Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

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Abstract. In this paper, we define the almost uniform convergence and the almost everywhere convergence for cone-valued functions with respect to an operator valued measure. We prove the Egoroff theorem for P-valued functions and operator valued measure $\theta: \mathcal{R} \to \mathcal{L}(P, Q)$, where $\mathcal{R}$ is a $\sigma$-ring of subsets of $X \neq \emptyset$, $(P, V)$ is a quasi-full locally convex cone and $(Q, W)$ is a locally convex complete lattice cone.

Keywords: Locally convex cones, Egoroff Theorem, Operator valued measure.

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1. INTRODUCTION

The theory of locally convex cones as developed in [7] and [9] uses an order theoretical concept or convex quasi-uniform structure to introduce a topological structure on a cone. For recent researches see [1, 2, 3, 4, 8].

A cone is a set $P$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in P$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $\alpha(0) = 0$.\footnote{Corresponding Author}

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\[(a + \beta)a = \alpha a + \beta a, \quad \alpha(a + b) = \alpha a + \alpha b, \quad 1a = a \text{ and } 0a = 0 \text{ for all } a, b \in \mathcal{P} \text{ and } \alpha, \beta \geq 0.\]

An ordered cone \(\mathcal{P}\) carries a reflexive transitive relation \(\leq\) such that \(a \leq b\) implies \(a + c \leq b + c\) and \(a \alpha a \leq ab\) for all \(a, b, c \in \mathcal{P}\) and \(\alpha \geq 0\). The extended real numbers \(\mathbb{R} = \mathbb{R} \cup \{+\infty\}\) is a natural example of an ordered cone with the usual order and algebraic operations in \(\mathbb{R}\), in particular \(0 \cdot (+\infty) = 0\).

A subset \(V\) of the ordered cone \(\mathcal{P}\) is called an abstract neighborhood system, if the following properties hold:

1. \(0 < v\) for all \(v \in V\);
2. for all \(u, v \in V\) there is a \(w \in V\) with \(w \leq u\) and \(w \leq v\);
3. \(u + v \in V\) and \(\alpha v \in V\) whenever \(u, v \in V\) and \(\alpha > 0\).

For every \(a \in \mathcal{P}\) and \(v \in V\) we define

\[v(a) = \{b \in \mathcal{P} | b \leq a + v\} \quad \text{resp.} \quad (a)v = \{b \in \mathcal{P} | a \leq b + v\},\]

to be a neighborhood of \(a\) in the upper, resp. lower topologies on \(\mathcal{P}\). Their common refinement is called the symmetric topology generated by the neighborhoods \(v^\delta(a) = v(a) \cap (a)v\). If we suppose that all elements of \(\mathcal{P}\) are bounded below, that is for every \(a \in \mathcal{P}\) and \(v \in V\) we have \(0 \leq a + \lambda v\) for some \(\lambda > 0\), then the pair \((\mathcal{P}, V)\) is called a full locally convex cone. A locally convex cone \((\mathcal{P}, V)\) is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system \(V\). For example, the extended real number system \(\mathbb{R} = \mathbb{R} \cup \{+\infty\}\) endowed with the usual order and algebraic operations and the neighborhood system \(V = \{\varepsilon \in \mathbb{R} | \varepsilon > 0\}\) is a full locally convex cone.

A subset \(B\) of the locally convex cone \((\mathcal{P}, V)\) is called bounded below whenever for every \(v \in V\) there is \(\lambda > 0\), such that \(0 \leq b + \lambda v\) for all \(b \in B\).

For cones \(\mathcal{P}\) and \(\mathcal{Q}\) a mapping \(T : \mathcal{P} \to \mathcal{Q}\) is called a linear operator if \(T(a + b) = T(a) + T(b)\) and \(T(\alpha a) = \alpha T(a)\) hold for all \(a, b \in \mathcal{P}\) and \(\alpha \geq 0\). If both \(\mathcal{P}\) and \(\mathcal{Q}\) are ordered, then \(T\) is called monotone, if \(a \leq b\) implies \(T(a) \leq T(b)\). If both \((\mathcal{P}, V)\) and \((\mathcal{Q}, W)\) are locally convex cones, the operator \(T\) is called (uniformly) continuous if for every \(w \in W\) one can find \(v \in V\) such that \(T(a) \leq T(b) + w\) whenever \(a \leq b + v\) for \(a, b \in \mathcal{P}\).

A linear functional on \(\mathcal{P}\) is a linear operator \(\mu : \mathcal{P} \to \mathbb{R} = \mathbb{R} \cup \{+\infty\}\). The dual cone \(\mathcal{P}^\ast\) of a locally convex cone \((\mathcal{P}, V)\) consists of all continuous linear functionals on \(\mathcal{P}\) and is the union of all polars \(v^\circ\) of neighborhoods \(v \in V\), where \(\mu \in v^\circ\) means that \(\mu(a) \leq \mu(b) + 1\), whenever \(a \leq b + v\) for \(a, b \in \mathcal{P}\).

In addition to the given order \(\leq\) on the locally convex cone \((\mathcal{P}, V)\), the weak preorder \(\preceq\) is defined for \(a, b \in \mathcal{P}\) by

\[a \preceq b \quad \text{if} \quad a \leq \gamma b + \varepsilon v\]

for all \(v \in V\) and \(\varepsilon > 0\) with some \(1 \leq \gamma \leq 1 + \varepsilon\) (for details, see [9], I.3). It is obviously coarser than the given order, that is \(a \leq b\) implies \(a \preceq b\) for \(a, b \in \mathcal{P}\).
Given a neighborhood \( v \in \mathcal{V} \) and \( \varepsilon > 0 \), the corresponding upper and lower relative neighborhoods \( v_x(a) \) and \( (a)v_x \) for an element \( a \in \mathcal{P} \) are defined by

\[
v_x(a) = \{ b \in \mathcal{P} | b \leq \gamma a + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon \},
\]

\[
(a)v_x = \{ b \in \mathcal{P} | a \leq \gamma b + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon \}.
\]

Their intersection \( v_x^*(a) = v_x(a) \cap (a)v_x \) is the corresponding symmetric relative neighborhood. Suppose \( v \in \mathcal{V} \). If we consider the abstract neighborhood system \( \mathcal{V}_\alpha = \{ \alpha v : \alpha > 0 \} \) on \( \mathcal{P} \), then the corresponding upper (lower or symmetric) relative topology on \( \mathcal{P} \) is called upper (lower or symmetric) relative \( v \)-topology.

We shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is a locally convex \( \lor \)-semilattice cone if its order is antisymmetric and if for any two elements \( a, b \in \mathcal{P} \) their supremum \( a \lor b \) exists in \( \mathcal{P} \) and if

\[
(\lor 1) \ (a + c) \lor (b + c) = a \lor b + c \text{ holds for all } a, b, c \in \mathcal{P},
\]

\[
(\lor 2) \ a \leq c + v \text{ and } b \leq c + w \text{ for } a, b, c \in \mathcal{P} \text{ and } v, w \in \mathcal{V} \text{ imply that } a \lor b \leq c + (v + w).
\]

Likewise, \((\mathcal{P}, \mathcal{V})\) is a locally convex \( \land \)-semilattice cone if its order is antisymmetric and if for any two elements \( a, b \in \mathcal{P} \) their infimum \( a \land b \) exists in \( \mathcal{P} \) and if

\[
(\land 1) \ (a + c) \land (b + c) = a \land b + c \text{ holds for all } a, b, c \in \mathcal{P},
\]

\[
(\land 2) \ c \leq a + v \text{ and } c \leq b + w \text{ for } a, b, c \in \mathcal{P} \text{ and } v, w \in \mathcal{V} \text{ imply that } c \leq a \land b + (v + w).
\]

If both sets of the above conditions hold, then \((\mathcal{P}, \mathcal{V})\) is called a locally convex lattice cone (cf. [9]).

We shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is a locally convex \( \lor \)-semilattice cone if \( \mathcal{P} \) carries the weak preorder (that is the given order coincides with the weak preorder for the elements and the neighborhoods in \( \mathcal{P} \)), this order is antisymmetric and if

\[
(\lor \lambda 1) \text{ every non-empty subset } A \subseteq \mathcal{P} \text{ has a supremum sup } A \in \mathcal{P} \text{ and sup}(A + b) = \text{sup } A + b \text{ holds for all } b \in \mathcal{P},
\]

\[
(\lor \lambda 2) \text{ let } \emptyset \neq A \subseteq \mathcal{P}, \ b \in \mathcal{P} \text{ and } v \in \mathcal{V}. \text{ If } a \leq b + v \text{ for all } a \in A, \text{ then sup } A \leq b + v.
\]

Likewise, \((\mathcal{P}, \mathcal{V})\) is said to be a locally convex \( \land \)-semilattice cone if \( \mathcal{P} \) carries the weak preorder, this order is antisymmetric and if

\[
(\land \lambda 1) \text{ every bounded below subset } A \subseteq \mathcal{P} \text{ has an infimum inf } A \in \mathcal{P} \text{ and inf}(A + b) = \text{inf } A + b \text{ holds for all } b \in \mathcal{P},
\]

\[
(\land \lambda 2) \text{ let } A \subseteq \mathcal{P} \text{ be bounded below, } b \in \mathcal{P} \text{ and } v \in \mathcal{V}. \text{ If } b \leq a + v \text{ for all } a \in A, \text{ then } b \leq \text{inf } A + v.
\]

Combining both of the above notions, we shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is a locally convex complete lattice cone if \( \mathcal{P} \) is both a \( \lor \)-semilattice cone and a \( \land \)-semilattice cone.
As a simple example, the locally convex cone \((\mathbb{R}, V)\), where \(\mathbb{R} = \mathbb{R} \cup \{\infty\}\) and \(V = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}\), is a locally convex lattice cone and a locally convex complete lattice cone.

Suppose \((P, V)\) is a locally convex complete lattice cone. A net \((a_i)_{i \in I}\) in \(P\) is called bounded below if there is \(i_0 \in I\) such that \(\{a_i \mid i \geq i_0\}\) is bounded below. We define the superior and the inferior limits of a bounded below net \((a_i)_{i \in I}\) in \(P\) by

\[
\liminf_{i \in I} a_i = \sup \{\inf_{k \geq i} a_k \mid i \in I\} \quad \text{and} \quad \limsup_{i \in I} a_i = \inf \{\sup_{k \geq i} a_k \mid i \in I\}.
\]

If \(\liminf_{i \in I} a_i\) and \(\limsup_{i \in I} a_i\) coincide, then we denote their common value by \(\lim_{i \in I} a_i\), and say that the net \((a_i)_{i \in I}\) is order convergent. A series \(\sum_{i=1}^{\infty} a_i\) in \((P, V)\) is said to be order convergent to \(s \in P\) if the sequence \(s_n = \sum_{i=1}^{n} a_i\) is order convergent to \(s\).

2. Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

The classical Egoroff theorem states that almost everywhere convergent sequences of measurable functions on a finite measure space converge almost uniformly. In this paper, we prove the Egoroff theorem for operator-valued measures in locally convex cones.

We shall say that a locally convex cone \((P, V)\) is quasi-full if

(QF1) \(a \leq b + v\) for \(a, b \in P\) and \(v \in V\) if and only if \(a \leq b + s\) for some \(s \in P\) such that \(s \leq v\), and

(QF2) \(a \leq u + v\) for \(a \in P\) and \(u, v \in V\) if and only if \(a \leq s + t\) for some \(s, t \in P\) such that \(s \leq u\) and \(t \leq v\).

The collection \(\mathcal{R}\) of subsets of a set \(X\) is called a (weak) \(\sigma\)-ring whenever:

(R1) \(\emptyset \in \mathcal{R}\),

(R2) If \(E_1, E_2 \in \mathcal{R}\), then \(E_1 \cup E_2 \in \mathcal{R}\) and \(E_1 \setminus E_2 \in \mathcal{R}\),

(R3) If \(E_n \in \mathcal{R}\) for \(n \in \mathbb{N}\) and \(E_n \subseteq E\) for some \(E \in \mathcal{R}\), then \(\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{R}\) (see [9]).

Any \(\sigma\)-algebra is a \(\sigma\)-ring and a \(\sigma\)-ring \(\mathcal{R}\) is a \(\sigma\)-algebra if and only if \(X \in \mathcal{R}\). However, we can associate with \(\mathcal{R}\) in a canonical way the \(\sigma\)-algebra

\[
\mathcal{U}_\mathcal{R} = \{A \subset X : A \cap E \in \mathcal{R} \text{ for all } E \in \mathcal{R}\}.
\]

A subset \(A\) of \(X\) is said to be measurable whenever \(A \in \mathcal{U}_\mathcal{R}\).

We consider the symmetric relative topology on \(P\). The function \(f : X \to P\) is measurable with respect to the \(\sigma\)-ring \(\mathcal{R}\) if for every \(v \in V\),

(M1) \(f^{-1}(O) \cap E \in \mathcal{R}\) for every open subset \(O\) of \(P\) and every \(E \in \mathcal{R}\),

(M2) \(f(E)\) is separable in \(P\) for every \(E \in \mathcal{R}\).

The operator–valued measures in locally convex cones have been defined in [9]. Let \((P, V)\) be a quasi-full locally convex cone and let \((Q, W)\) be a locally convex complete lattice cone. Let \(L(P, Q)\) denote the cone of all (uniformly)
continuous linear operators from \( P \) to \( Q \). Recall from Section 3 in Chapter I from [9] that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because \( Q \) carries its weak preorder, this implies monotonicity with respect to the given orders of \( P \) and \( Q \) as well. Let \( X \) be a set and \( \mathfrak{A} \) a \( \sigma \)-ring of subsets of \( X \). An \( \mathcal{L}(P, Q) \)-valued measure \( \theta \) on \( \mathfrak{A} \) is a set function

\[
E \to \theta_E : \mathfrak{A} \to \mathcal{L}(P, Q)
\]

such that \( \theta_{\emptyset} = 0 \) and

\[
\theta(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \theta_{E_n}
\]

holds whenever the sets \( E_n \in \mathfrak{A} \) are disjoint and \( \bigcup_{n=1}^{\infty} E_n \in \mathfrak{A} \). Convergence for the series on the right-hand side is meant in the following way: For every \( a \in P \) the series \( \sum_{n \in \mathbb{N}} \theta_{E_n}(a) \) is order convergent in \( Q \). We note that the order convergence is implied by convergence in the symmetric relative topology.

Let \( (P, V) \) be a quasi-full locally convex cone and let \( (Q, W) \) be a locally convex complete lattice cone. Suppose \( \theta \) is a fixed \( \mathcal{L}(P, Q) \)-valued measure on \( \mathfrak{A} \). For a neighborhood \( v \in V \) and a set \( E \in \mathfrak{A} \), semivariation of \( \theta \) is defined as follows:

\[
|\theta|(E, v) = \sup \left\{ \sum_{i \in \mathbb{N}} \theta_{E_i}(s_i) : s_i \in P, \ s_i \leq v, \ E_i \in \mathfrak{A} \text{ disjoint subsets of } E \right\}.
\]

It is proved in Lemma 3.3 chapter II from [9], that if \( v \in P \), then \( |\theta|(E, v) = \theta_E(v) \).

**Proposition 2.1.** Let \( (P, V) \) be a quasi-full locally convex cone, \( (Q, W) \) be a locally convex complete lattice cone and \( \theta \) be a fixed \( \mathcal{L}(P, Q) \)-valued measure on \( \mathfrak{A} \).

(a) If for \( E \in \mathfrak{A} \), \( \theta_E = 0 \), then for every \( v \in V \), \( |\theta|(E, v) = 0 \),

(b) If for every \( v \in V \), \( |\theta|(E, v) = 0 \), then \( \theta_E(a) = 0 \) for every bounded element \( a \) of \( P \).

**Proof.** For (a), let \( \theta_E = 0 \) and \( F_1, \ldots, F_n, \ n \in \mathbb{N} \) be a partition of \( E \). Then for \( 0 \leq s_i \leq v, \ i = 1, \ldots, n \), we have \( 0 \leq \theta_{F_i}(s_i) \leq \theta_E(s_i) = 0 \). Since the order of \( Q \) is antisymmetric, for every \( i \in \{1, \ldots, n\} \), we have \( \theta_{F_i}(s_i) = 0 \). Then \( |\theta|(E, v) = 0 \).

For (b), let \( a \in P \) and for every \( v \in V \), \( |\theta|(E, v) = 0 \). Since \( a \) is bounded, for \( v \in V \), there is \( \lambda > 0 \) such that \( 0 \leq a + \lambda v \) and \( a \leq \lambda v \). Now we have

\[
0 \leq \theta_E(a) + |\theta|(E, \lambda v) \quad \text{and} \quad \theta_E(a) \leq |\theta|(E, \lambda v)
\]

by Lemma II.3.4 of [9]. This shows that \( 0 \leq \theta_E(a) \) and \( \theta_E(a) \leq 0 \). Since the order of \( Q \) is antisymmetric, we have \( \theta_E(a) = 0 \). \( \square \)

**Corollary 2.2.** Let \( (P, V) \) be a quasi-full locally convex cone, \( (Q, W) \) be a locally convex complete lattice cone and \( \theta \) be a fixed \( \mathcal{L}(P, Q) \)-valued measure on
If all elements of $\mathcal{P}$ are bounded, then for $E \in \mathcal{R}$, $\theta_E = 0$ if and only if $|\theta|(E, v) = 0$ for all $v \in \mathcal{V}$.

**Definition 2.3.** Let $\mathcal{R}$ be a $\sigma$-ring of subsets of $X$. The set $A \in \mathcal{R}$ is said to be of positive $v$-semivariation of the measure $\theta$ if $|\theta|(A, v) > 0$. Also, we say that the set $A$ has bounded $v$-semivariation of the measure $\theta$, if $|\theta|(A, v)$ is bounded in $(\mathcal{Q}, \mathcal{W})$.

**Definition 2.4.** Let $\theta$ be an operator-valued measure on $X$. We shall say that $\theta$ is *generalized strongly $v$-continuous* ($\text{GS}_v$-continuous, for short) if for every set of bounded $v$-semivariation $E \in \mathcal{R}$ and every monotone sequence of sets $(E_n)_{n \in \mathbb{N}} \in \mathcal{R}$, $E_n \subset E$, $n \in \mathbb{N}$ the following holds

$$\lim_{n \in \mathbb{N}} |\theta|(E_n, v) = |\theta|(\lim_{n \in \mathbb{N}} E_n, v) \quad v \in \mathcal{V},$$

where the limit in the left hand side of the equality means convergence with respect to the symmetric relative topology of $(\mathcal{Q}, \mathcal{W})$.

**Example 2.5.** Let $X = \mathbb{N} \cup \{+\infty\}$ and $\mathcal{P} = \mathcal{Q} = \mathbb{R}$. We consider on $\mathbb{R}$ the abstract neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$. Then $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ contains all nonnegative reals and the linear functional $0$ acting as

$$\tilde{\theta}(x) = \begin{cases} +\infty & x = +\infty \\ 0 & \text{else.} \end{cases}$$

We set $\mathcal{R} = \{E \subset X : E \text{ is finite}\}$. Then $\mathcal{R}$ is a $\sigma$-ring on $X$. We define the set function $\theta$ on $\mathcal{R}$ as following: for $x \in X$, $\theta_0 = 0$, $\theta_{\{n\}}(x) = nx$ for $n \in \mathbb{N}$ and $\theta_{\{+\infty\}}(x) = 0(x)$. For $E = \{a_1, \cdots, a_n\} \in \mathcal{R}$, $n \in \mathbb{N}$, we define $\theta_E(x) = \sum_{i=1}^{n} \theta_{\{a_i\}}(x)$ for $x \in X$. Then $\theta$ is clearly an operator-valued measure on $\mathcal{R}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $|\theta|(\{n\}, \varepsilon) = \theta_{\{n\}}(\varepsilon) = n\varepsilon$ and $|\theta|(\{+\infty\}, \varepsilon) = \theta_{\{+\infty\}}(\varepsilon) = \tilde{\theta}(\varepsilon) = 0$. Therefore each $E \in \mathcal{R}$ has finite $\varepsilon$-semivariation for all $\varepsilon > 0$. Let $E \in \mathcal{R}$. If $(E_n)_{n \in \mathbb{N}} \subset \mathcal{R}$ is a monotone sequence of subsets of $E$ such that $\lim_{n \in \mathbb{N}} E_n = F$, then there is $n_0 \in \mathbb{N}$ such that $E_n = F$ for all $n \geq n_0$. Then $\theta$ is clearly $\text{GS}_v$-continuous for each $\varepsilon > 0$.

**Definition 2.6.** A sequence $(f_n)_{n \in \mathbb{N}}$ of measurable functions is said to be $\theta$-almost uniformly convergent to a measurable function $f$ on $E \in \mathcal{R}$ if for every $\varepsilon > 0$, $w \in \mathcal{W}$ and $v \in \mathcal{V}$ there exists a subset $F = F(\varepsilon, v, w)$ of $E$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$, $f_n(x) \in v^s_\varepsilon(f(x))$ and $|\theta|(F, v) \in w^s_\varepsilon(0)$, for all $x \in E \setminus F$.

**Theorem 2.7** (Egoroff Theorem). Let $\mathcal{R}$ be a $\sigma$-ring of subsets of $X$, $(\mathcal{P}, \mathcal{V})$ be a full locally convex cone and $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone. For $v \in \mathcal{V}$, suppose $\theta : \mathcal{R} \to \mathcal{L}(\mathcal{P}, \mathcal{Q})$ be a $\text{GS}_v$-continuous operator valued measure, and $E \in \mathcal{R}$ has bounded $v$-semivariation. If $f : X \to \mathcal{P}$ is a measurable function, and $(f_n : X \to \mathcal{P})_{n \in \mathbb{N}}$ is a sequence of measurable
functions, such that for every \( t \in E \), \( f_n(t) \to f(t) \) with respect to the symmetric relative \( v \)-topology of \( (P, V) \), then \( (f_n)_{n \in \mathbb{N}} \) is \( \theta \)-almost uniformly convergent to \( f \) on \( E \), with respect to the symmetric relative \( v \)-topology of \( (P, V) \).

Proof. We identify \( v \in V \) with the constant function \( x \to v \) from \( X \) into \( P \). For \( m, n \in \mathbb{N} \), we set

\[
B_n^m = \bigcap_{i=n}^{\infty} \{ x \in E : f_i(x) \preceq_v f(x) + \frac{1}{m}v \text{ and } f(x) \preceq_v f_i(x) + \frac{1}{m}v \}.
\]

For every \( n, m \in \mathbb{N} \) we have \( B_n^m \subseteq R \) by Theorem II.1.6 from [9]. Clearly, \( B_n^m \subseteq B_{n+1}^m \) for all \( n, m \in \mathbb{N} \). We claim that \( E = \bigcup_{n=1}^{\infty} B_n^m \). Let \( x \in E \) and \( m \in \mathbb{N} \). Then \( (f_n(x))_{n \in \mathbb{N}} \) is convergent to \( f(x) \) with respect to the symmetric relative \( v \)-topology. This shows that for each \( \epsilon > 0 \) there is \( n_0 \in \mathbb{N} \) such that \( f_n(x) \in \left( \frac{1}{m}v \right)_s f(x) \) for all \( n \geq n_0 \). Therefore \( f_n(x) \leq \gamma f(x) + \epsilon \left( \frac{1}{m}v \right) \) and \( f_n(x) \leq \gamma f(x) + \epsilon \left( \frac{1}{m}v \right) \) for all \( n \geq n_0 \) and some \( 1 \leq \gamma \leq 1 + \epsilon \). This yields that \( f_n(x) \leq \gamma f(x) + (1 + \epsilon) \frac{1}{m}v \) and \( f_n(x) \leq \gamma f(x) + (1 + \epsilon) \frac{1}{m}v \) for all \( n \geq n_0 \) and some \( 1 \leq \gamma \leq 1 + \epsilon \). Now Lemma I.3.1 from [9] shows that \( f_n(x) \preceq_v f(x) + \frac{1}{m}v \) and \( f(x) \preceq_v f_n(x) + \frac{1}{m}v \) for all \( n \geq n_0 \). Thus \( x \in B_n^m \).

Then \( (E \setminus B_n^m) \) is a decreasing sequence of subsets of \( E \), such that \( \lim_{n \to \infty} E \setminus B_n^m = \emptyset \). Therefore for every \( m \in \mathbb{N} \), \( \theta|(E \setminus B_n^m \setminus \nu) \) is convergent to \( \theta|0 \) with respect to the symmetric relative topology of \( (Q, W) \) by the assumption. For \( \epsilon > 0 \) and \( m \in \mathbb{N} \) we choose \( n_m \) such that \( \theta|(E \setminus B_n^m \setminus \nu) \leq \frac{\epsilon}{2m} \). We set

\[
F = \bigcup_{m=1}^{\infty} E \setminus B_n^m.
\]

Then we have

\[
\theta|(F, \nu) \leq \sum_{m=1}^{\infty} \theta|(B_n^m, \nu) \leq \sum_{m=1}^{\infty} \frac{\epsilon}{2m} = \epsilon \nu.
\]

Also, we have \( 0 \leq \theta|(F, \nu) \leq \epsilon \nu \). Then \( \theta|(F, \nu) \in w_s^*(0) \).

Now, we show that the convergence on \( E \setminus F \) is uniform. Let \( \delta > 0 \). There is \( k \in \mathbb{N} \) such that \( \frac{2}{k} + \frac{1}{k^2} \leq \delta \). We have

\[
E \setminus F = E \setminus \left( \bigcup_{m=1}^{\infty} E \setminus B_n^m \right) = \bigcap_{m=1}^{\infty} B_n^m \subset B_{nk}^k.
\]

Now for each \( n \geq n_k \) and every \( x \in E \setminus F \) we have \( f_n(x) \preceq_v f(x) + \frac{1}{k}v \) and \( f(x) \preceq_v f_n(x) + \frac{1}{k}v \). The definition of \( \preceq_v \) shows that for \( \epsilon = \frac{1}{k} \) there is \( 1 \leq \gamma \leq
$1 + \frac{1}{k}$ such that $f_n(x) \leq \gamma(f(x) + \frac{1}{k}v) + \frac{1}{k}v$ and $f(x) \leq \gamma(f_n(x) + \frac{1}{k}v) + \frac{1}{k}v$.

Therefore $f_n(x) \leq \gamma f(x) + (\frac{2}{k} + \frac{1}{k^2})v \leq \gamma f(x) + \delta v$ and $f(x) \leq \gamma f_n(x) + (\frac{2}{k} + \frac{1}{k^2})v \leq \gamma f_n(x) + \delta v$. Since $1 \leq \gamma \leq 1 + \frac{1}{k} \leq 1 + \frac{2}{k} + \frac{1}{k^2} \leq 1 + \delta$, we realize that $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent to $f$ on $E \setminus F$, with respect to the symmetric relative topology.

\[ \square \]

Remark 2.8. If in the assumptions of Theorem 2.7, $(\mathcal{P}, \mathcal{V})$ is a quasi-full locally convex cone, then the theorem holds again. In fact every quasi-full locally convex cone can be embedded in a full locally convex cone as elaborated in ([9], I, 6.2).

Definition 2.9. W say that a sequence $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$ of measurable functions is $\theta$-almost everywhere convergent (with respect to the symmetric topology of $(\mathcal{P}, \mathcal{V})$) to $f$, if the set $\{x \in X : f_n(x) \not\rightarrow f(x)\}$ is contained in a subset $E$ of $X$ with $\theta_E = 0$.

Definition 2.10. Let $v \in \mathcal{V}$. We say that the sequence $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$ of measurable functions is $|\theta|_v$-almost everywhere convergent (with respect to symmetric topology of $(\mathcal{P}, \mathcal{V})$) to $f$, if the set $\{x \in X : f_n(x) \not\rightarrow f(x)\}$ is contained in a subset $E$ of $X$ with $|\theta|(E, v) = 0$.

Lemma 2.11. Let $\mathcal{R}$ be a $\sigma$-ring of subsets of $X$, $(\mathcal{P}, \mathcal{V})$ be a full locally convex cone and $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone. Then
(a) $\theta$-almost everywhere convergence implies $|\theta|_v$-almost everywhere convergence for each $v \in \mathcal{V}$.
(b) If all elements of $(\mathcal{P}, \mathcal{V})$ are bounded and a sequence $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$ is $|\theta|_v$-almost everywhere convergent to $f$ for each $v \in \mathcal{V}$, then $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$ is $\theta$-almost everywhere convergent to $f$.

Proof. The assertions are proved by the help of Proposition 2.1. \[ \square \]

Theorem 2.12. If in the Egoroff theorem (2.7), $f_n \rightarrow f$, $\theta$-almost everywhere or $|\theta|_v$-almost everywhere, then the assertion of theorem holds.

Proof. Suppose $f_n \rightarrow f$, $\theta$-almost everywhere, then there is a subset $A$ of $E$, which is contained in some $B \in \mathcal{R}$ with $\theta_B = 0$. Now $E \setminus B \in \mathcal{R}$ and it has bounded $v$-semivariation. We apply the theorem 2.7 for $E \setminus B$ and obtain a subset $F$ satisfying in definition 2.6. Now clearly $f_n$ is $\theta$-almost uniformly convergent to $f$ on $E \setminus (F \cap B)$. A similar argument yields our claim for $|\theta|_v$-almost everywhere convergence. \[ \square \]

Theorem 2.13. Let the symmetric relative $w$-topology of $(\mathcal{Q}, \mathcal{W})$ be Hausdorff for each $w \in \mathcal{W}$ and let $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$ be a sequence of measurable functions which converges to $f$, $\theta$-almost uniformly on $E \in \mathcal{R}$. Then $(f_n)_{n \in \mathbb{N}}$, is $|\theta|_v$-almost everywhere convergent to $f$ for each $v \in \mathcal{V}$.
Proof. For each $n \in \mathbb{N}$, $v \in V$ and $w \in W$ there is $F_n = F_n(v, w) \in \mathfrak{R}$ such that $F_n \subset E$ and $|\theta(F_n, v)\|_w \in w^*_n(0)$ and $(f_n)$ is convergent to $f$ on $E \setminus F_n$. Now, we set $F = \bigcap_{n=1}^{\infty} F_n$. Since $(Q, W)$ is separated, we have $|\theta(F, v)\| = 0$. Clearly, $(f_n(x))_{n \in \mathbb{N}}$ is convergent to $f(x)$ for each $x \in E \setminus F = \bigcup_{n=1}^{\infty} E \setminus F_n$. □

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REFERENCES