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On Generalizations of Hadamard Inequalities for Fractional Integrals

Ghulam Farid*, Atiq Ur Rehman, Moquddsa Zahra

Department of Mathematics
COMSATS University Islamabad
Attock Campus, Pakistan.

E-mail: faridphdsms@hotmail.com
E-mail: atiq@mathcity.org
E-mail: moquddsazahra@gmail.com

Abstract. Fejér Hadamard inequality is generalization of Hadamard inequality. In this paper we prove certain Fejér Hadamard inequalities for k-fractional integrals. We deduce Fejér Hadamard-type inequalities for Riemann-Liouville fractional integrals. Also as a special case Hadamard inequalities for k-fractional as well as fractional integrals are given.

Keywords: Convex functions, Hadamard inequalities, Fejér Hadamard inequality, Riemann-Liouville fractional integrals.

2000 Mathematics subject classification: 26A51, 26A33, 26D10.

1. Introduction

Fractional calculus is a branch of mathematics that deals with the equations concerning integrals and derivatives of fractional orders. The history of fractional calculus is as old as the history of differential calculus. In fact, fractional calculus is a natural extension of standard mathematics. In view of the fact that the commencement of the hypothesis of differential and integral calculus,

*Corresponding Author

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mathematicians such as Euler and Liouville developed their ideas on the calculation of derivatives and integral of non-integers. Possibly the topic would be more suitably called "integration and differentiation of arbitrary order" (see, [4]).

Fractional calculus has lots of applications in the fields of science counting rheology, fluid flow, diffusive transport, electrical networks, electromagnetic theory and probability (see, [3]).

In mathematical analysis inequalities have been fascinating for researchers of all ages (see, [1, 2, 5, 11, 15] and references there in). Now a days fractional integral inequalities especially diverse versions of Hadamard and Ostrowski fractional inequalities have been established (see, [6, 7, 8, 12] and references there in). We are interested to give generalizations of fractional Hadamard inequalities via k-fractional integrals.

In [9] k-fractional Riemann-Liouville integrals are defined.

Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as:

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b, \tag{1.2}$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$$

and $I_{a+}^{0,1} f(x) = I_{b-}^{0,1} f(x) = f(x)$.

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

Following results for k-fractional integrals hold [7].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a,b]$, then the following inequalities for k-fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha, k > 0$.

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Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for k-fractional integral holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(b) + I_{b-}^{\alpha,k}f(a)\right]\right| \tag{1.4}$$
$$\leq \frac{b-a}{2(\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)[|f'(a)|+|f'(b)|]$$

with $\alpha, k > 0$.

There in [7] we remark that for $k = 1$ in above theorems we get the results of [12], and for $\alpha = 1$ along with $k = 1$ we get the classical Hadamard inequality. In [8] the following results related to Fejér Hadamard-type inequalities for fractional integrals are given.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a convex function with $a < b$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\left[I_{a+}^{\alpha}g(b) + I_{b-}^{\alpha}g(a)\right]$$
$$\leq [I_{a+}^{\alpha}(fg)(b) + I_{b-}^{\alpha}(fg)(a)]$$
$$\leq \frac{f(a)+f(b)}{2}\left[I_{a+}^{\alpha}g(b) + I_{b-}^{\alpha}g(a)\right]$$

with $\alpha > 0$.

Theorem 1.4. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)[I_{a+}^{\alpha}g(b) + I_{b-}^{\alpha}g(a)] - [I_{a+}^{\alpha}(fg)(b) + I_{b-}^{\alpha}(fg)(a)]\right|$$
$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[|f'(a)|+|f'(b)|]$$

with $\alpha > 0$.

Theorem 1.5. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I_{a+}^{\alpha}g(b) + I_{b-}^{\alpha}g(a)\right] - [I_{a+}^{\alpha}(fg)(b) + I_{b-}^{\alpha}(fg)(a)]\right|$$
$$\leq \frac{2(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)(b-a)^{\frac{1}{q}}}\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

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Theorem 1.6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequalities for fractional integrals hold:*

$$(i)\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I^{\alpha}_{a+}g(b)+I^{\alpha}_{b-}g(a)\right]-\left[I^{\alpha}_{a+}(fg)(b)+I^{\alpha}_{b-}(fg)(a)\right]\right|$$
$$\leq \frac{2^{\frac{1}{p}}(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

with $\alpha > 0$.

$$(ii)\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I^{\alpha}_{a+}g(b)+I^{\alpha}_{b-}g(a)\right]-\left[I^{\alpha}_{a+}(fg)(b)+I^{\alpha}_{b-}(fg)(a)\right]\right|$$
$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

with $0 < \alpha \leq 1$, where $\frac{1}{p}+\frac{1}{q} = 1$.

In this paper we give Fejér Hadamard inequality for k-fractional integrals and note that the results in [8] are special cases of these results. Also we present new results as generalizations of Hadamard inequalities for fractional integrals and deduce some results of [12, 7].

2. Main Results

The following lemma is given in [14].

Lemma 2.1. *For $0 < \lambda \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\lambda} - b^{\lambda}| \leq (b-a)^{\lambda}.$$

Here first we prove the following result.

Lemma 2.2. *If $g : [a, b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$ with $a < b$, then*

$$I^{\alpha,k}_{a+}g(b) = I^{\alpha,k}_{b-}g(a) = \frac{1}{2}[I^{\alpha,k}_{a+}g(b)+I^{\alpha,k}_{b-}g(a)]$$

with $\alpha, k > 0$.

Proof. By symmetricity of g we have $g(a+b-x) = g(x)$, where $x \in [a, b]$. Setting $x = a+b-x$ in the following integral we have

$$\begin{aligned} I^{\alpha,k}_{a+}g(b) &= \frac{1}{k\Gamma_k(\alpha)}\int_a^b (b-x)^{\frac{\alpha}{k}-1}g(x)dx \\ &= \frac{1}{k\Gamma_k(\alpha)}\int_a^b (x-a)^{\frac{\alpha}{k}-1}g(a+b-x)dx \\ &= \frac{1}{k\Gamma_k(\alpha)}\int_a^b (x-a)^{\frac{\alpha}{k}-1}g(x)dx \\ &= I^{\alpha,k}_{b-}g(a). \end{aligned}$$

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□

Using the above lemma we prove the following results.

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a convex function with $a < b$. If $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following inequalities for k-fractional integrals hold:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[I_{a+}^{\alpha,k}g(b) + I_{b-}^{\alpha,k}g(a)\right] \\ & \leq [I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a)] \\ & \leq \frac{f(a)+f(b)}{2}\left[I_{a+}^{\alpha,k}g(b) + I_{b-}^{\alpha,k}g(a)\right] \end{aligned} \tag{2.1}$$

with $\alpha, k > 0$.

Proof. By convexity of f we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \\ &\leq \frac{f(ta+(1-t)b)+f(tb+(1-t)a)}{2}, \end{aligned} \tag{2.2}$$

where $t \in [0, 1]$. Multiplying both sides of the above inequality with $2t^{\frac{\alpha}{k}-1}g(tb+(1-t)a)$ and integrating the resulting inequality over $[0, 1]$ we have,

$$\begin{aligned} & 2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}g(tb+(1-t)a)dt \\ & \leq \int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)b)g(tb+(1-t)a)dt \\ & + \int_0^1 t^{\frac{\alpha}{k}-1}f(tb+(1-t)a)g(tb+(1-t)a)dt. \end{aligned}$$

Putting $x = tb + (1-t)a$ we get

$$\begin{aligned} & \frac{2}{(b-a)^{\frac{\alpha}{k}}}f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\frac{\alpha}{k}-1}g(x)dx \\ & \leq \frac{1}{(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (x-a)^{\frac{\alpha}{k}-1}f(a+b-x)g(x)dx + \int_a^b (x-a)^{\frac{\alpha}{k}-1}f(x)g(x)dx\right] \\ & = \frac{1}{(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1}f(x)g(a+b-x)dx + \int_a^b (x-a)^{\frac{\alpha}{k}-1}f(x)g(x)dx\right] \\ & = \frac{1}{(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1}f(x)g(x)dx + \int_a^b (x-a)^{\frac{\alpha}{k}-1}f(x)g(x)dx\right]. \end{aligned}$$

By using Lemma 2.2, we get the first inequality of (2.1).

For the second inequality of (2.1) convexity of f gives

$$f(ta+(1-t)b) + f(tb+(1-t)a) \leq f(a) + f(b),$$

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where $t \in [0,1]$. Multiplying both sides of the above inequality with $t^{\frac{\alpha}{k}-1}g(tb+(1-t)a)$ and integrating the resulting inequality over $[0,1]$ we get

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1} f(ta+(1-t)b)g(tb+(1-t)a)dt \\
&+ \int_0^1 t^{\frac{\alpha}{k}-1} f(tb+(1-t)a)g(tb+(1-t)a)dt \\
&\leq (f(a)+f(b)) \int_0^1 t^{\frac{\alpha}{k}-1} g(tb+(1-t)a)dt,
\end{aligned}$$

which after some computations the result follows. □

Remark 2.4. If we take $k=1$ in Theorem 2.3, then we get Theorem 1.3. If we take $\alpha = 1$ along with $k=1$, then we get [8, Theorem 1]. If we take $g(x)=1$ in Theorem 2.3, then we get inequality (1.3). If we take $g(x)=1$ along with $k=1$ in Theorem 2.3, then we get [12, Theorem 2].

Next we need the following lemma.

Lemma 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f' \in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then the following equality for k-fractional integrals hold:*

$$\begin{aligned}
&\left(\frac{f(a)+f(b)}{2}\right)[I_{a+}^{\alpha,k}g(b)+I_{b-}^{\alpha,k}g(a)] - [I_{a+}^{\alpha,k}(fg)(b)+I_{b-}^{\alpha,k}(fg)(a)] \\
&= \frac{1}{k\Gamma_k(\alpha)}\int_a^b \left[\int_a^t (b-s)^{\frac{\alpha}{k}-1}g(s)ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1}g(s)ds\right] f'(t)dt
\end{aligned}$$

with $\alpha, k > 0$.

Proof. Note that

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}\int_a^b \left[\int_a^t (b-s)^{\frac{\alpha}{k}-1}g(s)ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1}g(s)ds\right] f'(t)dt \\
&= \frac{1}{k\Gamma_k(\alpha)}\left[\int_a^b \left(\int_a^t (b-s)^{\frac{\alpha}{k}-1}g(s)ds\right) f'(t)dt \right. \\
&\left. + \int_a^b \left(-\int_t^b (s-a)^{\frac{\alpha}{k}-1}g(s)ds\right) f'(t)dt\right].
\end{aligned} \tag{2.3}$$

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By simple calculations one can get

$$\begin{aligned}
&\int_a^b \left(\int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds \right) f'(t)dt \\
&= \left[\left(\int_a^b (b-s)^{\frac{\alpha}{k}-1} g(s)ds \right) f(b) - \int_a^b (b-t)^{\frac{\alpha}{k}-1} fg(t)dt \right] \\
&= k\Gamma_k(\alpha) \left[f(b) I_{a+}^{\alpha,k} g(b) - I_{a+}^{\alpha,k}(fg)(b) \right] \\
&= k\Gamma_k(\alpha) \left[\frac{f(b)}{2} [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - I_{a+}^{\alpha,k} fg(b) \right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(-\int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right) f'(t)dt \\
&= \left(\int_a^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right) f(a) - \int_a^b (t-a)^{\frac{\alpha}{k}-1} (fg)(t)dt \\
&= k\Gamma_k(\alpha) \left[\frac{f(a)}{2} [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - I_{a+}^{\alpha,k} fg(a)) \right].
\end{aligned}$$

Hence using (2.3) implies the result. □

Theorem 2.6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o the interior of I, and $f' \in L[a,b], a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for k-fractional integrals holds:*

$$\begin{aligned}
&\left| \left(\frac{f(a)+f(b)}{2} \right) [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - [I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a)] \right| \\
&\leq \frac{(b-a)^{\frac{\alpha}{k}+1} \|g\|_\infty}{(\frac{\alpha}{k}+1)\Gamma_k(\alpha+k)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) [|f'(a)| + |f'(b)|]
\end{aligned}$$

with $\alpha, k > 0$.

Proof. Using Lemma 2.5 we have

$$\begin{aligned}
&\left| \left(\frac{f(a)+f(b)}{2} \right) [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - [I_{a+}^{\alpha,k}(fg)(b) + I_{b-}^{\alpha,k}(fg)(a)] \right| \\
&\leq \frac{1}{k\Gamma_k(\alpha)} \int_a^b \left| \int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right| |f'(t)|dt. \quad (2.4)
\end{aligned}$$

Using convexity of $|f'|$ we have

$$|f'(t)| \leq \frac{b-t}{t-a}|f'(a)| + \frac{t-a}{b-a}|f'(b)|, \quad (2.5)$$

where $t \in [a,b]$.

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From symmetricity of g we have

$$\int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds = \int_a^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(a+b-s)ds$$
$$= \int_a^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(s)ds.$$

This gives

$$\left| \int_a^t (b-s)^{\frac{\alpha}{k}-1} g(s)ds - \int_t^b (s-a)^{\frac{\alpha}{k}-1} g(s)ds \right| \tag{2.6}$$
$$= \left| \int_t^{a+b-t} (b-s)^{\frac{\alpha}{k}-1} g(s)ds \right| \leq \begin{cases} \int_t^{a+b-t} |(b-s)^{\frac{\alpha}{k}-1} g(s)|ds, & t \in [a, \frac{a+b}{2}] \\ \int_{a+b-t}^t |(b-s)^{\frac{\alpha}{k}-1} g(s)|ds, & t \in [\frac{a+b}{2}, b]. \end{cases}$$

By virtue of (2.4), (2.5), (2.6), we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a) \right] - \left[I_{a+}^{\alpha,k} (fg)(b) + I_{b-}^{\alpha,k} (fg)(a) \right] \right|$$
$$\leq \frac{1}{k\Gamma_k(\alpha)} \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \left| (b-s)^{\frac{\alpha}{k}-1} g(s) \right| ds \right) \left(\frac{b-t}{t-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \right.$$
$$\left. + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \left| (b-s)^{\frac{\alpha}{k}-1} g(s) ds \right| \right) \left(\frac{b-t}{t-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \right]$$
$$\leq \frac{\|g\|_\infty}{\Gamma_k(\alpha+k)(b-a)} \left[\int_a^{\frac{a+b}{2}} \left((b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}} \right) \left((b-t)|f'(a)| + (t-a)|f'(b)| \right) dt \right.$$
$$\left. + \int_{\frac{a+b}{2}}^b \left((t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}} \right) \left((b-t)|f'(a)| + (t-a)|f'(b)| \right) dt \right]. \tag{2.7}$$

One can have

$$\int_a^{\frac{a+b}{2}} ((b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}})(b-t)dt = \int_{\frac{a+b}{2}}^b ((t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}})(t-a)dt$$
$$= \frac{(b-a)^{\frac{\alpha}{k}+2}}{\frac{\alpha}{k}+1} \left(\frac{\frac{\alpha}{k}+1}{\frac{\alpha}{k}+1} - \frac{1}{2^{\frac{\alpha}{k}+1}} \right) \tag{2.8}$$

and

$$\int_a^{\frac{a+b}{2}} ((b-t)^{\frac{\alpha}{k}} - (t-a)^{\frac{\alpha}{k}})(t-a)dt = \int_{\frac{a+b}{2}}^b ((t-a)^{\frac{\alpha}{k}} - (b-t)^{\frac{\alpha}{k}})(b-t)dt$$
$$= \frac{(b-a)^{\frac{\alpha}{k}+2}}{(\frac{\alpha}{k}+1)} \left(\frac{1}{\frac{\alpha}{k}+2} - \frac{1}{2^{\frac{\alpha}{k}+1}} \right). \tag{2.9}$$

Using (2.8), (2.9) in (2.7) we get required result. □

Remark 2.7. If we take $k = 1$, then we get Theorem 1.4. If we take $g(x) = 1$ in the above theorem, then we get inequality (1.4). If we take $g(x) = 1$ along with $k = 1$ in above theorem, then we get [12, Theorem 3].

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Theorem 2.8. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o the interior of I, and $f' \in L[a,b], a,b \in I^o$ with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for k-fractional integrals holds:*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I^{\alpha,k}_{a+}g(b)+I^{\alpha,k}_{b-}g(a)\right]-\left[I^{\alpha,k}_{a+}(fg)(b)+I^{\alpha,k}_{b-}(fg)(a)\right]\right|$$
$$\leq \frac{2(b-a)^{\frac{\alpha}{k}+1}\|g\|_\infty}{(\frac{\alpha}{k}+1)\Gamma_k(\alpha+k)(b-a)^{\frac{1}{q}}}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

with $\alpha, k > 0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Using Lemma 2.5, Hölder inequality, inequality (2.6) and convexity of $|f'|^q$ we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I^{\alpha,k}_{a+}g(b)+I^{\alpha,k}_{b-}g(a)\right]-\left[I^{\alpha,k}_{a+}(fg)(b)+I^{\alpha,k}_{b-}(fg)(a)\right]\right|$$
$$\leq \frac{1}{k\Gamma_k(\alpha)}\left[\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|dt\right]^{1-\frac{1}{q}}$$
$$\left[\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right||f'(t)|^q dt\right]^{\frac{1}{q}}$$
$$\leq \frac{1}{k\Gamma_k(\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right)dt\right]^{1-\frac{1}{q}}$$
$$\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds\right)|f'(t)|^q dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right)|f'(t)|^q dt\right]^{\frac{1}{q}}$$
$$\leq \frac{\|g\|_\infty}{k\Gamma_k(\alpha)}\left[\left(\frac{2k(b-a)^{\frac{\alpha}{k}+1}}{\alpha(\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right)^{1-\frac{1}{q}}\right.$$
$$\left.\left(\frac{k(|f'(a)|^q+|f'(b)|^q)(b-a)^{\frac{\alpha}{k}+1}}{\alpha(\frac{\alpha}{k}+1)(b-a)^{\frac{1}{q}}}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right)^{\frac{1}{q}}\right].$$

From which after a little computation one can have required result. □

Theorem 2.9. *Let $f : I \to \mathbb{R}$ be a differential mapping on I^o the interior of I, and $f' \in L[a,b], a,b \in I^o$ with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a,b]$ and*

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$g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then following inequalities for k-fractional integrals hold:

$$(i)\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I_{a+}^{\alpha,k}g(b)+I_{b-}^{\alpha,k}g(a)\right]-\left[I_{a+}^{\alpha,k}(fg)(b)+I_{b-}^{\alpha,k}(fg)(a)\right]\right|$$
$$\leq\frac{2^{\frac{1}{p}}(b-a)^{\frac{\alpha}{k}+1}\|g\|_\infty}{(\frac{\alpha p}{k}+1)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

with $\alpha,k>0$.

$$(ii)\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I_{a+}^{\alpha,k}g(b)+I_{b-}^{\alpha,k}g(a)\right]-[I_{a+}^{\alpha,k}(fg)(b)+I_{b-}^{\alpha,k}(fg)(a)]\right|$$
$$\leq\frac{(b-a)^{\frac{\alpha}{k}+1}\|g\|_\infty}{(\frac{\alpha p}{k}+1)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\tag{2.10}$$

with $0<\alpha\leq 1$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Using Lemma 2.5, Hölder inequality, inequality (2.6) and convexity of $|f'|^q$ we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I_{a+}^{\alpha,k}g(b)+I_{b-}^{\alpha,k}g(a)\right]-[I_{a+}^{\alpha,k}(fg)(b)+I_{b-}^{\alpha,k}(fg)(a)]\right|$$
$$\leq\frac{1}{k\Gamma_k(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|f'(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{1}{k\Gamma_k(\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right)dt\right]^{\frac{1}{p}}\left[\int_a^b\left(\frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$
$$\leq\frac{\|g\|_\infty}{k\Gamma_k(\alpha+k)}\left[\int_a^{\frac{a+b}{2}}\left((b-t)^{\frac{\alpha}{k}}-(t-a)^{\frac{\alpha}{k}}\right)^p dt+\int_{\frac{a+b}{2}}^b\left((t-a)^{\frac{\alpha}{k}}-(b-t)^{\frac{\alpha}{k}}\right)^p dt\right]^{\frac{1}{p}}$$
$$\left[\int_a^b\left(\frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}.\tag{2.11}$$

Now

$$(A-B)^q\leq A^q-B^q,\quad A\geq B\geq 0$$

gives

$$[(b-t)^{\frac{\alpha}{k}}-(t-a)^{\frac{\alpha}{k}}]^p\leq(b-t)^{\frac{\alpha p}{k}}-(t-a)^{\frac{\alpha p}{k}}\tag{2.12}$$

for $t\in[a,\frac{a+b}{2}]$, and

$$[(t-a)^{\frac{\alpha}{k}}-(b-t)^{\frac{\alpha}{k}}]^p\leq(t-a)^{\frac{\alpha p}{k}}-(b-t)^{\frac{\alpha p}{k}}\tag{2.13}$$

for $t\in[\frac{a+b}{2},b]$.

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Using (2.12) and (2.13) in inequality (2.11) and solving we get required result.

For (2.10) use (2.11) and Lemma 2.1. □

Remark 2.10. If we take $k = 1$ in above theorem we get Theorem 1.6.

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References

1. Y.-M. Chu, M. Adil Khan, T. Khan, T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.*, **9**(6), (2016), 4305-4316.
2. Y.-M. Chu, G.-D. Wang, X.-H. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.*, **13**(4), (2010), 725-731.
3. M. Dalir, M. Bashour, Applications of fractional calculus, *Appl. Math. Sci.*, **4**(21), (2010), 1021–1032.
4. S. A. David, J. L. Linares, E. M. J. A. Pallone, Fractional order calculus: Historical apologia, basic concepts and some applications, Rev. Bras. Ensino Fís. **33**(4), (2011), Article ID 4302, 1–7.
5. S. S. Dragomir, New Jensen and Ostrowski type inequalities for general Lebesgue integral with applications, *Iran. J. Math. Sci. Inform*, **11**(2), (2016), 1–22.
6. G. Farid, Some new Ostrowski type inequalities via fractional integrals, *Int. J. Anal. App.*, **14**(1), (2017), 64-68.
7. G. Farid, A. U. Rehman, M. Zahra, On Hadamard inequalities for k-Fractional integrals, *Nonlinear Funct. Anal. Appl.*, **21**(3), (2016), 463–478.
8. I. İşcan, Hermite Hadamard Fejér type inequalities for convex functions via fractional integrals, *Stud. Univ. Babeş-Bolyai Math.*, **60**(3), (2015), 355–366.
9. S. Mubeen, G. M. Habibullah, k-Fractional integrals and applications, *Int. J. Contemp. Math. Sci.*, **7**, (2012), 89–94.
10. J. Pečarić, F. Proechan, Y. L. Tong, Convex functions, partial ordering and statistical applications, Academic Press, New York, 1991.
11. M. Z. Sarikaya, A. Saglamb, H. Yildirim, On generalization of Čebyšev type inequalities, *Iran. J. Math. Sci. Inform.*, **5**(1), (2010), 41–48.
12. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *J. Math. Comput. Model.*, **57**, (2013), 2403–2407.
13. L. Fejér, Uberdie fourierreihen II, *Math. Naturwise. Anz Ungar. Akad.*, Wiss, **24**, (1906), 369–390. (In Hungarian)
14. J. Wang, C. Zhu, Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, *J. Inequal. Appl.*, **2013**, (2013), Article ID 325, 15pp.
15. X-M. Zhang, Y.-M. Chu, X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, *J. Inequal. Appl.*, **2010** (2010), Article ID 507560, 11 pages.

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