quasi-full locally convex cone, $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone and $\theta$ be a fixed $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measure on $\mathfrak{A}$.

(a) If for $E \in \mathfrak{A}$, $\theta_E = 0$, then for every $v \in \mathcal{V}$, $|\theta|(E, v) = 0$,

(b) If for every $v \in \mathcal{V}$, $|\theta|(E, v) = 0$, then $\theta_E(a) = 0$ for every bounded element $a \in \mathcal{P}$.

**Proof.** For (a), let $\theta_E = 0$ and $F_1, \ldots, F_n$, $n \in \mathbb{N}$ be a partition of $E$. Then for $0 \leq s_i \leq v$, $i = 1, \ldots, n$, we have $0 \leq \theta_{F_i}(s_i) \leq \theta_E(s_i) = 0$. Since the order of $\mathcal{Q}$ is antisymmetric, for every $i \in \{1, \ldots, n\}$, we have $\theta_{F_i}(s_i) = 0$. Then $|\theta|(E, v) = 0$.

For (b), let $a \in \mathcal{P}$ and for every $v \in \mathcal{V}$, $|\theta|(E, v) = 0$. Since $a$ is bounded, for $v \in \mathcal{V}$, there is $\lambda > 0$ such that $0 \leq a + \lambda v$ and $a \leq \lambda v$. Now we have $0 \leq \theta_E(a) + |\theta|(E, \lambda v)$ and $\theta_E(a) \leq |\theta|(E, \lambda v)$ by Lemma II.3.4 of [9]. This shows that $0 \leq \theta_E(a)$ and $\theta_E(a) \leq 0$. Since the order of $\mathcal{Q}$ is antisymmetric, we have $\theta_E(a) = 0$.

**Corollary 2.2.** Let $(\mathcal{P}, \mathcal{V})$ be a quasi-full locally convex cone, $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone and $\theta$ be a fixed $\mathcal{L}(\mathcal{P}, \mathcal{Q})$-valued measure on
\(\mathfrak{R}\). If all elements of \(\mathcal{P}\) are bounded, then for \(E \in \mathfrak{R}\), \(\theta_E = 0\) if and only if \(|\theta|(E, v) = 0\) for all \(v \in \mathcal{V}\).

**Definition 2.3.** Let \(\mathfrak{R}\) be a \(\sigma\)-ring of subsets of \(X\). The set \(A \in \mathfrak{R}\) is said to be of positive \(v\)-semivariation of the measure \(\theta\) if \(|\theta|(A, v) > 0\). Also, we say that the set \(A\) has bounded \(v\)-semivariation of the measure \(\theta\), if \(|\theta|(A, v)\) is bounded in \((\mathcal{Q}, \mathcal{W})\).

**Definition 2.4.** Let \(\theta\) be an operator-valued measure on \(X\). We shall say that \(\theta\) is generalized strongly \(v\)-continuous (GS\(_v\)-continuous, for short) if for every set of bounded \(v\)-semivariation \(E \in \mathfrak{R}\) and every monotone sequence of sets \((E_n)_{n \in \mathbb{N}} \subseteq \mathfrak{R}\), \(E_n \subseteq E\), \(n \in \mathbb{N}\) the following holds

\[
\lim_{n \in \mathbb{N}} |\theta|(E_n, v) = |\theta|(\lim_{n \in \mathbb{N}} E_n, v) \quad v \in \mathcal{V},
\]

where the limit in the left hand side of the equality means convergence with respect to the symmetric relative topology of \((\mathcal{Q}, \mathcal{W})\).

**Example 2.5.** Let \(X = \mathbb{N} \cup \{+\infty\}\) and \(\mathcal{P} = \mathcal{Q} = \mathfrak{R}\). We consider on \(\mathfrak{R}\) the abstract neighborhood system \(\mathcal{V} = \{\varepsilon \in \mathfrak{R}: \varepsilon > 0\}\). Then \(\mathcal{L}(\mathcal{P}, \mathcal{Q})\) contains all nonnegative reals and the linear functional \(\bar{0}\) acting as

\[
\bar{0}(x) = \begin{cases} 1 & x = +\infty \\ 0 & \text{else.} \end{cases}
\]

We set \(\mathfrak{R} = \{E \subset X : E\text{ is finite}\}\). Then \(\mathfrak{R}\) is a \(\sigma\)-ring on \(X\). We define the set function \(\theta\) on \(\mathfrak{R}\) as following: for \(x \in X\), \(\theta_0 = 0\), \(\theta_{(n)}(x) = nx\) for \(n \in \mathbb{N}\) and \(\theta_{(+\infty)}(x) = 0(x)\). For \(E = \{a_1, \cdots, a_n\} \subseteq \mathfrak{R}\), \(n \in \mathbb{N}\), we define \(\theta_E(x) = \sum_{i=1}^n \theta_{(n)}(x)\) for \(x \in X\). Then \(\theta\) is clearly an operator-valued measure on \(\mathfrak{R}\).

For \(n \in \mathbb{N}\) and \(\varepsilon > 0\), we have \(|\theta|\{\varepsilon\} = \theta_{(\varepsilon)}(\varepsilon) = n\varepsilon\) and \(|\theta|\{+\infty\}, \varepsilon\) = \(\theta_{(+\infty)}(\varepsilon) = \bar{0}(\varepsilon) = 0\). Therefore each \(E \in \mathfrak{R}\) has finite \(\varepsilon\)-semivariation for all \(\varepsilon > 0\). Let \(E \in \mathfrak{R}\). If \((E_n)_{n \in \mathbb{N}} \subseteq \mathfrak{R}\) is a monotone sequence of subsets of \(E\) such that \(\lim_{n \in \mathbb{N}} E_n = F\), then there is \(n_0 \in \mathbb{N}\) such that \(E_n = F\) for all \(n \geq n_0\). Then \(\theta\) is clearly GS\(_v\)-continuous for each \(\varepsilon > 0\).

**Definition 2.6.** A sequence \((f_n)_{n \in \mathbb{N}}\) of measurable functions is said to be \(\theta\)-almost uniformly convergent to a measurable function \(f\) on \(E \in \mathfrak{R}\) if for every \(\varepsilon > 0\), \(w \in \mathcal{W}\) and \(v \in \mathcal{V}\) there exists a subset \(F = F(\varepsilon, v, w)\) of \(E\) and \(n_0 \in \mathbb{N}\) such that for every \(n > n_0\),

\[
|\theta|(F, v) \in w^*_\varepsilon(0),
\]

for all \(x \in E \setminus F\).

**Theorem 2.7** (Egoroff Theorem). Let \(\mathfrak{R}\) be a \(\sigma\)-ring of subsets of \(X\), \((\mathcal{P}, \mathcal{V})\) be a full locally convex cone and \((\mathcal{Q}, \mathcal{W})\) be a locally convex complete lattice cone. For \(v \in \mathcal{V}\), suppose \(\theta: \mathfrak{R} \rightarrow \mathcal{L}(\mathcal{P}, \mathcal{Q})\) be a GS\(_v\)-continuous operator valued measure, and \(E \in \mathfrak{R}\) has bounded \(v\)-semivariation. If \(f: X \rightarrow \mathcal{P}\) is a measurable function, and \((f_n: X \rightarrow \mathcal{P})_{n \in \mathbb{N}}\) is a sequence of measurable
functions, such that for every $t \in E$, $f_n(t) \to f(t)$ with respect to the symmetric relative $\nu$-topology of $(\mathcal{P}, \mathcal{V})$, then $(f_n)_{n \in \mathbb{N}}$ is $\theta$-almost uniformly convergent to $f$ on $E$, with respect to the symmetric relative $\nu$-topology of $(\mathcal{P}, \mathcal{V})$.

Proof. We identify $\nu \in \mathcal{V}$ with the constant function $x \to \nu$ from $X$ into $\mathcal{P}$.

For $m, n \in \mathbb{N}$, we set

$$B_n^m = \bigcap_{i=n}^\infty \{x \in E : f_i(x) \preceq_\nu f(x) + \frac{1}{m} \nu \text{ and } f(x) \preceq_\nu f_i(x) + \frac{1}{m} \nu\}.$$  

For every $n, m \in \mathbb{N}$ we have $B_n^m \in \mathcal{B}$ by Theorem II.1.6 from [9]. Clearly, $B_n^m \subset B_n^{m+1}$ for all $n, m \in \mathbb{N}$. We claim that $E = \bigcup_{n=1}^\infty B_n^m$. Let $x \in E$ and $m \in \mathbb{N}$. Then $(f_n(x))_{n \in \mathbb{N}}$ is convergent to $f(x)$ with respect to the symmetric relative $\nu$-topology. This shows that for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $f_n(x) \in (\frac{1}{m})^\varepsilon f(x)$ for all $n \geq n_0$. Therefore $f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m})^\varepsilon$ and $f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m})^\varepsilon$ for all $n \geq n_0$ and some $1 \leq \gamma \leq 1 + \varepsilon$. This yields that $f_n(x) \leq \gamma f(x) + (1 + \varepsilon)(\frac{1}{m})^\varepsilon$ and $f_n(x) \leq \gamma f(x) + (1 + \varepsilon)(\frac{1}{m})^\varepsilon$ for all $n \geq n_0$ and some $1 \leq \gamma \leq 1 + \varepsilon$. Now Lemma I.3.1 from [9] shows that $f_n(x) \preceq_\nu f(x) + \frac{1}{m} \nu$ and $f(x) \preceq_\nu f_n(x) + \frac{1}{m} \nu$ for all $n \geq n_0$. Thus $x \in B_n^m$.

Then $(E \setminus B_n^m)_{m \in \mathbb{N}}$ is a decreasing sequence of subsets of $E$, such that $\lim_{m \to \infty} E \setminus B_n^m = 0$. Therefore for every $m \in \mathbb{N}$, $|\theta|(E \setminus B_n^m, v) = 0$ with respect to the symmetric relative topology of $(\mathcal{Q}, \mathcal{W})$ by the assumption. For $\varepsilon > 0$ and $m \in \mathbb{N}$ we choose $n_m$ such that $|\theta|(E \setminus B_n^m, v) \leq \frac{\varepsilon}{2m}$. We set

$$F = \bigcup_{n=1}^\infty E \setminus B_n^{n_m}.$$  

Then we have

$$|\theta|(F, v) = \sum_{m=1}^\infty |\theta|(B_n^{n_m}, v) \leq \sum_{m=1}^\infty \frac{\varepsilon}{2m} w = \varepsilon w.$$  

Also, we have $0 \leq |\theta|(F, v) + \varepsilon w$. Then $|\theta|(F, v) \in w_+^c(0)$.

Now, we show that the convergence on $E \setminus F$ is uniform. Let $\delta > 0$. There is $k \in \mathbb{N}$ such that $\frac{2}{k} + \frac{1}{k^2} \leq \delta$. We have

$$E \setminus F = E \setminus (\bigcup_{m=1}^\infty E \setminus B_n^{n_m}) = \bigcap_{m=1}^\infty B_n^{n_m} \subset B_n^{n_k}.$$  

Now for each $n \geq n_k$ and every $x \in E \setminus F$ we have $f_n(x) \preceq_\nu f(x) + \frac{1}{k} \nu$ and $f(x) \preceq_\nu f_n(x) + \frac{1}{k} \nu$. The definition of $\preceq_\nu$ shows that for $\varepsilon = \frac{1}{k}$ there is $1 \leq \gamma \leq 1 + \frac{1}{k}$, and
1 + \frac{1}{k} such that \( f_n(x) \leq \gamma(f(x) + \frac{1}{k}v) + \frac{1}{k}v \) and \( f(x) \leq \gamma(f_n(x) + \frac{1}{k}v) + \frac{1}{k}v \).

Therefore \( f_n(x) \leq \gamma f(x) + \left(\frac{2}{k} + \frac{1}{k^2}\right)v \leq \gamma f(x) + \delta v \) and \( f(x) \leq \gamma f_n(x) + \left(\frac{2}{k} + \frac{1}{k^2}\right)v \leq \gamma f_n(x) + \delta v \). Since \( 1 \leq \gamma \leq 1 + \frac{1}{k} \leq 1 + \frac{2}{k} + \frac{1}{k^2} \leq 1 + \delta \), we realize that \( (f_n)_{n \in \mathbb{N}} \) is uniformly convergent to \( f \) on \( E \setminus F \), with respect to the symmetric relative topology. \( \square \)

Remark 2.8. If in the assumptions of Theorem 2.7, \( (\mathcal{P}, \mathcal{V}) \) is a quasi-full locally convex cone, then the theorem holds again. In fact every quasi-full locally convex cone can be embedded in a full locally convex cone as elaborated in ([9], I, 6.2).

Definition 2.9. W say that a sequence \( (f_n : X \to \mathcal{P})_{n \in \mathbb{N}} \) of measurable functions is \( \theta \)-almost everywhere convergent (with respect to the symmetric topology of \( (\mathcal{P}, \mathcal{V}) \)) to \( f \), if the set \( \{ x \in X : f_n(x) \not\to f(x) \} \) is contained in a subset \( E \) of \( X \) with \( \theta_E = 0 \).

Definition 2.10. Let \( v \in \mathcal{V} \). We say that the sequence \( (f_n : X \to \mathcal{P})_{n \in \mathbb{N}} \) of measurable functions is \( |\theta|_v \)-almost everywhere convergent (with respect to symmetric topology of \( (\mathcal{P}, \mathcal{V}) \)) to \( f \), if the set \( \{ x \in X : f_n(x) \not\to f(x) \} \) is contained in a subset \( E \) of \( X \) with \( |\theta|(E, v) = 0 \).

Lemma 2.11. Let \( \mathfrak{R} \) be a \( \sigma \)-ring of subsets of \( X \), \( (\mathcal{P}, \mathcal{V}) \) be a full locally convex cone and \( (\mathcal{Q}, \mathcal{W}) \) be a locally convex complete lattice cone. Then

(a) \( \theta \)-almost everywhere convergence implies \( |\theta|_v \)-almost everywhere convergence for each \( v \in \mathcal{V} \).

(b) If all elements of \( (\mathcal{P}, \mathcal{V}) \) are bounded and a sequence \( (f_n : X \to \mathcal{P})_{n \in \mathbb{N}} \) is \( |\theta|_v \)-almost everywhere convergent to \( f \) for each \( v \in \mathcal{V} \), then \( (f_n : X \to \mathcal{P})_{n \in \mathbb{N}} \) is \( \theta \)-almost everywhere convergent to \( f \).

Proof. The assertions are proved by the help of Proposition 2.1. \( \square \)

Theorem 2.12. If in the Egoroff theorem (2.7), \( f_n \rightarrow f \), \( \theta \)-almost everywhere or \( |\theta|_v \)-almost everywhere, then the assertion of theorem holds.

Proof. Suppose \( f_n \rightarrow f \), \( \theta \)-almost everywhere, then there is a subset \( A \) of \( E \), which is contained in some \( B \in \mathfrak{R} \) with \( \theta_B = 0 \). Now \( E \setminus B \in \mathfrak{R} \) and it has bounded \( v \)-semivariation. We apply the theorem 2.7 for \( E \setminus B \) and obtain a subset \( F \) satisfying in definition 2.6. Now clearly \( f_n \) is \( \theta \)-almost uniformly convergent to \( f \) on \( E \setminus (F \cap B) \). A similar argument yields our claim for \( |\theta|_v \)-almost everywhere convergence. \( \square \)

Theorem 2.13. Let the symmetric relative \( w \)-topology of \( (\mathcal{Q}, \mathcal{W}) \) be Hausdorff for each \( w \in \mathcal{W} \) and let \( (f_n : X \to \mathcal{P})_{n \in \mathbb{N}} \) be a sequence of measurable functions which converges to \( f \), \( \theta \)-almost uniformly on \( E \in \mathfrak{R} \). Then \( \{f_n\}_{n \in \mathbb{N}} \), is \( |\theta|_v \)-almost everywhere convergent to \( f \) for each \( v \in \mathcal{V} \).
Proof. For each $n \in \mathbb{N}$, $v \in V$ and $w \in W$ there is $F_n = F_n(v, w) \in \mathfrak{N}$ such that $F_n \subseteq E$ and $\theta(F_n, v) \in w^*_n(0)$ and $(f_n)$ is convergent to $f$ on $E \setminus F_n$. Now, we set $F = \bigcap_{n=1}^{\infty} F_n$. Since $(Q, W)$ is separated, we have $\theta(F, v) = 0$. Clearly, $(f_n(x))_{n \in \mathbb{N}}$ is convergent to $f(x)$ for each $x \in E \setminus F = \bigcup_{n=1}^{\infty} E \setminus F_n$. \hfill \Box

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References