Balanced Degree-Magic Labelings of Complete Bipartite Graphs under Binary Operations

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ABSTRACT. A graph is called supermagic if there is a labeling of edges where the edges are labeled with consecutive distinct positive integers such that the sum of the labels of all edges incident with any vertex is constant. A graph \( G \) is called degree-magic if there is a labeling of the edges by integers 1, 2, ..., \( |E(G)| \) such that the sum of the labels of the edges incident with any vertex \( v \) is equal to \( (1 + |E(G)|)\deg(v)/2 \). Degree-magic graphs extend supermagic regular graphs. In this paper we find the necessary and sufficient conditions for the existence of balanced degree-magic labelings of graphs obtained by taking the join, composition, Cartesian product, tensor product and strong product of complete bipartite graphs.

Keywords: Complete bipartite graphs, Supermagic graphs, Degree-magic graphs, Balanced degree-magic graphs.

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1. Introduction

We consider simple graphs without isolated vertices. If $G$ is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of $G$, respectively. Cardinalities of these sets are called the order and size of $G$.

Let a graph $G$ and a mapping $f$ from $E(G)$ into positive integers be given. The index mapping of $f$ is the mapping $f^*$ from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every} \quad v \in V(G),$$

where $\eta(v, e)$ is equal to 1 when $e$ is an edge incident with a vertex $v$, and 0 otherwise. An injective mapping $f$ from $E(G)$ into positive integers is called a magic labeling of $G$ for an index $\lambda$ if its index mapping $f^*$ satisfies

$$f^*(v) = \lambda \quad \text{for all} \quad v \in V(G).$$

A magic labeling $f$ of a graph $G$ is called a supermagic labeling if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph $G$ is supermagic (magic) whenever a supermagic (magic) labeling of $G$ exists.

A bijective mapping $f$ from $E(G)$ into $\{1, 2, \ldots, |E(G)|\}$ is called a degree-magic labeling (or only $d$-magic labeling) of a graph $G$ if its index mapping $f^*$ satisfies

$$f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all} \quad v \in V(G).$$

A $d$-magic labeling $f$ of a graph $G$ is called balanced if for all $v \in V(G)$, the following equation is satisfied

$$|\{e \in E(G) : \eta(v, e) = 1, f(e) \leq \lfloor|E(G)|/2\rfloor\}| = |\{e \in E(G) : \eta(v, e) = 1, f(e) > \lfloor|E(G)|/2\rfloor\}|.$$

We say that a graph $G$ is degree-magic (balanced degree-magic) or only $d$-magic when a $d$-magic (balanced $d$-magic) labeling of $G$ exists.

The concept of magic graphs was introduced by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. There are now many papers published on magic and supermagic graphs; see [6, 7, 10] for more comprehensive references. The concept of degree-magic graphs was then introduced by Bezegová and Ivančo [2] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degree-magic and balanced degree-magic complete bipartite graphs in [2]. They also characterized degree-magic complete tripartite graphs in [4]. Some of these concepts are investigated in [1, 3, 5]. We will hereinafter use the auxiliary results from these studies.
Theorem 1.1. [2] Let $G$ be a regular graph. Then $G$ is supermagic if and only if it is $d$-magic.

Theorem 1.2. [2] Let $G$ be a $d$-magic graph of even size. Then every vertex of $G$ has an even degree and every component of $G$ has an even size.

Theorem 1.3. [2] Let $G$ be a balanced $d$-magic graph. Then $G$ has an even number of edges and every vertex has an even degree.

Theorem 1.4. [2] Let $G$ be a $d$-magic graph having a half-factor. Then $2G$ is a balanced $d$-magic graph.

Theorem 1.5. [2] Let $H_1$ and $H_2$ be edge-disjoint subgraphs of a graph $G$ which form its decomposition. If $H_1$ is $d$-magic and $H_2$ is balanced $d$-magic, then $G$ is a balanced $d$-magic graph. Moreover, if $H_1$ and $H_2$ are both balanced $d$-magic, then $G$ is a balanced $d$-magic graph.

Proposition 1.6. [2] For $p, q > 1$, the complete bipartite graph $K_{p,q}$ is $d$-magic if and only if $p \equiv q \equiv 0 \pmod{2}$ and $(p, q) \neq (2, 2)$.

Theorem 1.7. [2] The complete bipartite graph $K_{p,q}$ is balanced $d$-magic if and only if the following statements hold:

(i) $p \equiv q \equiv 0 \pmod{2}$;

(ii) if $p \equiv q \equiv 2 \pmod{4}$, then $\min\{p, q\} \geq 6$.

Lemma 1.8. [4] Let $m, n$ and $o$ be even positive integers. Then the complete tripartite graph $K_{m,n,o}$ is balanced $d$-magic.

2. Balanced Degree-Magic Labelings in the Join of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the join of graphs $G$ and $H$, denoted by $G \cup H$, consists of $G \cup H$ and all edges joining a vertex of $G$ and a vertex of $H$. For any positive integers $p$ and $q$, we consider the join $K_{p,q} + K_{p,q}$ of complete bipartite graphs. Let $K_{p,q} + K_{p,q}$ be a $d$-magic graph. Since $\deg(v) = p + 2q$ or $p + q$ and $f^*(v) = (2pq + (p + q)^2 + 1)\deg(v)/2$ for any $v \in V(K_{p,q} + K_{p,q})$, we have

Proposition 2.1. Let $K_{p,q} + K_{p,q}$ be a $d$-magic graph. Then $p$ or $q$ is even.

Proposition 2.2. Let $K_{p,q} + K_{p,q}$ be a balanced $d$-magic graph. Then both $p$ and $q$ are even.

Proposition 2.3. Let $p$ and $q$ be even positive integers. Then $K_{p+q,p+q}$ is a balanced $d$-magic graph.

Proof. Applying Theorem 1.7, $K_{p+q,p+q}$ is a balanced $d$-magic graph. \qed

In the next result we show a sufficient condition for the existence of balanced $d$-magic labelings of the join of complete bipartite graphs $K_{p,q} + K_{p,q}$. 
Figure 1. A balanced d-magic graph $K_{2,6} + K_{2,6}$ with 16 vertices and 88 edges.

Theorem 2.4. Let $p$ and $q$ be even positive integers. Then $K_{p,q} + K_{p,q}$ is a balanced d-magic graph.

Proof. Let $p$ and $q$ be even positive integers. We consider the following two cases:

Case I. If $(p, q) = (2, 2)$, the graph $K_{2,2} + K_{2,2}$ is decomposable into three balanced d-magic subgraphs isomorphic to $K_{2,4}$. According to Theorem 1.5, $K_{2,2} + K_{2,2}$ is a balanced d-magic graph.

Case II. If $(p, q) \neq (2, 2)$, then $K_{p,q} + K_{p,q}$ is balanced d-magic by Proposition 2.3, and $2K_{p,q}$ is balanced d-magic by Theorem 1.4. Since $K_{p,q} + K_{p,q}$ is the graph such that $K_{p+q,p+q}$ and $2K_{p,q}$ form its decomposition, by Theorem 1.5, $K_{p,q} + K_{p,q}$ is a balanced d-magic graph. \qed

We know that $K_{2,6}$ is d-magic, but it is not balanced d-magic. Applying Theorem 2.4, we can construct a balanced d-magic graph $K_{2,6} + K_{2,6}$ (see Figure 1) with the labels on edges of $K_{2,6} + K_{2,6}$ in Table 2.

We will now generalize to find the necessary and sufficient conditions for the existence of balanced d-magic labelings of the join of complete bipartite graphs in a general form. For any positive integers $p, q, r$ and $s$, we consider the join $K_{p,q} + K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} + K_{r,s}$ be a d-magic graph. Since $\deg(v) = p + r + s, q + r + s, p + q + r$ or $p + q + s$ and $f^*(v) = (pq + (p + q)(r + s) + rs + 1) \deg(v)/2$ for any $v \in V(K_{p,q} + K_{r,s})$, we have

Proposition 2.5. Let $K_{p,q} + K_{r,s}$ be a d-magic graph. Then the following conditions hold:

(i) only one of $p, q, r$ and $s$ is even or
(ii) only two of $p, q, r$ and $s$ are even or
(iii) all of $p, q, r$ and $s$ are even.
Table 1. The labels on edges of balanced d-magic graph $K_{2,6} + K_{2,6}$.

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**Proposition 2.6.** Let $K_{p,q} + K_{r,s}$ be a balanced d-magic graph. Then $p, q, r$ and $s$ are even.

Now we are able to show a sufficient condition for the existence of balanced d-magic labelings of the join of complete bipartite graphs $K_{p,q} + K_{r,s}$.

**Theorem 2.7.** Let $p, q, r$ and $s$ be even positive integers. Then $K_{p,q} + K_{r,s}$ is a balanced d-magic graph.

**Proof.** Let $p, q, r$ and $s$ be even positive integers. We consider the following two cases:

**Case I.** If at least one of $p, q, r$ and $s$ is not congruent to 2 modulo 4. Suppose that $p$ is not congruent to 2 modulo 4. Thus, $K_{p,q}$ is balanced d-magic by Theorem 1.7. Since $r, s$ and $p + q$ are even, $K_{r,s,p+q}$ is balanced d-magic by Lemma 1.8. The graph $K_{p,q} + K_{r,s}$ is decomposable into two balanced d-magic subgraphs isomorphic to $K_{p,q}$ and $K_{r,s,p+q}$. According to Theorem 1.5, $K_{p,q} + K_{r,s}$ is a balanced d-magic graph.

**Case II.** If $p, q, r$ and $s$ are congruent to 2 modulo 4. Thus $q + r, q + s$ and $p + q$ are not congruent to 2 modulo 4. By Theorem 1.7, $K_{p,q+r}, K_{r,q+s}$ and $K_{s,p+q}$ are balanced d-magic. The graph $K_{p,q} + K_{r,s}$ is decomposable into three balanced d-magic subgraphs isomorphic to $K_{p,q+r}, K_{r,q+s}$ and $K_{s,p+q}$. According to Theorem 1.5, $K_{p,q} + K_{r,s}$ is a balanced d-magic graph. □

**Corollary 2.8.** Let $p, q, r$ and $s$ be even positive integers. If $p = q = r = s$, then $K_{p,q} + K_{r,s}$ is a supermagic graph.

**Proof.** Applying Theorems 1.1 and 2.7. □
Since 4 is not congruent to 2 modulo 4, applying Theorem 2.7, a balanced d-magic graph $K_{2,4} + K_{2,10}$ is constructed (see Figure 2), and the labels on edges of $K_{2,4} + K_{2,10}$ are shown in Table 2.

3. Balanced Degree-Magic Labelings in the Composition of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the composition of graphs $G$ and $H$, denoted by $G \cdot H$, is a graph such that the vertex set of $G \cdot H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \cdot H$ if and only if either $u$ is adjacent with $x$ in $G$ or $u = x$ and $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the composition $K_{p,q} \cdot K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \cdot K_{r,s}$ be a d-magic graph. Since $\text{deg}(v)$ is $(r + s)p + r$, $(r + s)p + s$, $(r + s)q + r$ or $(r + s)q + s$ and

Table 2. The labels on edges of balanced d-magic graph $K_{2,4} + K_{2,10}$.

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\[ f^*(v) = (pq(r + s)^2 + rs(p + q) + 1) \deg(v)/2 \]

for any \( v \in V(K_{p,q} \times K_{r,s}) \), we have

**Proposition 3.1.** Let \( K_{p,q} \cdot K_{r,s} \) be a d-magic graph. Then the following conditions hold:

(i) only one of \( p, q, r \) and \( s \) is even or

(ii) at least both \( r \) and \( s \) are even.

**Proposition 3.2.** Let \( K_{p,q} \cdot K_{r,s} \) be a balanced d-magic graph. Then at least both \( r \) and \( s \) are even.

In the next result we find a sufficient condition for the existence of balanced d-magic labelings of the composition of complete bipartite graphs \( K_{p,q} \cdot K_{r,s} \).

**Theorem 3.3.** Let \( p \) and \( q \) be positive integers, and let \( r \) and \( s \) be even positive integers. Then \( K_{p,q} \cdot K_{r,s} \) is a balanced d-magic graph.

**Proof.** Let \( p \) and \( q \) be positive integers, and let \( k = \min\{p, q\} \) and \( h = \max\{p, q\} \). It is clear that \( K_{r+s,r+s}, K_{r,s} \) and \( K_{r,s,r+s} \) are balanced d-magic by Proposition 2.3, Theorem 2.4 and Lemma 1.8, respectively. The graph \( K_{p,q} \cdot K_{r,s} \) is decomposable into \( k \) balanced d-magic subgraphs isomorphic to \( K_{r+s,r+s}, h(k-1) \) balanced d-magic subgraphs isomorphic to \( K_{r+s,r+s} \) and \( h-k \) balanced d-magic subgraphs isomorphic to \( K_{r,s,r+s} \). According to Theorem 1.5, \( K_{p,q} \cdot K_{r,s} \) is a balanced d-magic graph.

Notice that the graph composition \( K_{p,q} \cdot K_{r,s} \) is naturally nonisomorphic to \( K_{r,s} \cdot K_{p,q} \) except for the case \((p, q) = (r, s)\).

**Corollary 3.4.** Let \( p \) and \( q \) be positive integers, and let \( r \) and \( s \) be even positive integers. If \( p = q \) and \( r = s \), then \( K_{p,q} \cdot K_{r,s} \) is a supermagic graph.

**Proof.** Applying Theorems 1.1 and 3.3.

The following example is a balanced d-magic graph \( K_{1,2} \cdot K_{2,2} \) (see Figure 3) with the labels on edges of \( K_{1,2} \cdot K_{2,2} \) in Table 3.

4. Balanced Degree-Magic Labelings in the Cartesian Product of Complete Bipartite Graphs

For two vertex-disjoint graphs \( G \) and \( H \), the Cartesian product of graphs \( G \) and \( H \), denoted by \( G \times H \), is a graph such that the vertex set of \( G \times H \) is the Cartesian product \( V(G) \times V(H) \) and any two vertices \((u, v)\) and \((x, y)\) are adjacent in \( G \times H \) if and only if either \( u = x \) and \( v \) is adjacent with \( y \) in \( H \) or \( v = y \) and \( u \) is adjacent with \( x \) in \( G \). For any positive integers \( p, q, r \) and \( s \), we consider the Cartesian product \( K_{p,q} \times K_{r,s} \) of complete bipartite graphs. Let \( K_{p,q} \times K_{r,s} \) be a d-magic graph. Since \( \deg(v) \) is \( p + r, p + s, q + r \) or \( q + s \) and \( f^*(v) = (pq(r + s) + rs(p + q) + 1) \deg(v)/2 \) for any \( v \in V(K_{p,q} \times K_{r,s}) \), we have
Figure 3. A balanced d-magic graph \( K_{1,2} \cdot K_{2,2} \) with 12 vertices and 44 edges.

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<td>3</td>
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<td>( f_2 )</td>
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<td>-</td>
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</tr>
</tbody>
</table>

Table 3. The labels on edges of balanced d-magic graph \( K_{1,2} \cdot K_{2,2} \).

**Proposition 4.1.** Let \( K_{p,q} \times K_{r,s} \) be a d-magic graph. Then the following conditions hold:
(i) only one of \( p, q, r \) and \( s \) is even or
(ii) all of \( p, q, r \) and \( s \) are either odd or even.

**Proposition 4.2.** Let \( K_{p,q} \times K_{r,s} \) be a balanced d-magic graph. Then \( p, q, r \) and \( s \) are either odd or even.

In the next result we are able to find a sufficient condition for the existence of balanced d-magic labelings of the Cartesian product of complete bipartite graphs \( K_{p,q} \times K_{r,s} \).

**Theorem 4.3.** Let \( p, q, r \) and \( s \) be even positive integers with \( (p, q) \neq (2, 2) \) and \( (r, s) \neq (2, 2) \). Then \( K_{p,q} \times K_{r,s} \) is a balanced d-magic graph.

**Proof.** Let \( p, q, r \) and \( s \) be even positive integers with \( (p, q) \neq (2, 2) \) and \( (r, s) \neq (2, 2) \). Since \( K_{p,q} \) and \( K_{r,s} \) are d-magic by Proposition 1.6, \( 2K_{p,q} \) and \( 2K_{r,s} \) are balanced d-magic by Theorem 1.4. The graph \( K_{p,q} \times K_{r,s} \) is decomposable into \((r + s)/2\) balanced d-magic subgraphs isomorphic to \( 2K_{p,q} \) and \((p + q)/2\)
balanced d-magic subgraphs isomorphic to $2K_{r,s}$. According to Theorem 1.5, $K_{p,q} \times K_{r,s}$ is a balanced d-magic graph.

Observe that the Cartesian product graph $K_{p,q} \times K_{r,s}$ is naturally isomorphic to $K_{r,s} \times K_{p,q}$.

**Corollary 4.4.** Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. If $p = q$ and $r = s$, then $K_{p,q} \times K_{r,s}$ is a supermagic graph.

**Proof.** Applying Theorems 1.1 and 4.3.

The following example is a balanced d-magic graph $K_{2,4} \times K_{2,4}$ (see Figure 4), and the labels on edges of $K_{2,4} \times K_{2,4}$ are shown in Table 4.

<table>
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<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
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<th>$c_4$</th>
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<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_9$</th>
<th>$a_{10}$</th>
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5. Balanced Degree-Magic Labelings in the Tensor Product of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the tensor product of graphs $G$ and $H$, denoted by $G \oplus H$, is a graph such that the vertex set of $G \oplus H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \oplus H$ if and only if $u$ is adjacent with $x$ in $G$ and $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the tensor product $K_{p,q} \oplus K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \oplus K_{r,s}$ be a d-magic graph. Since $\deg(v)$ is $pr$, $ps$, $qr$ or $qs$ and $f^*(v) = (2pqr + 1) \deg(v)/2$ for any $v \in V(K_{p,q} \oplus K_{r,s})$, we have

**Proposition 5.1.** Let $K_{p,q} \oplus K_{r,s}$ be a balanced d-magic graph. Then $p$ and $q$ are even or $r$ and $s$ are even.

Now we can prove a sufficient condition for the existence of balanced d-magic labelings of the tensor product of complete bipartite graphs $K_{p,q} \oplus K_{r,s}$.

**Theorem 5.2.** Let $p$ and $q$ be positive integers with $(p, q) \neq (1, 1)$. Then $K_{p,q} \oplus K_{2,2}$ is a balanced d-magic graph.

**Proof.** Let $p$ and $q$ be positive integers with $(p, q) \neq (1, 1)$. Let $k = \min\{p, q\}$ and $h = \max\{p, q\}$. Since $K_{2,2h}$ is d-magic by Proposition 1.6, $2K_{2,2h}$ is balanced d-magic by Theorem 1.4. The graph $K_{p,q} \oplus K_{2,2}$ is decomposable into $k$ balanced d-magic subgraphs isomorphic to $2K_{2,2h}$. According to Theorem 1.5, $K_{p,q} \oplus K_{2,2}$ is a balanced d-magic graph. \(\square\)
Figure 5. A balanced d-magic graph $K_{1,3} \oplus K_{2,2}$ with 16 vertices and 24 edges.

Table 5. The labels on edges of balanced d-magic graph $K_{1,3} \oplus K_{2,2}$.

<table>
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<tr>
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<th>$b_4$</th>
<th>$b_5$</th>
<th>$b_6$</th>
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<td>6</td>
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</table>

Theorem 5.3. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers with $(r, s) \neq (2, 2)$. Then $K_{p,q} \oplus K_{r,s}$ is a balanced d-magic graph.

Proof. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers with $(r, s) \neq (2, 2)$. Since $K_{r,s}$ is d-magic by Proposition 1.6, $2K_{r,s}$ is balanced d-magic by Theorem 1.4. The graph $K_{p,q} \oplus K_{r,s}$ is decomposable into $pq$ balanced d-magic subgraphs isomorphic to $2K_{r,s}$. According to Theorem 1.5, $K_{p,q} \oplus K_{r,s}$ is a balanced d-magic graph.

It is clear that the tensor product graph $K_{p,q} \oplus K_{r,s}$ is isomorphic to $K_{r,s} \oplus K_{p,q}$.

Corollary 5.4. Let $p, q$ be positive integers with $(p, q) \neq (1, 1)$, and let $r, s$ be even positive integers. If $p = q$ and $r = s$, then $K_{p,q} \oplus K_{r,s}$ is a supermagic graph.

Proof. Applying Theorems 1.1, 5.2 and 5.3.

Below is an example of balanced d-magic graph $K_{1,3} \oplus K_{2,2}$ (see Figure 5), and the labels on edges of $K_{1,3} \oplus K_{2,2}$ are shown in Table 5.


For two vertex-disjoint graphs $G$ and $H$, the strong product of graphs $G$ and $H$, denoted by $G \otimes H$, is a graph such that the vertex set of $G \otimes H$ is...
the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are
adjacent in $G \otimes H$ if and only if $u = x$ and $v$ is adjacent with $y$ in $H$, or $v = y$
and $u$ is adjacent with $x$ in $G$, or $u$ is adjacent with $x$ in $G$ and $v$ is adjacent
with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the strong
product $K_{p,q} \otimes K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \otimes K_{r,s}$ be a d-magic
graph. Since $\deg(v) = p + r + pr$, $p + s + ps$, $q + r + qr$ or $q + s + qs$ and
$f^*(v) = (pq(r + s) + rs(p + q) + 2pqr + 1) \deg(v)/2$ for any $v \in V(K_{p,q} \otimes K_{r,s})$,
we have

**Proposition 6.1.** Let $K_{p,q} \otimes K_{r,s}$ be a d-magic graph. Then the following
conditions hold:
(i) only one of $p, q, r$ and $s$ is even or
(ii) all of $p, q, r$ and $s$ are even.

**Proposition 6.2.** Let $K_{p,q} \otimes K_{r,s}$ be a balanced d-magic graph. Then $p, q, r$
and $s$ are even.

We conclude this paper with an identification of the sufficient condition for
the existence of balanced d-magic labelings of the strong product of complete
bipartite graphs $K_{p,q} \otimes K_{r,s}$.

**Theorem 6.3.** Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$
and $(r, s) \neq (2, 2)$. Then $K_{p,q} \otimes K_{r,s}$ is a balanced d-magic graph.

**Proof.** Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq
(2, 2)$. Thus, $K_{p,q} \times K_{r,s}$ is balanced d-magic by Theorem 4.3, and $K_{p,q} \otimes K_{r,s}$
is balanced d-magic by Theorem 5.3. Since $K_{p,q} \otimes K_{r,s}$ is the graph such that
$K_{p,q} \times K_{r,s}$ and $K_{p,q} \oplus K_{r,s}$ form its decomposition, by Theorem 1.5, $K_{p,q} \otimes K_{r,s}$
is a balanced d-magic graph. □

It is clear that the strong product graph $K_{p,q} \otimes K_{r,s}$ is isomorphic to $K_{r,s} \otimes
K_{p,q}$.

**Corollary 6.4.** Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$
and $(r, s) \neq (2, 2)$. If $p = q$ and $r = s$, then $K_{p,q} \otimes K_{r,s}$ is a supermagic graph.

**Proof.** Applying Theorems 1.1 and 6.3. □

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**REFERENCES**

Balanced degree-magic labelings of complete bipartite graphs under binary operations