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Egoroff Theorem for Operator-Valued Measures in Locally Convex Cones

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Abstract. In this paper, we define the almost uniform convergence and the almost everywhere convergence for cone-valued functions with respect to an operator valued measure. We prove the Egoroff theorem for $\mathcal{P}$-valued functions and operator valued measure $\theta : \mathfrak{R} \to \mathcal{L}(\mathcal{P}, \mathcal{Q})$, where $\mathfrak{R}$ is a σ-ring of subsets of $X \neq \emptyset$, $(\mathcal{P}, \mathcal{V})$ is a quasi-full locally convex cone and $(\mathcal{Q}, \mathcal{W})$ is a locally convex complete lattice cone.

Keywords: Locally convex cones, Egoroff Theorem, Operator valued measure.

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1. Introduction

The theory of locally convex cones as developed in [7] and [9] uses an order theoretical concept or convex quasi-uniform structure to introduce a topological structure on a cone. For recent researches see [1, 2, 3, 4, 8].

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$,

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$(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.

An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. The extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a natural example of an ordered cone with the usual order and algebraic operations in $\overline{\mathbb{R}}$, in particular $0 \cdot (+\infty) = 0$.

A subset $\mathcal{V}$ of the ordered cone $\mathcal{P}$ is called an *abstract neighborhood system,* if the following properties hold:
(1) $0 < v$ for all $v \in \mathcal{V}$;
(2) for all $u, v \in \mathcal{V}$ there is a $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;
(3) $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we define

$$v(a) = \{b \in \mathcal{P} | b \leq a + v\} \quad \textit{resp.} \quad (a)v = \{b \in \mathcal{P} | a \leq b + v\},$$

to be a neighborhood of a in the upper, resp. lower topologies on $\mathcal{P}$. Their common refinement is called the symmetric topology generated by the neighborhoods $v^s(a) = v(a) \cap (a)v$. If we suppose that all elements of $\mathcal{P}$ are bounded below, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda > 0$, then the pair $(\mathcal{P}, \mathcal{V})$ is called a *full locally convex cone.* A *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$. For example, the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ endowed with the usual order and algebraic operations and the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} | \varepsilon > 0\}$ is a full locally convex cone.

A subset B of the locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *bounded below* whenever for every $v \in \mathcal{V}$ there is $\lambda > 0$, such that $0 \leq b + \lambda v$ for all $b \in B$.

For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $\mathcal{P}$ and $\mathcal{Q}$ are ordered, then T is called *monotone,* if $a \leq b$ implies $T(a) \leq T(b)$. If both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars v° of neighborhoods $v \in \mathcal{V}$, where $\mu \in v^\circ$ means that $\mu(a) \leq \mu(b) + 1$, whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. In addition to the given order $\leq$ on the locally convex cone $(\mathcal{P}, \mathcal{V})$, the *weak pereorder* $\preccurlyeq$ is defined for $a, b \in \mathcal{P}$ by

$$a \preccurlyeq b \quad \textit{if} \quad a \leq \gamma b + \varepsilon v$$

for all $v \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1 + \varepsilon$ (for details, see [9], $I.3$). It is obviously coarser than the given order, that is $a \leq b$ implies $a \preccurlyeq b$ for $a, b \in \mathcal{P}$.

Given a neighborhood $v \in \mathcal{V}$ and $\varepsilon > 0$, the corresponding upper and lower relative neighborhoods $v_\varepsilon(a)$ and $(a)v_\varepsilon$ for an element $a \in \mathcal{P}$ are defined by

$$v_\varepsilon(a) = \{b \in \mathcal{P} | b \leq \gamma a + \varepsilon v \quad \text{for some} \quad 1 \leq \gamma \leq 1+\varepsilon\},$$

$$(a)v_\varepsilon = \{b \in \mathcal{P} | a \leq \gamma b + \varepsilon v \quad \text{for some} \quad 1 \leq \gamma \leq 1+\varepsilon\}.$$

Their intersection $v^s_\varepsilon(a) = v_\varepsilon(a) \cap (a)v_\varepsilon$ is the corresponding symmetric relative neighborhood. Suppose $v \in \mathcal{V}$. If we consider the abstract neighborhood system $\mathcal{V}_v = \{\alpha v : \alpha > 0\}$ on $\mathcal{P}$, then the corresponding upper (lower or symmetric) relative topology on $\mathcal{P}$ is called *upper (lower or symmetric) relative v-topology.*

We shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a *locally convex $\vee$-semilattice cone* if its order is antisymmetric and if for any two elements $a, b \in \mathcal{P}$ their supremum $a \vee b$ exists in $\mathcal{P}$ and if
($\vee$1) $(a + c) \vee (b + c) = a \vee b + c$ holds for all $a, b, c \in \mathcal{P}$,
($\vee$2) $a \leq c + v$ and $b \leq c + w$ for $a, b, c \in \mathcal{P}$ and $v, w \in \mathcal{V}$ imply that $a \vee b \leq c + (v + w)$.
Likewise, $(\mathcal{P}, \mathcal{V})$ is a *locally convex $\wedge$-semilattice cone* if its order is antisymmetric and if for any two elements $a, b \in \mathcal{P}$ their infimum $a \wedge b$ exists in $\mathcal{P}$ and if
($\wedge$1) $(a + c) \wedge (b + c) = a \wedge b + c$ holds for all $a, b, c \in \mathcal{P}$,
($\wedge$2) $c \leq a + v$ and $c \leq b + w$ for $a, b, c \in \mathcal{P}$ and $v, w \in \mathcal{V}$ imply that $c \leq a \wedge b + (v + w)$.
If both sets of the above conditions hold, then $(\mathcal{P}, \mathcal{V})$ is called a *locally convex lattice cone* (cf. [9]).

We shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a locally convex $\vee^c$- *semilattice cone* if $\mathcal{P}$ carries the weak preorder (that is the given order coincides with the weak preorder for the elements and the neighborhoods in $\mathcal{P}$), this order is antisymmetric and if
($\vee^c_1$) every non-empty subset $A \subseteq \mathcal{P}$ has a supremum $\sup A \in \mathcal{P}$ and $\sup(A + b) = \sup A + b$ holds for all $b \in \mathcal{P}$,
($\vee^c_2$) let $\emptyset \neq A \subseteq \mathcal{P}$, $b \in \mathcal{P}$ and $v \in \mathcal{V}$. If $a \leq b + v$ for all $a \in A$, then $\sup A \leq b + v$.

Likewise, $(\mathcal{P}, \mathcal{V})$ is said to be a locally convex $\wedge^c$-*semilattice cone* if $\mathcal{P}$ carries the weak preorder, this order is antisymmetric and if
($\wedge^c_1$) every bounded below subset $A \subset \mathcal{P}$ has an infimum $\inf A \in \mathcal{P}$ and $\inf(A + b) = \inf A + b$ holds for all $b \in \mathcal{P}$,
($\wedge^c_2$) let $A \subset \mathcal{P}$ be bounded below, $b \in \mathcal{P}$ and $v \in \mathcal{V}$. If $b \leq a + v$ for all $a \in A$, then $b \leq inf A + v$.
Combining both of the above notions, we shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a *locally convex complete lattice cone* if $\mathcal{P}$ is both a $\vee^c$-semilattice cone and a $\wedge^c$-semilattice cone.

As a simple, example the locally convex cone $(\overline{\mathbb{R}}, \mathcal{V})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$, is a locally convex lattice cone and a locally convex complete lattice cone.

Suppose $(\mathcal{P}, \mathcal{V})$ is a locally convex complete lattice cone. A net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ is called *bounded below* if there is $i_0 \in \mathcal{I}$ such that the set $\{a_i \mid i \geq i_0\}$ is bounded below. We define the superior and the inferior limits of a bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ by

$$\liminf_{i\in\mathcal{I}} a_i = \sup_{i\in\mathcal{I}}(\inf_{k\geq i} a_k) \text{ and } \limsup_{i\in\mathcal{I}} a_i = \inf_{i\in\mathcal{I}}(\sup_{k\geq i} a_k).$$

If $\liminf_{i\in\mathcal{I}} a_i$ and $\limsup_{i\in\mathcal{I}} a_i$ coincide, then we denote their common value by $\lim_{i\in\mathcal{I}} a_i$ and say that the net $(a_i)_{i\in\mathcal{I}}$ is order convergent. A series $\sum_{i=1}^{\infty} a_i$ in $(\mathcal{P}, \mathcal{V})$ is said to be *order convergent* to $s \in \mathcal{P}$ if the sequence $s_n = \sum_{i=1}^{n} a_i$ is order convergent to s.

2. Egoroff Theorem for Operator-valued Measures in Locally Convex Cones

The classical Egoroff theorem states that almost everywhere convergent sequences of measurable functions on a finite measure space converge almost uniformly. In this paper, we prove the Egoroff theorem for operator-valued measures in locally convex cones.

We shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is quasi-full if
$(QF1)$ $a \leq b + v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $a \leq b + s$ for some $s \in \mathcal{P}$ such that $s \leq v$, and
$(QF2)$ $a \leq u + v$ for $a \in \mathcal{P}$ and $u, v \in \mathcal{V}$ if and only if $a \leq s + t$ for some $s, t \in \mathcal{P}$ such that $s \leq u$ and $t \leq v$.

The collection $\mathfrak{R}$ of subsets of a set X is called a (weak) σ-ring whenever:
$(R1)$ $\emptyset \in \mathfrak{R}$,
$(R2)$ If $E_1, E_2 \in \mathfrak{R}$, then $E_1 \cup E_2 \in \mathfrak{R}$ and $E_1 \backslash E_2 \in \mathfrak{R}$,
$(R3)$ If $E_n \in \mathfrak{R}$ for $n \in \mathbb{N}$ and $E_n \subseteq E$ for some $E \in \mathfrak{R}$, then $\bigcup_{n\in\mathbb{N}} E_n \in \mathfrak{R}$ (see [9]).

Any σ-algebra is a σ-ring and a σ-ring $\mathfrak{R}$ is a σ-algebra if and only if $X \in \mathfrak{R}$. However, we can associate with $\mathfrak{R}$ in a canonical way the σ-algebra

$$\mathfrak{U}_{\mathfrak{R}} = \{A \subset X : A \cap E \in \mathfrak{R} \text{ for all } E \in \mathfrak{R}\}.$$

A subset A of X is said to be measurable whenever $A \in \mathfrak{U}_{\mathfrak{R}}$.

We consider the symmetric relative topology on $\mathcal{P}$. The function $f : X \to \mathcal{P}$ is measurable with respect to the σ-ring $\mathfrak{R}$ if for every $v \in \mathcal{V}$,
(M_1) $f^{-1}(O) \cap E \in \mathfrak{R}$ for every open subset O of $\mathcal{P}$ and every $E \in \mathfrak{R}$,
(M_2) $f(E)$ is separable in $\mathcal{P}$ for every $E \in \mathfrak{R}$.

The operator–valued measures in locally convex cones have been defined in [9]. Let $(\mathcal{P}, \mathcal{V})$ be a quasi-full locally convex cone and let $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone. Let $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ denote the cone of all (uniformly) continuous linear operators from $\mathcal{P}$ to $\mathcal{Q}$. Recall from Section 3 in Chapter I from [9] that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because $\mathcal{Q}$ carries its weak preorder, this implies monotonicity with respect to the given orders of $\mathcal{P}$ and $\mathcal{Q}$ as well. Let X be a set and $\mathfrak{R}$ a σ-ring of subsets of X. An $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure θ on $\mathfrak{R}$ is a set function

$$E \to \theta_E : \mathfrak{R} \to \mathcal{L}(\mathcal{P},\mathcal{Q})$$

such that $\theta_{\emptyset} = 0$ and

$$\theta_{(\bigcup_{n\in\mathbb{N}} E_n)} = \sum_{n\in\mathbb{N}} \theta_{E_n}$$

holds whenever the sets $E_n \in \mathfrak{R}$ are disjoint and $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{R}$. Convergence for the series on the right-hand side is meant in the following way: For every $a \in \mathcal{P}$ the series $\sum_{n\in\mathbb{N}} \theta_{E_n}(a)$ is order convergent in $\mathcal{Q}$. We note that the order convergence is implied by convergence in the symmetric relative topology.

Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone and let $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone. Suppose θ is a fixed $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure on $\mathfrak{R}$. For a neighborhood $v \in \mathcal{V}$ and a set $E \in \mathfrak{R}$, semivariation of θ is defined as follows:

$$|\theta|(E,v) = \sup\left\{\sum_{i\in\mathbb{N}} \theta_{E_i}(s_i) : s_i \in \mathcal{P}, s_i \le v, E_i \in \mathfrak{R} \text{ disjoint subsets of } E\right\}.$$

It is proved in Lemma 3.3 chapter II from [9], that if $v \in \mathcal{P}$, then $|\theta|(E,v) = \theta_E(v)$.

Proposition 2.1. *Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone, $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone and θ be a fixed $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure on $\mathfrak{R}$.*
(a) If for $E \in \mathfrak{R}$, $\theta_E = 0$, then for every $v \in \mathcal{V}$, $|\theta|(E,v) = 0$,
(b) If for every $v \in \mathcal{V}$, $|\theta|(E,v) = 0$, then $\theta_E(a) = 0$ for every bounded element a of $\mathcal{P}$.

Proof. For (a), let $\theta_E = 0$ and $F_1, \cdots, F_n$, $n \in \mathbb{N}$ be a partition of E. Then for $0 \le s_i \le v$, $i = 1, \cdots, n$, we have $0 \le \theta_{F_i}(s_i) \le \theta_E(s_i) = 0$. Since the order of $\mathcal{Q}$ is antisymmetric, for every $i \in \{1, \cdots, n\}$, we have $\theta_{F_i}(s_i) = 0$. Then $|\theta|(E,v) = 0$.

For (b), let $a \in \mathcal{P}$ and for every $v \in \mathcal{V}$, $|\theta|(E,v) = 0$. Since a is bounded, for $v \in \mathcal{V}$, there is $\lambda > 0$ such that $0 \le a + \lambda v$ and $a \le \lambda v$. Now we have $0 \le \theta_E(a) + |\theta|(E,\lambda v)$ and $\theta_E(a) \le |\theta|(E,\lambda v)$ by Lemma II,3.4 of [9]. This shows that $0 \le \theta_E(a)$ and $\theta_E(a) \le 0$. Since the order of $\mathcal{Q}$ is antisymmetric, we have $\theta_E(a) = 0$. □

Corollary 2.2. *Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone, $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone and θ be a fixed $\mathcal{L}(\mathcal{P},\mathcal{Q})$-valued measure on*

$\mathfrak{R}$. *If all elements of* $\mathcal{P}$ *are bounded, then for* $E \in \mathfrak{R}$, $\theta_E = 0$ *if and only if* $|\theta|(E, v) = 0$ *for all* $v \in \mathcal{V}$.

Definition 2.3. Let $\mathfrak{R}$ be a σ-ring of subsets of X. The set $A \in \mathfrak{R}$ is said to be of positive v-semivariation of the measure θ if $|\theta|(A, v) > 0$. Also, we say that the set A has bounded v-semivariation of the measure θ, if $|\theta|(A, v)$ is bounded in $(\mathcal{Q}, \mathcal{W})$.

Definition 2.4. Let θ be an operator-valued measure on X. We shall say that θ is *generalized strongly v-continuous* (GS_v-continuous, for short) if for every set of bounded v-semivariation $E \in \mathfrak{R}$ and every monotone sequence of sets $(E_n)_{n\in\mathbb{N}} \in \mathfrak{R}$, $E_n \subset E$, $n \in \mathbb{N}$ the following holds

$$\lim_{n\in\mathbb{N}} |\theta|(E_n, v) = |\theta|(\lim_{n\in\mathbb{N}} E_n, v) \quad v \in \mathcal{V},$$

where the limit in the left hand side of the equality means convergence with respect to the symmetric relative topology of $(\mathcal{Q}, \mathcal{W})$.

Example 2.5. Let $X = \mathbb{N} \cup \{+\infty\}$ and $\mathcal{P} = \mathcal{Q} = \bar{\mathbb{R}}$. We consider on $\overline{\mathbb{R}}$ the abstract neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$. Then $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ contains all nonnegative reals and the linear functional $\bar{0}$ acting as

$$\bar{0}(x) = \begin{cases} +\infty & x=+\infty \\ 0 & else. \end{cases}$$

We set $\mathfrak{R} = \{E \subset X : E \text{ is finite}\}$. Then $\mathfrak{R}$ is a σ-ring on X. We define the set function θ on $\mathfrak{R}$ as following: for $x \in X$, $\theta_\emptyset = 0$, $\theta_{\{n\}}(x) = nx$ for $n \in \mathbb{N}$ and $\theta_{\{+\infty\}}(x) = \bar{0}(x)$. For $E = \{a_1, \cdots, a_n\} \in \mathfrak{R}$, $n \in \mathbb{N}$, we define $\theta_E(x) = \sum_{i=1}^{n} \theta_{\{a_i\}}(x)$ for $x \in X$. Then θ is clearly an operator-valued measure on $\mathfrak{R}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $|\theta|(\{n\}, \varepsilon) = \theta_{\{n\}}(\varepsilon) = n\varepsilon$ and $|\theta|(\{+\infty\}, \varepsilon) = \theta_{\{\infty\}}(\varepsilon) = \bar{0}(\varepsilon) = 0$. Therefore each $E \in \mathfrak{R}$ has finite ε-semivariation for all $\varepsilon > 0$. Let $E \in \mathfrak{R}$. If $(E_n)_{n\in\mathbb{N}} \subset \mathfrak{R}$ is a monotone sequence of subsets of E such that $\lim_{n\in\mathbb{N}} E_n = F$, then there is $n_0 \in \mathbb{N}$ such that $E_n = F$ for all $n \geq n_0$. Then θ is clearly GS_ε-continuous for each $\varepsilon > 0$.

Definition 2.6. A sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions is said to be θ-almost uniformly convergent to a measurable function f on $E \in \mathfrak{R}$ if for every $\varepsilon > 0$, $w \in \mathcal{W}$ and $v \in \mathcal{V}$ there exists a subset $F = F(\varepsilon, v, w)$ of E and $n_0 \in \mathbb{N}$ such that for every $n > n_0$,

$$f_n(x) \in v_\varepsilon^s(f(x)) \text{ and } |\theta|(F, v) \in w_\varepsilon^s(0),$$

for all $x \in E \backslash F$.

Theorem 2.7 (Egoroff Theorem)**.** *Let* $\mathfrak{R}$ *be a* σ*-ring of subsets of* X, $(\mathcal{P}, \mathcal{V})$ *be a full locally convex cone and* $(\mathcal{Q}, \mathcal{W})$ *be a locally convex complete lattice cone. For* $v \in \mathcal{V}$, *suppose* $\theta : \mathfrak{R} \to \mathcal{L}(\mathcal{P}, \mathcal{Q})$ *is a* GS_v*-continuous operator valued measure, and* $E \in \mathfrak{R}$ *has bounded* v*-semivariation. If* $f : X \to \mathcal{P}$ *is a measurable function, and* $(f_n : X \to \mathcal{P})_{n\in\mathbb{N}}$ *is a sequence of measurable* *functions, such that for every $t \in E$, $f_n(t) \to f(t)$ with respect to the symmetric relative v-topology of $(\mathcal{P}, \mathcal{V})$, then $(f_n)_{n\in\mathbb{N}}$ is θ-almost uniformly convergent to f on E, with respect to the symmetric relative v-topology of $(\mathcal{P}, \mathcal{V})$.*

Proof. We identify $v \in \mathcal{V}$ with the constant function $x \to v$ from X into $\mathcal{P}$. For $m, n \in \mathbb{N}$, we set

$$B_n^m = \bigcap_{i=n}^{\infty} \{x \in E : f_i(x) \preceq_v f(x) + \frac{1}{m} v \text{ and } f(x) \preceq_v f_i(x) + \frac{1}{m} v\}.$$

For every $n, m \in \mathbb{N}$ we have $B_n^m \in \mathfrak{R}$ by Theorem II.1.6 from [9]. Clearly, $B_n^m \subset B_{n+1}^m$ for all $n, m \in \mathbb{N}$. We claim that $E = \bigcup_{n=1}^{\infty} B_n^m$. Let $x \in E$ and $m \in \mathbb{N}$. Then $(f_n(x))_{n\in\mathbb{N}}$ is convergent to $f(x)$ with respect to the symmetric relative v-topology. This shows that for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $f_n(x) \in (\frac{1}{m}v)^s_\varepsilon(f(x))$ for all $n \geq n_0$. Therefore $f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m}v)$ and $f_n(x) \leq \gamma f(x) + \varepsilon(\frac{1}{m}v)$ for all $n \geq n_0$ and some $1 \leq \gamma \leq 1+\varepsilon$. This yields that $f_n(x) \leq \gamma f(x) + (1+\varepsilon)(\frac{1}{m}v)$ and $f_n(x) \leq \gamma f(x) + (1+\varepsilon)(\frac{1}{m}v)$ for all $n \geq n_0$ and some $1 \leq \gamma \leq 1+\varepsilon$. Now Lemma I.3.1 from [9] shows that $f_n(x) \preceq_v f(x) + \frac{1}{m}v$ and $f(x) \preceq_v f_n(x) + \frac{1}{m}v$ for all $n \geq n_0$. Thus $x \in B_{n_0}^m$.

Then $(E \setminus B_n^m)_{n\in\mathbb{N}}$ is a decreasing sequence of subsets of E, such that $\lim_{n\to\infty} E \setminus B_n^m = \emptyset$. Therefore for every $m \in \mathbb{N}$, $|\theta|(E \setminus B_n^m, v)$ is convergent to $|\theta|(\emptyset, v) = 0$ with respect to the symmetric relative topology of $(\mathcal{Q}, \mathcal{W})$ by the assumption. For $\varepsilon > 0$ and $m \in \mathbb{N}$ we choose n_m such that $|\theta|(E \setminus B_{n_m}^m, v) \leq \frac{\varepsilon}{2^m} w$. We set

$$F = \bigcup_{m=1}^{\infty} E \setminus B_{n_m}^m.$$

Then we have

$$|\theta|(F, v) \leq \sum_{m=1}^{\infty} |\theta|(B_{n_m}^m, v) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} w = \varepsilon w.$$

Also, we have $0 \leq |\theta|(F, v) + \varepsilon w$. Then $|\theta|(F, v) \in w^s_\varepsilon(0)$.

Now, we show that the convergence on $E \setminus F$ is uniform. Let $\delta > 0$. There is $k \in \mathbb{N}$ such that $\frac{2}{k} + \frac{1}{k^2} \leq \delta$. We have

$$E \setminus F = E \setminus (\bigcup_{m=1}^{\infty} E \setminus B_{n_m}^m) = \bigcap_{m=1}^{\infty} B_{n_m}^m \subset B_{n_k}^k$$

Now for each $n \geq n_k$ and every $x \in E \setminus F$ we have $f_n(x) \preceq_v f(x) + \frac{1}{k}v$ and $f(x) \preceq_v f_n(x) + \frac{1}{k}v$. The definition of $\preceq_v$ shows that for $\varepsilon = \frac{1}{k}$ there is $1 \leq \gamma \leq$

$1+\frac{1}{k}$ such that $f_n(x) \leq \gamma(f(x)+\dfrac{1}{k}v)+\dfrac{1}{k}v$ and $f(x) \leq \gamma(f_n(x)+\dfrac{1}{k}v)+\dfrac{1}{k}v$. Therefore $f_n(x) \leq \gamma f(x)+(\dfrac{2}{k}+\dfrac{1}{k^2})v \leq \gamma f(x)+\delta v$ and $f(x) \leq \gamma f_n(x)+(\dfrac{2}{k}+\dfrac{1}{k^2})v \leq \gamma f_n(x)+\delta v$. Since $1 \leq \gamma \leq 1+\dfrac{1}{k} \leq 1+\dfrac{2}{k}+\dfrac{1}{k^2} \leq 1+\delta$, we realize that $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to f on $E \setminus F$, with respect to the symmetric relative topology. □

Remark 2.8. If in the assumptions of Theorem 2.7, $(\mathcal{P},\mathcal{V})$ is a quasi-full locally convex cone, then the theorem holds again. In fact every quasi-full locally convex cone can be embedded in a full locally convex cone as elaborated in ([9], I, 6.2).

Definition 2.9. W say that a sequence $(f_n : X \to \mathcal{P})_{n\in\mathbb{N}}$ of measurable functions is θ-almost everywhere convergent (with respect to the symmetric topology of $(\mathcal{P},\mathcal{V})$) to f, if the set $\{x \in X : f_n(x) \nrightarrow f(x)\}$ is contained in a subset E of X with $\theta_E = 0$.

Definition 2.10. Let $v \in \mathcal{V}$. We say that the sequence $(f_n : X \to \mathcal{P})_{n\in\mathbb{N}}$ of measurable functions is $|\theta|_v$-almost everywhere convergent (with respect to symmetric topology of $(\mathcal{P},\mathcal{V})$) to f, if the set $\{x \in X : f_n(x) \nrightarrow f(x)\}$ is contained in a subset E of X with $|\theta|(E,v) = 0$.

Lemma 2.11. *Let $\mathfrak{R}$ be a σ-ring of subsets of X, $(\mathcal{P},\mathcal{V})$ be a full locally convex cone and $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone. Then*
(a) θ-almost everywhere convergence implies $|\theta|_v$-almost everywhere convergence for each $v \in \mathcal{V}$.
(b) If all elements of $(\mathcal{P},\mathcal{V})$ are bounded and a sequence $(f_n : X \to \mathcal{P})_{n\in\mathbb{N}}$ is $|\theta|_v$-almost everywhere convergent to f for each $v \in \mathcal{V}$, then $(f_n : X \to \mathcal{P})_{n\in\mathbb{N}}$ is θ-almost everywhere convergent to f.

Proof. The assertions are proved by the help of Proposition 2.1. □

Theorem 2.12. *If in the Egoroff theorem (2.7), $f_n \to f$, θ-almost everywhere or $|\theta|_v$-almost everywhere, then the assertion of theorem holds.*

Proof. Suppose $f_n \to f$, θ-almost everywhere, then there is a subset A of E, which is contained in some $B \in \mathfrak{R}$ with $\theta_B = 0$. Now $E\backslash B \in \mathfrak{R}$ and it has bounded v-semivariation. We apply the theorem 2.7 for $E\backslash B$ and obtain a subset F satisfying in definition 2.6. Now clearly f_n is θ-almost uniformly convergent to f on $E\backslash(F \cap B)$. A similar argument yields our claim for $|\theta|_v$-almost everywhere convergence. □

Theorem 2.13. *Let the symmetric relative w-topology of $(\mathcal{Q},\mathcal{W})$ be Hausdorff for each $w \in \mathcal{W}$ and let $(f_n : X \to \mathcal{P})_{n\in\mathbb{N}}$ be a sequence of measurable functions which converges to f, θ-almost uniformly on $E \in \mathfrak{R}$. Then $\{f_n\}_{n\in\mathbb{N}}$, is $|\theta|_v$-almost every where convergent to f for each $v \in \mathcal{V}$.*

Proof. For each $n \in \mathbb{N}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$ there is $F_n = F_n(v, w) \in \mathfrak{R}$ such that $F_n \subset E$ and $|\theta|(F_n, v) \in w^s_{\frac{1}{n}}(0)$ and (f_n) is convergent to f on $E \backslash F_n$. Now, we set $F = \bigcap_{n=1}^{\infty} F_n$. Since $(\mathcal{Q}, \mathcal{W})$ is separated, we have $|\theta|(F, v) = 0$. Clearly, $(f_n(x))_{n \in \mathbb{N}}$ is convergent to $f(x)$ for each $x \in E \backslash F = \bigcup_{n=1}^{\infty} E \backslash F_n$. □

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