Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

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Abstract. In this paper, we define the almost uniform convergence and the almost everywhere convergence for cone-valued functions with respect to an operator valued measure. We prove the Egoroff theorem for \( P \)-valued functions and operator valued measure \( \theta : \mathfrak{R} \to \mathcal{L}(P, Q) \), where \( \mathfrak{R} \) is a \( \sigma \)-ring of subsets of \( X \neq \emptyset \), \((P, V)\) is a quasi-full locally convex cone and \((Q, W)\) is a locally convex complete lattice cone.

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1. Introduction

The theory of locally convex cones as developed in [7] and [9] uses an order theoretical concept or convex quasi-uniform structure to introduce a topological structure on a cone. For recent researches see [1, 2, 3, 4, 8].

A cone is a set \( \mathcal{P} \) endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element \( 0 \in \mathcal{P} \). For the scalar multiplication the usual associative and distributive properties hold, that is \( \alpha(\beta a) = (\alpha \beta)a \),
(α + β)a = αa + βa, α(a + b) = αa + αb, 1a = a and 0a = 0 for all a, b ∈ P and α, β ≥ 0.

An ordered cone P carries a reflexive transitive relation ≤ such that a ≤ b implies a + c ≤ b + c and αa ≤ αb for all a, b, c ∈ P and α ≥ 0. The extended real numbers \( \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \) is a natural example of an ordered cone with the usual order and algebraic operations in \( \overline{\mathbb{R}} \), in particular 0 · (+∞) = 0.

A subset V of the ordered cone P is called an abstract neighborhood system, if the following properties hold:

1. \( 0 < v \) for all \( v ∈ V \);
2. for all \( u, v ∈ V \), there is a \( w ∈ V \) with \( w ≤ u \) and \( w ≤ v \);
3. \( u + v ∈ V \) and \( αv ∈ V \) whenever \( u, v ∈ V \) and \( α > 0 \).

For every \( a ∈ P \) and \( v ∈ V \) we define

\[ v(a) = \{ b ∈ P | b ≤ a + v \} \text{ resp. } (a)v = \{ b ∈ P | a ≤ b + v \}, \]

to be a neighborhood of \( a \) in the upper, resp. lower topologies on P. Their common refinement is called the symmetric topology generated by the neighborhoods \( v^*(a) = v(a) ∩ (a)v \). If we suppose that all elements of P are bounded below, that is for every \( a ∈ P \) and \( v ∈ V \) we have \( 0 ≤ a + ϵv \) for some \( ϵ > 0 \), then the pair \((P, V)\) is called a full locally convex cone. A locally convex cone \((P, V)\) is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system V. For example, the extended real number system \( \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \) endowed with the usual order and algebraic operations and the neighborhood system \( V = \{ ε ∈ \mathbb{R} | ε > 0 \} \) is a full locally convex cone.

A subset B of the locally convex cone \((P, V)\) is called bounded below whenever for every \( v ∈ V \) there is \( λ > 0 \), such that \( 0 ≤ b + λv \) for all \( b ∈ B \).

If \( P \) and \( Q \) are ordered cones, \( T : P → Q \) is called a linear operator if \( T(a + b) = T(a) + T(b) \) and \( T(αa) = αT(a) \) hold for all \( a, b ∈ P \) and \( α ≥ 0 \). If both \( P \) and \( Q \) are ordered, then \( T \) is called monotone, if \( a ≤ b \) implies \( T(a) ≤ T(b) \). If both \((P, V)\) and \((Q, W)\) are locally convex cones, the operator \( T \) is called uniformly continuous if for every \( w ∈ W \) one can find \( v ∈ V \) such that \( T(a) ≤ T(b) + w \) whenever \( a ≤ b + v \) for \( a, b ∈ P \).

A linear functional on \( P \) is a linear operator \( μ : P → \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \). The dual cone \( П^* \) of a locally convex cone \((P, V)\) consists of all continuous linear functionals on \( P \) and is the union of all polars \( v^c \) of neighborhoods \( v ∈ V \), where \( μ ∈ v^c \) means that \( μ(a) ≤ μ(b) + 1 \), whenever \( a ≤ b + v \) for \( a, b ∈ P \).

In addition to the given order ≤ on the locally convex cone \((P, V)\), the weak preorder \( ≪ \) is defined for \( a, b ∈ P \) by

\[ a ≪ b \text{ if } a ≤ γb + ϵv \]

for all \( v ∈ V \) and \( ϵ > 0 \) with some \( 1 ≤ γ ≤ 1 + ϵ \) (for details, see [9], I.3). It is obviously coarser than the given order, that is \( a ≤ b \) implies \( a ≪ b \) for \( a, b ∈ P \).
Given a neighborhood \( v \in V \) and \( \varepsilon > 0 \), the corresponding upper and lower relative neighborhoods \( v_{\varepsilon}(a) \) and \( (a)v_{\varepsilon} \) for an element \( a \in P \) are defined by

\[
v_{\varepsilon}(a) = \{ b \in P | b \leq \gamma a + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon \},
\]

\[
(a)v_{\varepsilon} = \{ b \in P | a \leq \gamma b + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon \}.
\]

Their intersection \( v_{\varepsilon}^*(a) = v_{\varepsilon}(a) \cap (a)v_{\varepsilon} \) is the corresponding symmetric relative neighborhood. Suppose \( v \in V \). If we consider the abstract neighborhood system \( V_v = \{ \alpha v : \alpha > 0 \} \) on \( P \), then the corresponding upper (lower or symmetric) relative topology on \( P \) is called upper (lower or symmetric) relative \( v \)-topology.

We shall say that a locally convex cone \( (P, V) \) is a locally convex \( \vee \)-semilattice cone if its order is antisymmetric and if for any two elements \( a, b \in P \) their supremum \( a \vee b \) exists in \( P \) and if

(\( \vee 1 \)) \( (a + c) \vee (b + c) = a \vee b + c \) holds for all \( a, b, c \in P \),

(\( \vee 2 \)) \( a \leq c + v \) and \( b \leq c + w \) for \( a, b, c \in P \) and \( v, w \in V \) imply that \( a \vee b \leq c + (v + w) \).

Likewise, \( (P, V) \) is a locally convex \( \wedge \)-semilattice cone if its order is antisymmetric and if for any two elements \( a, b \in P \) their infimum \( a \wedge b \) exists in \( P \) and if

(\( \wedge 1 \)) \( (a + c) \wedge (b + c) = a \wedge b + c \) holds for all \( a, b, c \in P \),

(\( \wedge 2 \)) \( c \leq a + v \) and \( c \leq b + w \) for \( a, b, c \in P \) and \( v, w \in V \) imply that \( c \leq a \wedge b + (v + w) \).

If both sets of the above conditions hold, then \( (P, V) \) is called a locally convex lattice cone (cf. [9]).

We shall say that a locally convex cone \( (P, V) \) is a locally convex \( \vee^c \)-semilattice cone if \( P \) carries the weak preorder (that is the given order coincides with the weak preorder for the elements and the neighborhoods in \( P \)), this order is antisymmetric and if

(\( \vee^c 1 \)) every non-empty subset \( A \subseteq P \) has a supremum \( \sup A \in P \) and \( \sup(A + b) = \sup A + b \) holds for all \( b \in P \),

(\( \vee^c 2 \)) let \( \emptyset \neq A \subseteq P, b \in P \) and \( v \in V \). If \( a \leq b + v \) for all \( a \in A \), then \( \sup A \leq b + v \).

Likewise, \( (P, V) \) is said to be a locally convex \( \wedge^c \)-semilattice cone if \( P \) carries the weak preorder, this order is antisymmetric and if

(\( \wedge^c 1 \)) every bounded below subset \( A \subseteq P \) has an infimum \( \inf A \in P \) and \( \inf(A + b) = \inf A + b \) holds for all \( b \in P \),

(\( \wedge^c 2 \)) let \( A \subseteq P \) be bounded below, \( b \in P \) and \( v \in V \). If \( b \leq a + v \) for all \( a \in A \), then \( b \leq \inf A + v \).

Combining both of the above notions, we shall say that a locally convex cone \( (P, V) \) is a locally convex complete lattice cone if \( P \) is both a \( \vee^c \)-semilattice cone and a \( \wedge^c \)-semilattice cone.
As a simple example, the locally convex cone \((\mathbb{R}, V)\), where \(\mathbb{R} = \mathbb{R} \cup \{\infty\}\) and \(V = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}\), is a locally convex lattice cone and a locally convex complete lattice cone.

Suppose \((\mathcal{P}, \mathcal{V})\) is a locally convex complete lattice cone. A net \((a_i)_{i \in \mathcal{I}}\) in \(\mathcal{P}\) is called bounded below if there is \(i_0 \in \mathcal{I}\) such that the set \(\{a_i \mid i \geq i_0\}\) is bounded below. We define the superior and the inferior limits of a bounded below net \((a_i)_{i \in \mathcal{I}}\) in \(\mathcal{P}\) by

\[
\liminf_{i \in \mathcal{I}} a_i = \sup_{i \in \mathcal{I}} \inf_{k \geq i} a_k \quad \text{and} \quad \limsup_{i \in \mathcal{I}} a_i = \inf_{i \in \mathcal{I}} \sup_{k \geq i} a_k.
\]

If \(\liminf_{i \in \mathcal{I}} a_i\) and \(\limsup_{i \in \mathcal{I}} a_i\) coincide, then we denote their common value by \(\lim_{i \in \mathcal{I}} a_i\) and say that the net \((a_i)_{i \in \mathcal{I}}\) is order convergent. A series \(\sum_{i=1}^{\infty} a_i\) in \((\mathcal{P}, \mathcal{V})\) is said to be order convergent to \(s \in \mathcal{P}\) if the sequence \(s_n = \sum_{i=1}^{n} a_i\) is order convergent to \(s\).

2. Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

The classical Egoroff theorem states that almost everywhere convergent sequences of measurable functions on a finite measure space converge almost uniformly. In this paper, we prove the Egoroff theorem for operator-valued measures in locally convex cones.

We shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is quasi-full if

\((QF1)\) \(a \leq b + v\) for \(a, b \in \mathcal{P}\) and \(v \in \mathcal{V}\) if and only if \(a \leq b + s\) for some \(s \in \mathcal{P}\) such that \(s \leq v\), and

\((QF2)\) \(a \leq u + v\) for \(a \in \mathcal{P}\) and \(u, v \in \mathcal{V}\) if and only if \(a \leq s + t\) for some \(s, t \in \mathcal{P}\) such that \(s \leq u\) and \(t \leq v\).

The collection \(\mathfrak{M}\) of subsets of a set \(X\) is called a (weak) \(\sigma\)-ring whenever:

\((R1)\) \(\emptyset \in \mathfrak{M}\),

\((R2)\) If \(E_1, E_2 \in \mathfrak{M}\), then \(E_1 \cup E_2 \in \mathfrak{M}\) and \(E_1 \setminus E_2 \in \mathfrak{M}\),

\((R3)\) If \(E_n \in \mathfrak{M}\) for \(n \in \mathbb{N}\) and \(E_n \subseteq E\) for some \(E \in \mathfrak{M}\), then \(\bigcup_{n \in \mathbb{N}} E_n \in \mathfrak{M}\) (see [9]).

Any \(\sigma\)-algebra is a \(\sigma\)-ring and a \(\sigma\)-ring \(\mathfrak{M}\) is a \(\sigma\)-algebra if and only if \(X \in \mathfrak{M}\).

However, we can associate with \(\mathfrak{M}\) in a canonical way the \(\sigma\)-algebra

\[
\mathfrak{U}_\mathfrak{M} = \{A \subset X : A \cap E \in \mathfrak{M} \text{ for all } E \in \mathfrak{M}\}.
\]

A subset \(A\) of \(X\) is said to be measurable whenever \(A \in \mathfrak{U}_\mathfrak{M}\).

We consider the symmetric relative topology on \(\mathcal{P}\). The function \(f : X \to \mathcal{P}\) is measurable with respect to the \(\sigma\)-ring \(\mathfrak{M}\) if for every \(v \in \mathcal{V}\),

\[(M_1)\] \(f^{-1}(O) \cap E \in \mathfrak{M}\) for every open subset \(O\) of \(\mathcal{P}\) and every \(E \in \mathfrak{M}\),

\[(M_2)\] \(f(E)\) is separable in \(\mathcal{P}\) for every \(E \in \mathfrak{M}\).

The operator–valued measures in locally convex cones have been defined in [9]. Let \((\mathcal{P}, \mathcal{V})\) be a quasi-full locally convex cone and let \((\mathcal{Q}, \mathcal{W})\) be a locally convex complete lattice cone. Let \(L(\mathcal{P}, \mathcal{Q})\) denote the cone of all (uniformly)
continuous linear operators from \( \mathcal{P} \) to \( \mathcal{Q} \). Recall from Section 3 in Chapter I from [9] that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because \( \mathcal{Q} \) carries its weak preorder, this implies monotonicity with respect to the given orders of \( \mathcal{P} \) and \( \mathcal{Q} \) as well. Let \( X \) be a set and \( \mathfrak{R} \) a \( \sigma \)-ring of subsets of \( X \). An \( \mathcal{L}(\mathcal{P}, \mathcal{Q}) \)-valued measure \( \theta \) on \( \mathfrak{R} \) is a set function

\[
E \rightarrow \theta_E : \mathfrak{R} \rightarrow \mathcal{L}(\mathcal{P}, \mathcal{Q})
\]

such that \( \theta_0 = 0 \) and

\[
\theta(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \theta_{E_n}
\]

holds whenever the sets \( E_n \in \mathfrak{R} \) are disjoint and \( \bigcup_{n=1}^{\infty} E_n \in \mathfrak{R} \). Convergence for the series on the right-hand side is meant in the following way: For every \( a \in \mathcal{P} \) the series \( \sum_{n \in \mathbb{N}} \theta_{E_n}(a) \) is order convergent in \( \mathcal{Q} \). We note that the order convergence is implied by convergence in the symmetric relative topology.

Let \( (\mathcal{P}, \mathcal{V}) \) be a quasi-full locally convex cone and let \( (\mathcal{Q}, \mathcal{W}) \) be a locally convex complete lattice cone. Suppose \( \theta \) is a fixed \( \mathcal{L}(\mathcal{P}, \mathcal{Q}) \)-valued measure on \( \mathfrak{R} \). For a neighborhood \( v \in \mathcal{V} \) and a set \( E \in \mathfrak{R} \), semivariation of \( \theta \) is defined as follows:

\[
|\theta|(E, v) = \sup \left\{ \sum_{i \in \mathbb{N}} \theta_{E_i}(s_i) : s_i \in \mathcal{P}, s_i \leq v, E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}
\]

It is proved in Lemma 3.3 chapter II from [9], that if \( v \in \mathcal{P} \), then \( |\theta|(E, v) = \theta_E(v) \).

**Proposition 2.1.** Let \( (\mathcal{P}, \mathcal{V}) \) be a quasi-full locally convex cone, \( (\mathcal{Q}, \mathcal{W}) \) be a locally convex complete lattice cone and \( \theta \) be a fixed \( \mathcal{L}(\mathcal{P}, \mathcal{Q}) \)-valued measure on \( \mathfrak{R} \).

(a) If for \( E \in \mathfrak{R} \), \( \theta_E = 0 \), then for every \( v \in \mathcal{V} \), \( |\theta|(E, v) = 0 \).

(b) If for every \( v \in \mathcal{V} \), \( |\theta|(E, v) = 0 \), then \( \theta_E(a) = 0 \) for every bounded element \( a \) of \( \mathcal{P} \).

**Proof.** For (a), let \( \theta_E = 0 \) and \( F_1, \ldots, F_n, n \in \mathbb{N} \) be a partition of \( E \). Then for \( 0 \leq s_i \leq v, i = 1, \ldots, n \), we have \( 0 \leq \theta_{F_i}(s_i) \leq \theta_E(s_i) = 0 \). Since the order of \( \mathcal{Q} \) is antisymmetric, for every \( i \in \{1, \ldots, n\} \), we have \( \theta_{F_i}(s_i) = 0 \). Then \( |\theta|(E, v) = 0 \).

For (b), let \( a \in \mathcal{P} \) and for every \( v \in \mathcal{V} \), \( |\theta|(E, v) = 0 \). Since \( a \) is bounded, for \( v \in \mathcal{V} \), there is \( \lambda > 0 \) such that \( 0 \leq a + \lambda v \) and \( a \leq \lambda v \). Now we have \( 0 \leq \theta_E(a) + |\theta|(E, \lambda v) \) and \( \theta_E(a) \leq |\theta|(E, \lambda v) \) by Lemma II.3.4 of [9]. This shows that \( 0 \leq \theta_E(a) \) and \( \theta_E(a) \leq 0 \). Since the order of \( \mathcal{Q} \) is antisymmetric, we have \( \theta_E(a) = 0 \). \( \square \)

**Corollary 2.2.** Let \( (\mathcal{P}, \mathcal{V}) \) be a quasi-full locally convex cone, \( (\mathcal{Q}, \mathcal{W}) \) be a locally convex complete lattice cone and \( \theta \) be a fixed \( \mathcal{L}(\mathcal{P}, \mathcal{Q}) \)-valued measure on
If all elements of $\mathcal{P}$ are bounded, then for $E \in \mathfrak{R}$, $\theta_E = 0$ if and only if $|\theta|(E, v) = 0$ for all $v \in \mathcal{V}$.

**Definition 2.3.** Let $\mathfrak{R}$ be a $\sigma$-ring of subsets of $X$. The set $A \in \mathfrak{R}$ is said to be of positive $v$-semivariation of the measure $\theta$ if $|\theta|(A, v) > 0$. Also, we say that the set $A$ has bounded $v$-semivariation of the measure $\theta$, if $|\theta|(A, v)$ is bounded in $(\mathbb{Q}, \mathcal{W})$.

**Definition 2.4.** Let $\theta$ be an operator-valued measure on $X$. We shall say that $\theta$ is generalized strongly $v$-continuous ($GS_v$-continuous, for short) if for every set of bounded $v$-semivariation $E \in \mathfrak{R}$ and every monotone sequence of sets $(E_n)_{n \in \mathbb{N}} \in \mathfrak{R}$, $E_n \subset E$, $n \in \mathbb{N}$ the following holds:

$$\lim_{n \in \mathbb{N}} |\theta|(E_n, v) = |\theta|(\lim_{n \in \mathbb{N}} E_n, v), \quad v \in \mathcal{V},$$

where the limit in the left hand side of the equality means convergence with respect to the symmetric relative topology of $(\mathbb{Q}, \mathcal{W})$.

**Example 2.5.** Let $X = \mathbb{N} \cup \{+\infty\}$ and $\mathcal{P} = \mathbb{Q} = \mathbb{R}$. We consider on $\mathfrak{R}$ the abstract neighborhood system $\mathcal{V} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$. Then $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ contains all nonnegative reals and the linear functional $\overline{0}$ acting as

$$\overline{0}(x) = \begin{cases} +\infty & \text{if } x = +\infty \\ 0 & \text{else.} \end{cases}$$

We set $\mathfrak{R} = \{E \subset X : E \text{ is finite}\}$. Then $\mathfrak{R}$ is a $\sigma$-ring on $X$. We define the set function $\theta$ on $\mathfrak{R}$ as following: for $x \in X$, $\theta_0 = 0$, $\theta_{(n)}(x) = nx$ for $n \in \mathbb{N}$ and $\theta_{(+\infty)}(x) = 0(x)$. For $E = \{a_1, \cdots, a_n\} \in \mathfrak{R}$, $n \in \mathbb{N}$, we define $\theta_E(x) = \sum_{i=1}^n \theta_{(a_i)}(x)$ for $x \in X$. Then $\theta$ is clearly an operator-valued measure on $\mathfrak{R}$. For $n \in \mathbb{N}$ and $\epsilon > 0$, we have $|\theta|(\{n\}, \epsilon) = \theta_{(n)}(\epsilon) = ne$ and $|\theta|(+\infty, \epsilon) = \theta_{(+\infty)}(\epsilon) = \overline{0}(\epsilon) = 0$. Therefore each $E \in \mathfrak{R}$ has finite $\epsilon$-semivariation for all $\epsilon > 0$. Let $E \in \mathfrak{R}$. If $(E_n)_{n \in \mathbb{N}} \subset \mathfrak{R}$ is a monotone sequence of subsets of $E$ such that $\lim_{n \in \mathbb{N}} E_n = F$, then there is $n_0 \in \mathbb{N}$ such that $E_n = F$ for all $n \geq n_0$. Then $\theta$ is clearly $GS_v$-continuous for each $\epsilon > 0$.

**Definition 2.6.** A sequence $(f_n)_{n \in \mathbb{N}}$ of measurable functions is said to be $\theta$-almost uniformly convergent to a measurable function $f$ on $E \in \mathfrak{R}$ if for every $\epsilon > 0$, $w \in \mathcal{W}$ and $v \in \mathcal{V}$ there exists a subset $F = F(\epsilon, v, w)$ of $E$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$,

$$f_n(x) \in v^a_\epsilon(f(x)) \text{ and } |\theta|(f, v) \in w^a_\epsilon(0),$$

for all $x \in E \setminus F$.

**Theorem 2.7** (Egoroff Theorem). Let $\mathfrak{R}$ be a $\sigma$-ring of subsets of $X$, $(\mathcal{P}, \mathcal{V})$ be a full locally convex cone and $(\mathbb{Q}, \mathcal{W})$ be a locally convex complete lattice cone. For $v \in \mathcal{V}$, suppose $\theta : \mathfrak{R} \to \mathcal{L}(\mathcal{P}, \mathcal{Q})$ is a $GS_v$-continuous operator valued measure, and $E \in \mathfrak{R}$ has bounded $v$-semivariation. If $f : X \to \mathcal{P}$ is a measurable function, and $(f_n : X \to \mathcal{P})_{n \in \mathbb{N}}$ is a sequence of measurable
functions, such that for every \( t \in E \), \( f_n(t) \rightarrow f(t) \) with respect to the symmetric relative \( v \)-topology of \((P, V)\), then \((f_n)_{n \in \mathbb{N}}\) is \( \theta \)-almost uniformly convergent to \( f \) on \( E \), with respect to the symmetric relative \( v \)-topology of \((P, V)\).

**Proof.** We identify \( v \in V \) with the constant function \( x \rightarrow v \) from \( X \) into \( P \). For \( m, n \in \mathbb{N} \), we set

\[
B_n^m = \bigcap_{i=n}^{\infty} \{ x \in E : f_i(x) \preceq_v f(x) + \frac{1}{m} v \text{ and } f(x) \preceq_v f_i(x) + \frac{1}{m} v \}.
\]

For every \( n, m \in \mathbb{N} \) we have \( B_n^m \in \mathfrak{R} \) by Theorem II.1.6 from [9]. Clearly, \( B_n^m \subseteq B_{n+1}^m \) for all \( n, m \in \mathbb{N} \). We claim that \( E = \bigcup_{n=1}^{\infty} B_n^m \). Let \( x \in E \) and \( m \in \mathbb{N} \). Then \((f_n(x))_{n \in \mathbb{N}}\) is convergent to \( f(x) \) with respect to the symmetric relative \( v \)-topology. This shows that for each \( \varepsilon > 0 \) there is \( n_0 \in \mathbb{N} \) such that \( f_n(x) \in (\frac{1}{m} v)^v f(x) \) for all \( n \geq n_0 \). Therefore \( f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m} v) \) and \( f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m} v) \) for all \( n \geq n_0 \) and some 1 \( \leq \gamma \leq 1 + \varepsilon \). This yields that \( f_n(x) \leq \gamma f(x) + (1 + \varepsilon)(\frac{1}{m} v) \) and \( f_n(x) \leq \gamma f(x) + (1 + \varepsilon)(\frac{1}{m} v) \) for all \( n \geq n_0 \) and some 1 \( \leq \gamma \leq 1 + \varepsilon \). Now Lemma I.3.1 from [9] shows that \( f_n(x) \preceq_v f(x) + \frac{1}{m} v \) and \( f(x) \preceq_v f_n(x) + \frac{1}{m} v \) for all \( n \geq n_0 \). Thus \( x \in B_n^m \).

Then \((E \setminus B_n^m)_{n \in \mathbb{N}}\) is a decreasing sequence of subsets of \( E \), such that \( \lim_{m \rightarrow \infty} E \setminus B_n^m \neq \emptyset \). Therefore for every \( m \in \mathbb{N} \), \( |\theta|(E \setminus B_n^m, v) \) is convergent to \( |\theta|(0, v) = 0 \) with respect to the symmetric relative topology of \((Q, W)\) by the assumption. For \( \varepsilon > 0 \) and \( m \in \mathbb{N} \) we choose \( n_m \) such that \( |\theta|(E \setminus B_n^m, v) \leq \frac{\varepsilon}{2m} \). We set

\[
F = \bigcup_{m=1}^{\infty} E \setminus B_n^m.
\]

Then we have

\[
|\theta|(F, v) \leq \sum_{m=1}^{\infty} |\theta|(B_n^m, v) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2m} w = \varepsilon w.
\]

Also, we have \( 0 \leq |\theta|(F, v) + \varepsilon w \). Then \( |\theta|(F, v) \in W^v(0) \).

Now, we show that the convergence on \( E \setminus F \) is uniform. Let \( \delta > 0 \). There is \( k \in \mathbb{N} \) such that \( \frac{2}{k} + \frac{1}{k^2} \leq \delta \). We have

\[
E \setminus F = E \setminus \bigcup_{m=1}^{\infty} E \setminus B_n^m = \bigcap_{m=1}^{\infty} B_n^m = B_{n_k}^k
\]

Now for each \( n \geq n_k \) and every \( x \in E \setminus F \) we have \( f_n(x) \preceq_v f(x) + \frac{1}{k} v \) and \( f(x) \preceq_v f_n(x) + \frac{1}{k} v \). The definition of \( \preceq_v \) shows that for \( \varepsilon = \frac{1}{k} \) there is 1 \( \leq \gamma \leq \frac{2}{k} + \frac{1}{k^2} \leq \delta \).
1 + \frac{1}{k} \text{ such that } f_n(x) \leq \gamma(f(x) + \frac{1}{k}v) + \frac{1}{k}v \text{ and } f(x) \leq \gamma(f_n(x) + \frac{1}{k}v) + \frac{1}{k}v.

Therefore, $f_n(x) \leq \gamma f(x) + (\frac{2}{k} + \frac{1}{k^2})v \leq \gamma f(x) + \delta v$ and $f(x) \leq \gamma f_n(x) + (\frac{\gamma - 1}{k} + \frac{1}{k^2})v \leq \gamma f_n(x) + \delta v$. Since $1 \leq \gamma \leq 1 + \frac{1}{k} \leq 1 + \frac{2}{k} + \frac{1}{k^2} \leq 1 + \delta$, we realize that $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent to $f$ on $E \setminus F$, with respect to the symmetric relative topology.

\[ \square \]

Remark 2.8. If in the assumptions of Theorem 2.7, $(P, V)$ is a quasi-full locally convex cone, then the theorem holds again. In fact, every quasi-full locally convex cone can be embedded in a full locally convex cone as elaborated in ([9], I, 6.2).

Definition 2.9. W say that a sequence $(f_n : X \to P)_{n \in \mathbb{N}}$ of measurable functions is $\theta$-almost everywhere convergent (with respect to the symmetric topology of $(P, V)$) to $f$, if the set \( \{ x \in X : f_n(x) \not\rightarrow f(x) \} \) is contained in a subset $E$ of $X$ with $\theta_E = 0$.

Definition 2.10. Let $v \in V$. We say that the sequence $(f_n : X \to P)_{n \in \mathbb{N}}$ of measurable functions is $|\theta|_v$-almost everywhere convergent (with respect to the symmetric topology of $(P, V)$) to $f$, if the set \( \{ x \in X : f_n(x) \not\rightarrow f(x) \} \) is contained in a subset $E$ of $X$ with $|\theta|(E, v) = 0$.

Lemma 2.11. Let $\mathcal{R}$ be a $\sigma$-ring of subsets of $X$, $(P, V)$ be a full locally convex cone and $(Q, W)$ be a locally convex complete lattice cone. Then
(a) $\theta$-almost everywhere convergence implies $|\theta|_v$-almost everywhere convergence for each $v \in V$.
(b) If all elements of $(P, V)$ are bounded and a sequence $(f_n : X \to P)_{n \in \mathbb{N}}$ is $|\theta|_v$-almost everywhere convergent to $f$ for each $v \in V$, then $(f_n : X \to P)_{n \in \mathbb{N}}$ is $\theta$-almost everywhere convergent to $f$.

Proof. The assertions are proved by the help of Proposition 2.1. \[ \square \]

Theorem 2.12. \( f_n \to f, \theta \)-almost everywhere or $|\theta|_v$-almost everywhere, then the assertion of theorem holds.

Proof. Suppose $f_n \to f, \theta$-almost everywhere, then there is a subset $A$ of $E$, which is contained in some $B \in \mathcal{R}$ with $\theta_B = 0$. Now $E \setminus B \in \mathcal{R}$ and it has bounded r-semivariation. We apply the theorem 2.7 for $E \setminus B$ and obtain a subset $F$ satisfying in definition 2.6. Now clearly $f_n$ is $\theta$-almost uniformly convergent to $f$ on $E \setminus (F \cap B)$. A similar argument yields our claim for $|\theta|_v$-almost everywhere convergence.

\[ \square \]

Theorem 2.13. Let the symmetric relative $w$-topology of $(Q, W)$ be Hausdorff for each $w \in W$ and let $(f_n : X \to P)_{n \in \mathbb{N}}$ be a sequence of measurable functions which converges to $f$, $\theta$-almost uniformly on $E \in \mathcal{R}$. Then $(f_n)_{n \in \mathbb{N}}$, is $|\theta|_v$-almost everywhere convergent to $f$ for each $v \in V$. 
Proof. For each \( n \in \mathbb{N}, v \in V \) and \( w \in W \) there is \( F_n = F_n(v, w) \in \mathcal{R} \) such that \( F_n \subset E \) and \( |\theta(F_n, v)| \in W_+^* \) and \( (f_n) \) is convergent to \( f \) on \( E \setminus F_n \). Now, we set \( F = \bigcap_{n=1}^{\infty} F_n \). Since \((Q, W)\) is separated, we have \( |\theta(F, v)| = 0 \). Clearly, \( (f_n(x))_{n \in \mathbb{N}} \) is convergent to \( f(x) \) for each \( x \in E \setminus F = \bigcup_{n=1}^{\infty} E \setminus F_n \). □

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