

## Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

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ABSTRACT. In this paper, we define the almost uniform convergence and the almost everywhere convergence for cone-valued functions with respect to an operator valued measure. We prove the Egoroff theorem for  $\mathcal{P}$ -valued functions and operator valued measure  $\theta : \mathfrak{R} \rightarrow \mathcal{L}(\mathcal{P}, \mathcal{Q})$ , where  $\mathfrak{R}$  is a  $\sigma$ -ring of subsets of  $X \neq \emptyset$ ,  $(\mathcal{P}, \mathcal{V})$  is a quasi-full locally convex cone and  $(\mathcal{Q}, \mathcal{W})$  is a locally convex complete lattice cone.

**Keywords:** Locally convex cones, Egoroff Theorem, Operator valued measure.

**2000 Mathematics subject classification:** 28A12, 46A03.

### 1. INTRODUCTION

The theory of locally convex cones as developed in [7] and [9] uses an order theoretical concept or convex quasi-uniform structure to introduce a topological structure on a cone. For recent researches see [1, 2, 3, 4, 8].

A *cone* is a set  $\mathcal{P}$  endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element  $0 \in \mathcal{P}$ . For the scalar multiplication the usual associative and distributive properties hold, that is  $\alpha(\beta a) = (\alpha\beta)a$ ,

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$(\alpha + \beta)a = \alpha a + \beta a$ ,  $\alpha(a + b) = \alpha a + \alpha b$ ,  $1a = a$  and  $0a = 0$  for all  $a, b \in \mathcal{P}$  and  $\alpha, \beta \geq 0$ .

An *ordered cone*  $\mathcal{P}$  carries a reflexive transitive relation  $\leq$  such that  $a \leq b$  implies  $a + c \leq b + c$  and  $\alpha a \leq \alpha b$  for all  $a, b, c \in \mathcal{P}$  and  $\alpha \geq 0$ . The extended real numbers  $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  is a natural example of an ordered cone with the usual order and algebraic operations in  $\overline{\mathbb{R}}$ , in particular  $0 \cdot (+\infty) = 0$ .

A subset  $\mathcal{V}$  of the ordered cone  $\mathcal{P}$  is called an *abstract neighborhood system*, if the following properties hold:

- (1)  $0 < v$  for all  $v \in \mathcal{V}$ ;
- (2) for all  $u, v \in \mathcal{V}$  there is a  $w \in \mathcal{V}$  with  $w \leq u$  and  $w \leq v$ ;
- (3)  $u + v \in \mathcal{V}$  and  $\alpha v \in \mathcal{V}$  whenever  $u, v \in \mathcal{V}$  and  $\alpha > 0$ .

For every  $a \in \mathcal{P}$  and  $v \in \mathcal{V}$  we define

$$v(a) = \{b \in \mathcal{P} | b \leq a + v\} \quad \text{resp.} \quad (a)v = \{b \in \mathcal{P} | a \leq b + v\},$$

to be a neighborhood of  $a$  in the upper, resp. lower topologies on  $\mathcal{P}$ . Their common refinement is called the symmetric topology generated by the neighborhoods  $v^s(a) = v(a) \cap (a)v$ . If we suppose that all elements of  $\mathcal{P}$  are bounded below, that is for every  $a \in \mathcal{P}$  and  $v \in \mathcal{V}$  we have  $0 \leq a + \lambda v$  for some  $\lambda > 0$ , then the pair  $(\mathcal{P}, \mathcal{V})$  is called a *full locally convex cone*. A *locally convex cone*  $(\mathcal{P}, \mathcal{V})$  is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system  $\mathcal{V}$ . For example, the extended real number system  $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  endowed with the usual order and algebraic operations and the neighborhood system  $\mathcal{V} = \{\varepsilon \in \mathbb{R} | \varepsilon > 0\}$  is a full locally convex cone.

A subset  $B$  of the locally convex cone  $(\mathcal{P}, \mathcal{V})$  is called *bounded below* whenever for every  $v \in \mathcal{V}$  there is  $\lambda > 0$ , such that  $0 \leq b + \lambda v$  for all  $b \in B$ .

For cones  $\mathcal{P}$  and  $\mathcal{Q}$  a mapping  $T : \mathcal{P} \rightarrow \mathcal{Q}$  is called a *linear operator* if  $T(a + b) = T(a) + T(b)$  and  $T(\alpha a) = \alpha T(a)$  hold for all  $a, b \in \mathcal{P}$  and  $\alpha \geq 0$ . If both  $\mathcal{P}$  and  $\mathcal{Q}$  are ordered, then  $T$  is called *monotone*, if  $a \leq b$  implies  $T(a) \leq T(b)$ . If both  $(\mathcal{P}, \mathcal{V})$  and  $(\mathcal{Q}, \mathcal{W})$  are locally convex cones, the operator  $T$  is called (*uniformly*) *continuous* if for every  $w \in \mathcal{W}$  one can find  $v \in \mathcal{V}$  such that  $T(a) \leq T(b) + w$  whenever  $a \leq b + v$  for  $a, b \in \mathcal{P}$ .

A *linear functional* on  $\mathcal{P}$  is a linear operator  $\mu : \mathcal{P} \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ . The *dual cone*  $\mathcal{P}^*$  of a locally convex cone  $(\mathcal{P}, \mathcal{V})$  consists of all continuous linear functionals on  $\mathcal{P}$  and is the union of all polars  $v^\circ$  of neighborhoods  $v \in \mathcal{V}$ , where  $\mu \in v^\circ$  means that  $\mu(a) \leq \mu(b) + 1$ , whenever  $a \leq b + v$  for  $a, b \in \mathcal{P}$ . In addition to the given order  $\leq$  on the locally convex cone  $(\mathcal{P}, \mathcal{V})$ , the *weak preorder*  $\preceq$  is defined for  $a, b \in \mathcal{P}$  by

$$a \preceq b \quad \text{if} \quad a \leq \gamma b + \varepsilon v$$

for all  $v \in \mathcal{V}$  and  $\varepsilon > 0$  with some  $1 \leq \gamma \leq 1 + \varepsilon$  (for details, see [9], I.3). It is obviously coarser than the given order, that is  $a \leq b$  implies  $a \preceq b$  for  $a, b \in \mathcal{P}$ .

Given a neighborhood  $v \in \mathcal{V}$  and  $\varepsilon > 0$ , the corresponding upper and lower relative neighborhoods  $v_\varepsilon(a)$  and  $(a)v_\varepsilon$  for an element  $a \in \mathcal{P}$  are defined by

$$v_\varepsilon(a) = \{b \in \mathcal{P} \mid b \leq \gamma a + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon\},$$

$$(a)v_\varepsilon = \{b \in \mathcal{P} \mid a \leq \gamma b + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1 + \varepsilon\}.$$

Their intersection  $v_\varepsilon^s(a) = v_\varepsilon(a) \cap (a)v_\varepsilon$  is the corresponding symmetric relative neighborhood. Suppose  $v \in \mathcal{V}$ . If we consider the abstract neighborhood system  $\mathcal{V}_v = \{\alpha v : \alpha > 0\}$  on  $\mathcal{P}$ , then the corresponding upper (lower or symmetric) relative topology on  $\mathcal{P}$  is called *upper (lower or symmetric) relative  $v$ -topology*.

We shall say that a locally convex cone  $(\mathcal{P}, \mathcal{V})$  is a *locally convex  $\vee$ -semilattice cone* if its order is antisymmetric and if for any two elements  $a, b \in \mathcal{P}$  their supremum  $a \vee b$  exists in  $\mathcal{P}$  and if

$$(\vee 1) (a + c) \vee (b + c) = a \vee b + c \text{ holds for all } a, b, c \in \mathcal{P},$$

$$(\vee 2) a \leq c + v \text{ and } b \leq c + w \text{ for } a, b, c \in \mathcal{P} \text{ and } v, w \in \mathcal{V} \text{ imply that } a \vee b \leq c + (v + w).$$

Likewise,  $(\mathcal{P}, \mathcal{V})$  is a *locally convex  $\wedge$ -semilattice cone* if its order is antisymmetric and if for any two elements  $a, b \in \mathcal{P}$  their infimum  $a \wedge b$  exists in  $\mathcal{P}$  and if

$$(\wedge 1) (a + c) \wedge (b + c) = a \wedge b + c \text{ holds for all } a, b, c \in \mathcal{P},$$

$$(\wedge 2) c \leq a + v \text{ and } c \leq b + w \text{ for } a, b, c \in \mathcal{P} \text{ and } v, w \in \mathcal{V} \text{ imply that } c \leq a \wedge b + (v + w).$$

If both sets of the above conditions hold, then  $(\mathcal{P}, \mathcal{V})$  is called a *locally convex lattice cone* (cf. [9]).

We shall say that a locally convex cone  $(\mathcal{P}, \mathcal{V})$  is a *locally convex  $\vee^c$ -semilattice cone* if  $\mathcal{P}$  carries the weak preorder (that is the given order coincides with the weak preorder for the elements and the neighborhoods in  $\mathcal{P}$ ), this order is antisymmetric and if

$$(\vee_1^c) \text{ every non-empty subset } A \subseteq \mathcal{P} \text{ has a supremum } \sup A \in \mathcal{P} \text{ and } \sup(A + b) = \sup A + b \text{ holds for all } b \in \mathcal{P},$$

$$(\vee_2^c) \text{ let } \emptyset \neq A \subseteq \mathcal{P}, b \in \mathcal{P} \text{ and } v \in \mathcal{V}. \text{ If } a \leq b + v \text{ for all } a \in A, \text{ then } \sup A \leq b + v.$$

Likewise,  $(\mathcal{P}, \mathcal{V})$  is said to be a *locally convex  $\wedge^c$ -semilattice cone* if  $\mathcal{P}$  carries the weak preorder, this order is antisymmetric and if

$$(\wedge_1^c) \text{ every bounded below subset } A \subset \mathcal{P} \text{ has an infimum } \inf A \in \mathcal{P} \text{ and } \inf(A + b) = \inf A + b \text{ holds for all } b \in \mathcal{P},$$

$$(\wedge_2^c) \text{ let } A \subset \mathcal{P} \text{ be bounded below, } b \in \mathcal{P} \text{ and } v \in \mathcal{V}. \text{ If } b \leq a + v \text{ for all } a \in A, \text{ then } b \leq \inf A + v.$$

Combining both of the above notions, we shall say that a locally convex cone  $(\mathcal{P}, \mathcal{V})$  is a *locally convex complete lattice cone* if  $\mathcal{P}$  is both a  $\vee^c$ -semilattice cone and a  $\wedge^c$ -semilattice cone.

As a simple, example the locally convex cone  $(\overline{\mathbb{R}}, \mathcal{V})$ , where  $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  and  $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$ , is a locally convex lattice cone and a locally convex complete lattice cone.

Suppose  $(\mathcal{P}, \mathcal{V})$  is a locally convex complete lattice cone. A net  $(a_i)_{i \in \mathcal{I}}$  in  $\mathcal{P}$  is called *bounded below* if there is  $i_0 \in \mathcal{I}$  such that the set  $\{a_i \mid i \geq i_0\}$  is bounded below. We define the superior and the inferior limits of a bounded below net  $(a_i)_{i \in \mathcal{I}}$  in  $\mathcal{P}$  by

$$\liminf_{i \in \mathcal{I}} a_i = \sup_{i \in \mathcal{I}} (\inf_{k \geq i} a_k) \text{ and } \limsup_{i \in \mathcal{I}} a_i = \inf_{i \in \mathcal{I}} (\sup_{k \geq i} a_k).$$

If  $\liminf_{i \in \mathcal{I}} a_i$  and  $\limsup_{i \in \mathcal{I}} a_i$  coincide, then we denote their common value by  $\lim_{i \in \mathcal{I}} a_i$  and say that the net  $(a_i)_{i \in \mathcal{I}}$  is order convergent. A series  $\sum_{i=1}^{\infty} a_i$  in  $(\mathcal{P}, \mathcal{V})$  is said to be *order convergent* to  $s \in \mathcal{P}$  if the sequence  $s_n = \sum_{i=1}^n a_i$  is order convergent to  $s$ .

## 2. EGOROFF THEOREM FOR OPERATOR-VALUED MEASURES IN LOCALLY CONVEX CONES

The classical Egoroff theorem states that almost everywhere convergent sequences of measurable functions on a finite measure space converge almost uniformly. In this paper, we prove the Egoroff theorem for operator-valued measures in locally convex cones.

We shall say that a locally convex cone  $(\mathcal{P}, \mathcal{V})$  is quasi-full if

(QF1)  $a \leq b + v$  for  $a, b \in \mathcal{P}$  and  $v \in \mathcal{V}$  if and only if  $a \leq b + s$  for some  $s \in \mathcal{P}$  such that  $s \leq v$ , and

(QF2)  $a \leq u + v$  for  $a \in \mathcal{P}$  and  $u, v \in \mathcal{V}$  if and only if  $a \leq s + t$  for some  $s, t \in \mathcal{P}$  such that  $s \leq u$  and  $t \leq v$ .

The collection  $\mathfrak{A}$  of subsets of a set  $X$  is called a (weak)  $\sigma$ -ring whenever:

(R1)  $\emptyset \in \mathfrak{A}$ ,

(R2) If  $E_1, E_2 \in \mathfrak{A}$ , then  $E_1 \cup E_2 \in \mathfrak{A}$  and  $E_1 \setminus E_2 \in \mathfrak{A}$ ,

(R3) If  $E_n \in \mathfrak{A}$  for  $n \in \mathbb{N}$  and  $E_n \subseteq E$  for some  $E \in \mathfrak{A}$ , then  $\bigcup_{n \in \mathbb{N}} E_n \in \mathfrak{A}$  (see [9]).

Any  $\sigma$ -algebra is a  $\sigma$ -ring and a  $\sigma$ -ring  $\mathfrak{A}$  is a  $\sigma$ -algebra if and only if  $X \in \mathfrak{A}$ .

However, we can associate with  $\mathfrak{A}$  in a canonical way the  $\sigma$ -algebra

$$\mathfrak{M}_{\mathfrak{A}} = \{A \subset X : A \cap E \in \mathfrak{A} \text{ for all } E \in \mathfrak{A}\}.$$

A subset  $A$  of  $X$  is said to be measurable whenever  $A \in \mathfrak{M}_{\mathfrak{A}}$ .

We consider the symmetric relative topology on  $\mathcal{P}$ . The function  $f : X \rightarrow \mathcal{P}$  is measurable with respect to the  $\sigma$ -ring  $\mathfrak{A}$  if for every  $v \in \mathcal{V}$ ,

(M1)  $f^{-1}(O) \cap E \in \mathfrak{A}$  for every open subset  $O$  of  $\mathcal{P}$  and every  $E \in \mathfrak{A}$ ,

(M2)  $f(E)$  is separable in  $\mathcal{P}$  for every  $E \in \mathfrak{A}$ .

The operator-valued measures in locally convex cones have been defined in [9]. Let  $(\mathcal{P}, \mathcal{V})$  be a quasi-full locally convex cone and let  $(\mathcal{Q}, \mathcal{W})$  be a locally convex complete lattice cone. Let  $\mathcal{L}(\mathcal{P}, \mathcal{Q})$  denote the cone of all (uniformly)

continuous linear operators from  $\mathcal{P}$  to  $\mathcal{Q}$ . Recall from Section 3 in Chapter I from [9] that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because  $\mathcal{Q}$  carries its weak preorder, this implies monotonicity with respect to the given orders of  $\mathcal{P}$  and  $\mathcal{Q}$  as well. Let  $X$  be a set and  $\mathfrak{R}$  a  $\sigma$ -ring of subsets of  $X$ . An  $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ -valued measure  $\theta$  on  $\mathfrak{R}$  is a set function

$$E \rightarrow \theta_E : \mathfrak{R} \rightarrow \mathcal{L}(\mathcal{P}, \mathcal{Q})$$

such that  $\theta_\emptyset = 0$  and

$$\theta_{(\bigcup_{n \in \mathbb{N}} E_n)} = \sum_{n \in \mathbb{N}} \theta_{E_n}$$

holds whenever the sets  $E_n \in \mathfrak{R}$  are disjoint and  $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{R}$ . Convergence for the series on the right-hand side is meant in the following way: For every  $a \in \mathcal{P}$  the series  $\sum_{n \in \mathbb{N}} \theta_{E_n}(a)$  is order convergent in  $\mathcal{Q}$ . We note that the order convergence is implied by convergence in the symmetric relative topology.

Let  $(\mathcal{P}, \mathcal{V})$  be a quasi-full locally convex cone and let  $(\mathcal{Q}, \mathcal{W})$  be a locally convex complete lattice cone. Suppose  $\theta$  is a fixed  $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ -valued measure on  $\mathfrak{R}$ . For a neighborhood  $v \in \mathcal{V}$  and a set  $E \in \mathfrak{R}$ , semivariation of  $\theta$  is defined as follows:

$$|\theta|(E, v) = \sup \left\{ \sum_{i \in \mathbb{N}} \theta_{E_i}(s_i) : s_i \in \mathcal{P}, s_i \leq v, E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}.$$

It is proved in Lemma 3.3 chapter II from [9], that if  $v \in \mathcal{P}$ , then  $|\theta|(E, v) = \theta_E(v)$ .

**Proposition 2.1.** *Let  $(\mathcal{P}, \mathcal{V})$  be a quasi-full locally convex cone,  $(\mathcal{Q}, \mathcal{W})$  be a locally convex complete lattice cone and  $\theta$  be a fixed  $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ -valued measure on  $\mathfrak{R}$ .*

- (a) *If for  $E \in \mathfrak{R}$ ,  $\theta_E = 0$ , then for every  $v \in \mathcal{V}$ ,  $|\theta|(E, v) = 0$ ,*  
 (b) *If for every  $v \in \mathcal{V}$ ,  $|\theta|(E, v) = 0$ , then  $\theta_E(a) = 0$  for every bounded element  $a$  of  $\mathcal{P}$ .*

*Proof.* For (a), let  $\theta_E = 0$  and  $F_1, \dots, F_n, n \in \mathbb{N}$  be a partition of  $E$ . Then for  $0 \leq s_i \leq v, i = 1, \dots, n$ , we have  $0 \leq \theta_{F_i}(s_i) \leq \theta_E(s_i) = 0$ . Since the order of  $\mathcal{Q}$  is antisymmetric, for every  $i \in \{1, \dots, n\}$ , we have  $\theta_{F_i}(s_i) = 0$ . Then  $|\theta|(E, v) = 0$ .

For (b), let  $a \in \mathcal{P}$  and for every  $v \in \mathcal{V}$ ,  $|\theta|(E, v) = 0$ . Since  $a$  is bounded, for  $v \in \mathcal{V}$ , there is  $\lambda > 0$  such that  $0 \leq a + \lambda v$  and  $a \leq \lambda v$ . Now we have  $0 \leq \theta_E(a) + |\theta|(E, \lambda v)$  and  $\theta_E(a) \leq |\theta|(E, \lambda v)$  by Lemma II,3.4 of [9]. This shows that  $0 \leq \theta_E(a)$  and  $\theta_E(a) \leq 0$ . Since the order of  $\mathcal{Q}$  is antisymmetric, we have  $\theta_E(a) = 0$ .  $\square$

**Corollary 2.2.** *Let  $(\mathcal{P}, \mathcal{V})$  be a quasi-full locally convex cone,  $(\mathcal{Q}, \mathcal{W})$  be a locally convex complete lattice cone and  $\theta$  be a fixed  $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ -valued measure on*

$\mathfrak{A}$ . If all elements of  $\mathcal{P}$  are bounded, then for  $E \in \mathfrak{A}$ ,  $\theta_E = 0$  if and only if  $|\theta|(E, v) = 0$  for all  $v \in \mathcal{V}$ .

**Definition 2.3.** Let  $\mathfrak{A}$  be a  $\sigma$ -ring of subsets of  $X$ . The set  $A \in \mathfrak{A}$  is said to be of positive  $v$ -semivariation of the measure  $\theta$  if  $|\theta|(A, v) > 0$ . Also, we say that the set  $A$  has bounded  $v$ -semivariation of the measure  $\theta$ , if  $|\theta|(A, v)$  is bounded in  $(\mathcal{Q}, \mathcal{W})$ .

**Definition 2.4.** Let  $\theta$  be an operator-valued measure on  $X$ . We shall say that  $\theta$  is *generalized strongly  $v$ -continuous* ( $GS_v$ -continuous, for short) if for every set of bounded  $v$ -semivariation  $E \in \mathfrak{A}$  and every monotone sequence of sets  $(E_n)_{n \in \mathbb{N}} \in \mathfrak{A}$ ,  $E_n \subset E$ ,  $n \in \mathbb{N}$  the following holds

$$\lim_{n \in \mathbb{N}} |\theta|(E_n, v) = |\theta|(\lim_{n \in \mathbb{N}} E_n, v) \quad v \in \mathcal{V},$$

where the limit in the left hand side of the equality means convergence with respect to the symmetric relative topology of  $(\mathcal{Q}, \mathcal{W})$ .

EXAMPLE 2.5. Let  $X = \mathbb{N} \cup \{+\infty\}$  and  $\mathcal{P} = \mathcal{Q} = \bar{\mathbb{R}}$ . We consider on  $\bar{\mathbb{R}}$  the abstract neighborhood system  $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$ . Then  $\mathcal{L}(\mathcal{P}, \mathcal{Q})$  contains all nonnegative reals and the linear functional  $\bar{0}$  acting as

$$\bar{0}(x) = \begin{cases} +\infty & x = +\infty \\ 0 & \text{else.} \end{cases}$$

We set  $\mathfrak{A} = \{E \subset X : E \text{ is finite}\}$ . Then  $\mathfrak{A}$  is a  $\sigma$ -ring on  $X$ . We define the set function  $\theta$  on  $\mathfrak{A}$  as following: for  $x \in X$ ,  $\theta_\emptyset = 0$ ,  $\theta_{\{n\}}(x) = nx$  for  $n \in \mathbb{N}$  and  $\theta_{\{+\infty\}}(x) = \bar{0}(x)$ . For  $E = \{a_1, \dots, a_n\} \in \mathfrak{A}$ ,  $n \in \mathbb{N}$ , we define  $\theta_E(x) = \sum_{i=1}^n \theta_{\{a_i\}}(x)$  for  $x \in X$ . Then  $\theta$  is clearly an operator-valued measure on  $\mathfrak{A}$ . For  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , we have  $|\theta|(\{n\}, \varepsilon) = \theta_{\{n\}}(\varepsilon) = n\varepsilon$  and  $|\theta|(\{+\infty\}, \varepsilon) = \theta_{\{+\infty\}}(\varepsilon) = \bar{0}(\varepsilon) = 0$ . Therefore each  $E \in \mathfrak{A}$  has finite  $\varepsilon$ -semivariation for all  $\varepsilon > 0$ . Let  $E \in \mathfrak{A}$ . If  $(E_n)_{n \in \mathbb{N}} \subset \mathfrak{A}$  is a monotone sequence of subsets of  $E$  such that  $\lim_{n \in \mathbb{N}} E_n = F$ , then there is  $n_0 \in \mathbb{N}$  such that  $E_n = F$  for all  $n \geq n_0$ . Then  $\theta$  is clearly  $GS_\varepsilon$ -continuous for each  $\varepsilon > 0$ .

**Definition 2.6.** A sequence  $(f_n)_{n \in \mathbb{N}}$  of measurable functions is said to be  $\theta$ -almost uniformly convergent to a measurable function  $f$  on  $E \in \mathfrak{A}$  if for every  $\varepsilon > 0$ ,  $w \in \mathcal{W}$  and  $v \in \mathcal{V}$  there exists a subset  $F = F(\varepsilon, v, w)$  of  $E$  and  $n_0 \in \mathbb{N}$  such that for every  $n > n_0$ ,

$$f_n(x) \in v_\varepsilon^s(f(x)) \text{ and } |\theta|(F, v) \in w_\varepsilon^s(0),$$

for all  $x \in E \setminus F$ .

**Theorem 2.7** (Egoroff Theorem). Let  $\mathfrak{A}$  be a  $\sigma$ -ring of subsets of  $X$ ,  $(\mathcal{P}, \mathcal{V})$  be a full locally convex cone and  $(\mathcal{Q}, \mathcal{W})$  be a locally convex complete lattice cone. For  $v \in \mathcal{V}$ , suppose  $\theta : \mathfrak{A} \rightarrow \mathcal{L}(\mathcal{P}, \mathcal{Q})$  is a  $GS_v$ -continuous operator valued measure, and  $E \in \mathfrak{A}$  has bounded  $v$ -semivariation. If  $f : X \rightarrow \mathcal{P}$  is a measurable function, and  $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$  is a sequence of measurable

functions, such that for every  $t \in E$ ,  $f_n(t) \rightarrow f(t)$  with respect to the symmetric relative  $v$ -topology of  $(\mathcal{P}, \mathcal{V})$ , then  $(f_n)_{n \in \mathbb{N}}$  is  $\theta$ -almost uniformly convergent to  $f$  on  $E$ , with respect to the symmetric relative  $v$ -topology of  $(\mathcal{P}, \mathcal{V})$ .

*Proof.* We identify  $v \in \mathcal{V}$  with the constant function  $x \rightarrow v$  from  $X$  into  $\mathcal{P}$ . For  $m, n \in \mathbb{N}$ , we set

$$B_n^m = \bigcap_{i=n}^{\infty} \{x \in E : f_i(x) \preceq_v f(x) + \frac{1}{m}v \text{ and } f(x) \preceq_v f_i(x) + \frac{1}{m}v\}.$$

For every  $n, m \in \mathbb{N}$  we have  $B_n^m \in \mathfrak{R}$  by Theorem II.1.6 from [9]. Clearly,  $B_n^m \subset B_{n+1}^m$  for all  $n, m \in \mathbb{N}$ . We claim that  $E = \bigcup_{n=1}^{\infty} B_n^m$ . Let  $x \in E$  and  $m \in \mathbb{N}$ . Then  $(f_n(x))_{n \in \mathbb{N}}$  is convergent to  $f(x)$  with respect to the symmetric relative  $v$ -topology. This shows that for each  $\varepsilon > 0$  there is  $n_0 \in \mathbb{N}$  such that  $f_n(x) \in (\frac{1}{m}v)_{\varepsilon}^s(f(x))$  for all  $n \geq n_0$ . Therefore  $f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m}v)$  and  $f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m}v)$  for all  $n \geq n_0$  and some  $1 \leq \gamma \leq 1 + \varepsilon$ . This yields that  $f_n(x) \leq \gamma f(x) + (1 + \varepsilon)(\frac{1}{m}v)$  and  $f_n(x) \leq \gamma f(x) + (1 + \varepsilon)(\frac{1}{m}v)$  for all  $n \geq n_0$  and some  $1 \leq \gamma \leq 1 + \varepsilon$ . Now Lemma I.3.1 from [9] shows that  $f_n(x) \preceq_v f(x) + \frac{1}{m}v$  and  $f(x) \preceq_v f_n(x) + \frac{1}{m}v$  for all  $n \geq n_0$ . Thus  $x \in B_{n_0}^m$ .

Then  $(E \setminus B_n^m)_{n \in \mathbb{N}}$  is a decreasing sequence of subsets of  $E$ , such that  $\lim_{n \rightarrow \infty} E \setminus B_n^m = \emptyset$ . Therefore for every  $m \in \mathbb{N}$ ,  $|\theta|(E \setminus B_n^m, v)$  is convergent to  $|\theta|(\emptyset, v) = 0$  with respect to the symmetric relative topology of  $(\mathcal{Q}, \mathcal{W})$  by the assumption. For  $\varepsilon > 0$  and  $m \in \mathbb{N}$  we choose  $n_m$  such that  $|\theta|(E \setminus B_{n_m}^m, v) \leq \frac{\varepsilon}{2^m}w$ . We set

$$F = \bigcup_{m=1}^{\infty} E \setminus B_{n_m}^m.$$

Then we have

$$|\theta|(F, v) \leq \sum_{m=1}^{\infty} |\theta|(B_{n_m}^m, v) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m}w = \varepsilon w.$$

Also, we have  $0 \leq |\theta|(F, v) + \varepsilon w$ . Then  $|\theta|(F, v) \in w_{\varepsilon}^s(0)$ .

Now, we show that the convergence on  $E \setminus F$  is uniform. Let  $\delta > 0$ . There is  $k \in \mathbb{N}$  such that  $\frac{2}{k} + \frac{1}{k^2} \leq \delta$ . We have

$$E \setminus F = E \setminus \left( \bigcup_{m=1}^{\infty} E \setminus B_{n_m}^m \right) = \bigcap_{m=1}^{\infty} B_{n_m}^m \subset B_{n_k}^k$$

Now for each  $n \geq n_k$  and every  $x \in E \setminus F$  we have  $f_n(x) \preceq_v f(x) + \frac{1}{k}v$  and  $f(x) \preceq_v f_n(x) + \frac{1}{k}v$ . The definition of  $\preceq_v$  shows that for  $\varepsilon = \frac{1}{k}$  there is  $1 \leq \gamma \leq$

$1 + \frac{1}{k}$  such that  $f_n(x) \leq \gamma(f(x) + \frac{1}{k}v) + \frac{1}{k}v$  and  $f(x) \leq \gamma(f_n(x) + \frac{1}{k}v) + \frac{1}{k}v$ . Therefore  $f_n(x) \leq \gamma f(x) + (\frac{2}{k} + \frac{1}{k^2})v \leq \gamma f(x) + \delta v$  and  $f(x) \leq \gamma f_n(x) + (\frac{2}{k} + \frac{1}{k^2})v \leq \gamma f_n(x) + \delta v$ . Since  $1 \leq \gamma \leq 1 + \frac{1}{k} \leq 1 + \frac{2}{k} + \frac{1}{k^2} \leq 1 + \delta$ , we realize that  $(f_n)_{n \in \mathbb{N}}$  is uniformly convergent to  $f$  on  $E \setminus F$ , with respect to the symmetric relative topology.  $\square$

*Remark 2.8.* If in the assumptions of Theorem 2.7,  $(\mathcal{P}, \mathcal{V})$  is a quasi-full locally convex cone, then the theorem holds again. In fact every quasi-full locally convex cone can be embedded in a full locally convex cone as elaborated in ([9], I, 6.2).

**Definition 2.9.** We say that a sequence  $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$  of measurable functions is  $\theta$ -almost everywhere convergent (with respect to the symmetric topology of  $(\mathcal{P}, \mathcal{V})$ ) to  $f$ , if the set  $\{x \in X : f_n(x) \not\rightarrow f(x)\}$  is contained in a subset  $E$  of  $X$  with  $\theta_E = 0$ .

**Definition 2.10.** Let  $v \in \mathcal{V}$ . We say that the sequence  $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$  of measurable functions is  $|\theta|_v$ -almost everywhere convergent (with respect to symmetric topology of  $(\mathcal{P}, \mathcal{V})$ ) to  $f$ , if the set  $\{x \in X : f_n(x) \not\rightarrow f(x)\}$  is contained in a subset  $E$  of  $X$  with  $|\theta|(E, v) = 0$ .

**Lemma 2.11.** Let  $\mathfrak{A}$  be a  $\sigma$ -ring of subsets of  $X$ ,  $(\mathcal{P}, \mathcal{V})$  be a full locally convex cone and  $(\mathcal{Q}, \mathcal{W})$  be a locally convex complete lattice cone. Then

- (a)  $\theta$ -almost everywhere convergence implies  $|\theta|_v$ -almost everywhere convergence for each  $v \in \mathcal{V}$ .
- (b) If all elements of  $(\mathcal{P}, \mathcal{V})$  are bounded and a sequence  $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$  is  $|\theta|_v$ -almost everywhere convergent to  $f$  for each  $v \in \mathcal{V}$ , then  $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$  is  $\theta$ -almost everywhere convergent to  $f$ .

*Proof.* The assertions are proved by the help of Proposition 2.1.  $\square$

**Theorem 2.12.** If in the Egoroff theorem (2.7),  $f_n \rightarrow f$ ,  $\theta$ -almost everywhere or  $|\theta|_v$ -almost everywhere, then the assertion of theorem holds.

*Proof.* Suppose  $f_n \rightarrow f$ ,  $\theta$ -almost everywhere, then there is a subset  $A$  of  $E$ , which is contained in some  $B \in \mathfrak{A}$  with  $\theta_B = 0$ . Now  $E \setminus B \in \mathfrak{A}$  and it has bounded  $v$ -semivariation. We apply the theorem 2.7 for  $E \setminus B$  and obtain a subset  $F$  satisfying in definition 2.6. Now clearly  $f_n$  is  $\theta$ -almost uniformly convergent to  $f$  on  $E \setminus (F \cap B)$ . A similar argument yields our claim for  $|\theta|_v$ -almost everywhere convergence.  $\square$

**Theorem 2.13.** Let the symmetric relative  $w$ -topology of  $(\mathcal{Q}, \mathcal{W})$  be Hausdorff for each  $w \in \mathcal{W}$  and let  $(f_n : X \rightarrow \mathcal{P})_{n \in \mathbb{N}}$  be a sequence of measurable functions which converges to  $f$ ,  $\theta$ -almost uniformly on  $E \in \mathfrak{A}$ . Then  $\{f_n\}_{n \in \mathbb{N}}$  is  $|\theta|_v$ -almost everywhere convergent to  $f$  for each  $v \in \mathcal{V}$ .



*Proof.* For each  $n \in \mathbb{N}$ ,  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$  there is  $F_n = F_n(v, w) \in \mathfrak{A}$  such that  $F_n \subset E$  and  $|\theta|(F_n, v) \in w_{\frac{1}{n}}^s(0)$  and  $(f_n)$  is convergent to  $f$  on  $E \setminus F_n$ . Now, we set  $F = \bigcap_{n=1}^{\infty} F_n$ . Since  $(\mathcal{Q}, \mathcal{W})$  is separated, we have  $|\theta|(F, v) = 0$ . Clearly,  $(f_n(x))_{n \in \mathbb{N}}$  is convergent to  $f(x)$  for each  $x \in E \setminus F = \bigcup_{n=1}^{\infty} E \setminus F_n$ .  $\square$

## ACKNOWLEDGMENTS

The authors would like to thank the referees for their useful comments.

## REFERENCES

1. D. Ayaseh, A. Ranjbari, Bornological Convergence in Locally Convex Cones, *Mediterr. J. Math.*, **13**(4), (2016), 1921-1931.
2. D. Ayaseh, A. Ranjbari, Bornological Locally Convex Cones, *Le Matematiche*, **69**(2), (2014), 267-284.
3. D. Ayaseh, A. Ranjbari, Locally Convex Quotient Lattice Cones, *Math. Nachr.*, **287**(10), (2014), 1083-1092.
4. D. Ayaseh, A. Ranjbari, Some Notes on Bornological and Nonbornological Locally Convex Cones, *Le Matematiche*, **70**(2), (2015), 235-241.
5. N. Eftekhari, The Basic Theorem and its Consequences, *Iranian Journal of Mathematical Sciences and Informatics*, **4**(1), (2009), 27-35.
6. P. R. Halmos, *Measure Theory*, Van Nostrand, Princeton, N. J., 1950. MR 11, 504.
7. K. Keimel, W. Roth, Ordered Cones and Approximation, *Lecture Notes in Mathematics*, **1517**, 1992, Springer Verlag, Heidelberg-Berlin-New York.
8. A. Ranjbari, Strict Inductive Limits in Locally Convex Cones, *Positivity*, **15**(3), (2011), 465-471.
9. W. Roth, Operator-valued Measures and Integrals for Cone-valued Functions, *Lecture Notes in Mathematics*, **1964**, 2009, Springer Verlag, Heidelberg-Berlin-New York.
10. A. Taghavi, R. Hosseinzadeh, Uniform Boundedness Principle for Operators on Hyper-vector Spaces, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(2), (2012), 9-16.