Balanced Degree-Magic Labelings of Complete Bipartite Graphs under Binary Operations

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Abstract. A graph is called supermagic if there is a labeling of edges where the edges are labeled with consecutive distinct positive integers such that the sum of the labels of all edges incident with any vertex is constant. A graph $G$ is called degree-magic if there is a labeling of the edges by integers $1, 2, ..., |E(G)|$ such that the sum of the labels of the edges incident with any vertex $v$ is equal to $(1 + |E(G)|)\deg(v)/2$. Degree-magic graphs extend supermagic regular graphs. In this paper we find the necessary and sufficient conditions for the existence of balanced degree-magic labelings of graphs obtained by taking the join, composition, Cartesian product, tensor product and strong product of complete bipartite graphs.

Keywords: Complete bipartite graphs, Supermagic graphs, Degree-magic graphs, Balanced degree-magic graphs.

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1. Introduction

We consider simple graphs without isolated vertices. If \( G \) is a graph, then \( V(G) \) and \( E(G) \) stand for the vertex set and the edge set of \( G \), respectively. Cardinalities of these sets are called the order and size of \( G \).

Let a graph \( G \) and a mapping \( f \) from \( E(G) \) into positive integers be given. The index mapping of \( f \) is the mapping \( f^* \) from \( V(G) \) into positive integers defined by

\[
f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every } v \in V(G),
\]

where \( \eta(v, e) \) is equal to 1 when \( e \) is an edge incident with a vertex \( v \), and 0 otherwise. An injective mapping \( f \) from \( E(G) \) into positive integers is called a magic labeling of \( G \) for an index \( \lambda \) if its index mapping \( f^* \) satisfies

\[
f^*(v) = \lambda \quad \text{for all } v \in V(G).
\]

A magic labeling \( f \) of a graph \( G \) is called a supermagic labeling if the set \( \{ f(e) : e \in E(G) \} \) consists of consecutive positive integers. We say that a graph \( G \) is supermagic (magic) whenever a supermagic (magic) labeling of \( G \) exists.

A bijective mapping \( f \) from \( E(G) \) into \( \{ 1, 2, \ldots, |E(G)| \} \) is called a degree-magic labeling (or only d-magic labeling) of a graph \( G \) if its index mapping \( f^* \) satisfies

\[
f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all } v \in V(G).
\]

A d-magic labeling \( f \) of a graph \( G \) is called balanced if for all \( v \in V(G) \), the following equation is satisfied

\[
|\{ e \in E(G) : \eta(v, e) = 1, f(e) \leq |E(G)|/2 \}| = |\{ e \in E(G) : \eta(v, e) = 1, f(e) > |E(G)|/2 \}|.
\]

We say that a graph \( G \) is degree-magic (balanced degree-magic) or only d-magic when a d-magic (balanced d-magic) labeling of \( G \) exists.

The concept of magic graphs was introduced by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. There are now many papers published on magic and supermagic graphs; see [6, 7, 10] for more comprehensive references. The concept of degree-magic graphs was then introduced by Bezegová and Ivančo [2] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degree-magic and balanced degree-magic complete bipartite graphs in [2]. They also characterized degree-magic complete tripartite graphs in [4]. Some of these concepts are investigated in [1, 3, 5]. We will hereinafter use the auxiliary results from these studies.
Theorem 1.1. [2] Let \( G \) be a regular graph. Then \( G \) is supermagic if and only if it is \( d \)-magic.

Theorem 1.2. [2] Let \( G \) be a \( d \)-magic graph of even size. Then every vertex of \( G \) has an even degree and every component of \( G \) has an even size.

Theorem 1.3. [2] Let \( G \) be a balanced \( d \)-magic graph. Then \( G \) has an even number of edges and every vertex has an even degree.

Theorem 1.4. [2] Let \( G \) be a \( d \)-magic graph having a half-factor. Then \( 2G \) is a balanced \( d \)-magic graph.

Theorem 1.5. [2] Let \( H_1 \) and \( H_2 \) be edge-disjoint subgraphs of a graph \( G \) which form its decomposition. If \( H_1 \) is \( d \)-magic and \( H_2 \) is balanced \( d \)-magic, then \( G \) is a balanced \( d \)-magic graph. Moreover, if \( H_1 \) and \( H_2 \) are both balanced \( d \)-magic, then \( G \) is a balanced \( d \)-magic graph.

Proposition 1.6. [2] For \( p, q > 1 \), the complete bipartite graph \( K_{p,q} \) is \( d \)-magic if and only if \( p \equiv q \equiv 0 \pmod{2} \) and \( (p, q) \neq (2, 2) \).

Theorem 1.7. [2] The complete bipartite graph \( K_{p,q} \) is balanced \( d \)-magic if and only if the following statements hold:

(i) \( p \equiv q \equiv 0 \pmod{2} \);
(ii) if \( p \equiv q \equiv 2 \pmod{4} \), then \( \min\{p, q\} \geq 6 \).

Lemma 1.8. [4] Let \( m, n \) and \( o \) be even positive integers. Then the complete tripartite graph \( K_{m,n,o} \) is balanced \( d \)-magic.

2. Balanced Degree-Magic Labelings in the Join of Complete Bipartite Graphs

For two vertex-disjoint graphs \( G \) and \( H \), the join of graphs \( G \) and \( H \), denoted by \( G + H \), consists of \( G \cup H \) and all edges joining a vertex of \( G \) and a vertex of \( H \). For any positive integers \( p \) and \( q \), we consider the join \( K_{p,q} + K_{p,q} \) of complete bipartite graphs. Let \( K_{p,q} + K_{p,q} \) be a \( d \)-magic graph. Since \( \deg(v) = p + 2q \) or \( 2p + q \) and \( f^*(v) = (2pq + (p + q)^2 + 1) \deg(v)/2 \) for any \( v \in V(K_{p,q} + K_{p,q}) \), we have

Proposition 2.1. Let \( K_{p,q} + K_{p,q} \) be a \( d \)-magic graph. Then \( p \) or \( q \) is even.

Proposition 2.2. Let \( K_{p,q} + K_{p,q} \) be a balanced \( d \)-magic graph. Then both \( p \) and \( q \) are even.

Proposition 2.3. Let \( p \) and \( q \) be even positive integers. Then \( K_{p+q,p+q} \) is a balanced \( d \)-magic graph.

Proof. Applying Theorem 1.7, \( K_{p+q,p+q} \) is a balanced \( d \)-magic graph. \( \square \)

In the next result we show a sufficient condition for the existence of balanced \( d \)-magic labelings of the join of complete bipartite graphs \( K_{p,q} + K_{p,q} \).
Theorem 2.4. Let $p$ and $q$ be even positive integers. Then $K_{p,q} + K_{p,q}$ is a balanced $d$-magic graph.

Proof. Let $p$ and $q$ be even positive integers. We consider the following two cases:

Case I. If $(p, q) = (2, 2)$, the graph $K_{2,2} + K_{2,2}$ is decomposable into three balanced $d$-magic subgraphs isomorphic to $K_{2,4}$. According to Theorem 1.5, $K_{2,2} + K_{2,2}$ is a balanced $d$-magic graph.

Case II. If $(p, q) \neq (2, 2)$, then $K_{p,q} + K_{p,q}$ is balanced $d$-magic by Proposition 2.3, and $2K_{p,q}$ is balanced $d$-magic by Theorem 1.4. Since $K_{p,q} + K_{p,q}$ is the graph such that $K_{p,q} + K_{p,q}$ and $2K_{p,q}$ form its decomposition, by Theorem 1.5, $K_{p,q} + K_{p,q}$ is a balanced $d$-magic graph. □

We know that $K_{2,6}$ is $d$-magic, but it is not balanced $d$-magic. Applying Theorem 2.4, we can construct a balanced $d$-magic graph $K_{2,6} + K_{2,6}$ (see Figure 1) with the labels on edges of $K_{2,6} + K_{2,6}$ in Table 2.

![Figure 1. A balanced d-magic graph $K_{2,6} + K_{2,6}$ with 16 vertices and 88 edges.](image)

We will now generalize to find the necessary and sufficient conditions for the existence of balanced $d$-magic labelings of the join of complete bipartite graphs in a general form. For any positive integers $p, q, r$ and $s$, we consider the join $K_{p,q} + K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} + K_{r,s}$ be a $d$-magic graph. Since $\deg(v) = p + r + s, q + r + s, p + q + r$ or $p + q + s$ and $f^*(v) = (pq + (p + q)(r + s) + rs + 1) \deg(v)/2$ for any $v \in V(K_{p,q} + K_{r,s})$, we have

Proposition 2.5. Let $K_{p,q} + K_{r,s}$ be a $d$-magic graph. Then the following conditions hold:

(i) only one of $p, q, r$ and $s$ is even or
(ii) only two of $p, q, r$ and $s$ are even or
(iii) all of $p, q, r$ and $s$ are even.
Proposition 2.6. Let $K_{p,q} + K_{r,s}$ be a balanced d-magic graph. Then $p, q, r$ and $s$ are even.

Now we are able to show a sufficient condition for the existence of balanced d-magic labelings of the join of complete bipartite graphs $K_{p,q} + K_{r,s}$.

Theorem 2.7. Let $p, q, r$ and $s$ be even positive integers. Then $K_{p,q} + K_{r,s}$ is a balanced d-magic graph.

Proof. Let $p, q, r$ and $s$ be even positive integers. We consider the following two cases:

Case I. If at least one of $p, q, r$ and $s$ is not congruent to 2 modulo 4. Suppose that $p$ is not congruent to 2 modulo 4. Thus, $K_{p,q}$ is balanced d-magic by Theorem 1.7. Since $r, s$ and $p + q$ are even, $K_{r,s,p+q}$ is balanced d-magic by Lemma 1.8. The graph $K_{p,q} + K_{r,s}$ is decomposable into two balanced d-magic subgraphs isomorphic to $K_{p,q}$ and $K_{r,s,p+q}$. According to Theorem 1.5, $K_{p,q} + K_{r,s}$ is a balanced d-magic graph.

Case II. If $p, q, r$ and $s$ are congruent to 2 modulo 4. Thus $q + r, q + s$ and $p + q$ are not congruent to 2 modulo 4. By Theorem 1.7, $K_{p,q+r}, K_{r,q+s}$ and $K_{s,p+q}$ are balanced d-magic. The graph $K_{p,q} + K_{r,s}$ is decomposable into three balanced d-magic subgraphs isomorphic to $K_{p,q+r}, K_{r,q+s}$ and $K_{s,p+q}$. According to Theorem 1.5, $K_{p,q} + K_{r,s}$ is a balanced d-magic graph. □

Corollary 2.8. Let $p, q, r$ and $s$ be even positive integers. If $p = q = r = s$, then $K_{p,q} + K_{r,s}$ is a supermagic graph.

Proof. Applying Theorems 1.1 and 2.7. □
Since 4 is not congruent to 2 modulo 4, applying Theorem 2.7, a balanced d-magic graph $K_{2,4} + K_{2,10}$ is constructed (see Figure 2), and the labels on edges of $K_{2,4} + K_{2,10}$ are shown in Table 2.

### 3. Balanced Degree-Magic Labelings in the Composition of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the composition of graphs $G$ and $H$, denoted by $G \cdot H$, is a graph such that the vertex set of $G \cdot H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \cdot H$ if and only if either $u$ is adjacent with $x$ in $G$ or $u = x$ and $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the composition $K_{p,q} \cdot K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \cdot K_{r,s}$ be a d-magic graph. Since $\deg(v) = (r+s)p + r, (r+s)p + s, (r+s)q + r$ or $(r+s)q + s$ and...

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**Figure 2.** A balanced d-magic graph $K_{2,4} + K_{2,10}$ with 18 vertices and 100 edges.

**Table 2.** The labels on edges of balanced d-magic graph $K_{2,4} + K_{2,10}$.
Balanced degree-magic labelings of complete bipartite graphs under binary operations

Let $K_{p,q} \cdot K_{r,s}$ be a $d$-magic graph. Then the following conditions hold:
(i) only one of $p,q,r$ and $s$ is even or
(ii) at least both $r$ and $s$ are even.

Proposition 3.2. Let $K_{p,q} \cdot K_{r,s}$ be a balanced $d$-magic graph. Then at least both $r$ and $s$ are even.

In the next result we find a sufficient condition for the existence of balanced $d$-magic labelings of the composition of complete bipartite graphs $K_{p,q} \cdot K_{r,s}$.

Theorem 3.3. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers. Then $K_{p,q} \cdot K_{r,s}$ is a balanced $d$-magic graph.

Proof. Let $p$ and $q$ be positive integers, and let $k = \min\{p,q\}$ and $h = \max\{p,q\}$. It is clear that $K_{r+1,s+1}, K_{r,s} + K_{r,s}$ and $K_{r,s,r+1}$ are balanced $d$-magic by Proposition 2.3, Theorem 2.4 and Lemma 1.8, respectively. The graph $K_{p,q} \cdot K_{r,s}$ is decomposable into $k$ balanced $d$-magic subgraphs isomorphic to $K_{r+1,s+1} + K_{r,s}$, $h(k-1)$ balanced $d$-magic subgraphs isomorphic to $K_{r+s,r+1}$ and $h-k$ balanced $d$-magic subgraphs isomorphic to $K_{r,s,r+1}$. According to Theorem 1.5, $K_{p,q} \cdot K_{r,s}$ is a balanced $d$-magic graph. □

Notice that the graph composition $K_{p,q} \cdot K_{r,s}$ is naturally nonisomorphic to $K_{r,s} \cdot K_{p,q}$ except for the case $(p,q) = (r,s)$.

Corollary 3.4. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers. If $p = q$ and $r = s$, then $K_{p,q} \cdot K_{r,s}$ is a supermagic graph.

Proof. Applying Theorems 1.1 and 3.3. □

The following example is a balanced $d$-magic graph $K_{1,2} \cdot K_{2,2}$ (see Figure 3) with the labels on edges of $K_{1,2} \cdot K_{2,2}$ in Table 3.

4. BALANCED DEGREE-MAGIC LABELINGS IN THE CARTESIAN PRODUCT OF COMPLETE BIPARTITE GRAPHS

For two vertex-disjoint graphs $G$ and $H$, the Cartesian product of graphs $G$ and $H$, denoted by $G \times H$, is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u,v)$ and $(x,y)$ are adjacent in $G \times H$ if and only if either $u = x$ and $v$ is adjacent with $y$ in $H$ or $v = y$ and $u$ is adjacent with $x$ in $G$. For any positive integers $p,q,r$ and $s$, we consider the Cartesian product $K_{p,q} \times K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \times K_{r,s}$ be a $d$-magic graph. Since $\deg(v)$ is $p+r$, $p+s$, $q+r$ or $q+s$ and $f^*(v) = (pq(r+s) + rs(p+q) + 1) \deg(v)/2$ for any $v \in V(K_{p,q} \times K_{r,s})$, we have

$$f^*(v) = (pq(r+s)^2 + rs(p+q) + 1) \deg(v)/2$$ for any $v \in V(K_{p,q} \times K_{r,s})$, we have
Figure 3. A balanced d-magic graph $K_{1,2} \cdot K_{2,2}$ with 12 vertices and 44 edges.

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Table 3. The labels on edges of balanced d-magic graph $K_{1,2} \cdot K_{2,2}$.

Proposition 4.1. Let $K_{p,q} \times K_{r,s}$ be a d-magic graph. Then the following conditions hold:
(i) only one of $p, q, r$ and $s$ is even or
(ii) all of $p, q, r$ and $s$ are either odd or even.

Proposition 4.2. Let $K_{p,q} \times K_{r,s}$ be a balanced d-magic graph. Then $p, q, r$ and $s$ are either odd or even.

In the next result we are able to find a sufficient condition for the existence of balanced d-magic labelings of the Cartesian product of complete bipartite graphs $K_{p,q} \times K_{r,s}$.

Theorem 4.3. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Then $K_{p,q} \times K_{r,s}$ is a balanced d-magic graph.

Proof. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Since $K_{p,q}$ and $K_{r,s}$ are d-magic by Proposition 1.6, $2K_{p,q}$ and $2K_{r,s}$ are balanced d-magic by Theorem 1.4. The graph $K_{p,q} \times K_{r,s}$ is decomposable into $(r + s)/2$ balanced d-magic subgraphs isomorphic to $2K_{p,q}$ and $(p + q)/2$.
Balanced degree-magic labelings of complete bipartite graphs under binary operations

Figure 4. A balanced d-magic graph $K_{2,4} \times K_{2,4}$ with 36 vertices and 96 edges.

balanced d-magic subgraphs isomorphic to $2K_{r,s}$. According to Theorem 1.5, $K_{p,q} \times K_{r,s}$ is a balanced d-magic graph. □

Observe that the Cartesian product graph $K_{p,q} \times K_{r,s}$ is naturally isomorphic to $K_{r,s} \times K_{p,q}$.

Corollary 4.4. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. If $p = q$ and $r = s$, then $K_{p,q} \times K_{r,s}$ is a supermagic graph.

Proof. Applying Theorems 1.1 and 4.3. □

The following example is a balanced d-magic graph $K_{2,4} \times K_{2,4}$ (see Figure 4), and the labels on edges of $K_{2,4} \times K_{2,4}$ are shown in Table 4.

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<td>-</td>
<td>-</td>
<td>-</td>
<td>13</td>
</tr>
</tbody>
</table>

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5. Balanced Degree-Magic Labelings in the Tensor Product of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the tensor product of graphs $G$ and $H$, denoted by $G \oplus H$, is a graph such that the vertex set of $G \oplus H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \oplus H$ if and only if $u$ is adjacent with $x$ in $G$ and $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the tensor product $K_{p,q} \oplus K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \oplus K_{r,s}$ be a d-magic graph. Since $\deg(v)$ is $pr, ps, qr$ or $qs$ and $f^*(v) = (2pqrs + 1) \deg(v)/2$ for any $v \in V(K_{p,q} \oplus K_{r,s})$, we have

**Proposition 5.1.** Let $K_{p,q} \oplus K_{r,s}$ be a balanced d-magic graph. Then $p$ and $q$ are even or $r$ and $s$ are even.

Now we can prove a sufficient condition for the existence of balanced d-magic labelings of the tensor product of complete bipartite graphs $K_{p,q} \oplus K_{r,s}$.

**Theorem 5.2.** Let $p$ and $q$ be positive integers with $(p, q) \neq (1, 1)$. Then $K_{p,q} \oplus K_{2,2}$ is a balanced d-magic graph.

**Proof.** Let $p$ and $q$ be positive integers with $(p, q) \neq (1, 1)$. Let $k = \min\{p, q\}$ and $h = \max\{p, q\}$. Since $K_{2,2h}$ is d-magic by Proposition 1.6, $2K_{2,2h}$ is balanced d-magic by Theorem 1.4. The graph $K_{p,q} \oplus K_{2,2}$ is decomposable into $k$ balanced d-magic subgraphs isomorphic to $2K_{2,2h}$. According to Theorem 1.5, $K_{p,q} \oplus K_{2,2}$ is a balanced d-magic graph. □
Balanced degree-magic labelings of complete bipartite graphs under binary operations

Figure 5. A balanced d-magic graph \( K_{1,3} \oplus K_{2,2} \) with 16 vertices and 24 edges.

Table 5. The labels on edges of balanced d-magic graph \( K_{1,3} \oplus K_{2,2} \).

<table>
<thead>
<tr>
<th>Vertices</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
<th>( b_4 )</th>
<th>( b_5 )</th>
<th>( b_6 )</th>
<th>( b_7 )</th>
<th>( b_8 )</th>
<th>( b_9 )</th>
<th>( b_{10} )</th>
<th>( b_{11} )</th>
<th>( b_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>11</td>
<td>-</td>
<td>-</td>
<td>3</td>
<td>21</td>
<td>-</td>
<td>-</td>
<td>20</td>
<td>19</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>-</td>
<td>-</td>
<td>24</td>
<td>14</td>
<td>-</td>
<td>-</td>
<td>22</td>
<td>4</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>13</td>
<td>23</td>
<td>-</td>
<td>-</td>
<td>15</td>
<td>9</td>
<td>-</td>
<td>-</td>
<td>8</td>
<td>7</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>12</td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>10</td>
<td>16</td>
<td>-</td>
<td>-</td>
<td>17</td>
<td>18</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Theorem 5.3. Let \( p \) and \( q \) be positive integers, and let \( r \) and \( s \) be even positive integers with \( (r, s) \neq (2, 2) \). Then \( K_{p,q} \oplus K_{r,s} \) is a balanced d-magic graph.

Proof. Let \( p \) and \( q \) be positive integers, and let \( r \) and \( s \) be even positive integers with \( (r, s) \neq (2, 2) \). Since \( K_{r,s} \) is d-magic by Proposition 1.6, \( 2K_{r,s} \) is balanced d-magic by Theorem 1.4. The graph \( K_{p,q} \oplus K_{r,s} \) is decomposable into \( pq \) balanced d-magic subgraphs isomorphic to \( 2K_{r,s} \). According to Theorem 1.5, \( K_{p,q} \oplus K_{r,s} \) is a balanced d-magic graph. \( \square \)

It is clear that the tensor product graph \( K_{p,q} \oplus K_{r,s} \) is isomorphic to \( K_{r,s} \oplus K_{p,q} \).

Corollary 5.4. Let \( p, q \) be positive integers with \( (p, q) \neq (1, 1) \), and let \( r, s \) be even positive integers. If \( p = q \) and \( r = s \), then \( K_{p,q} \oplus K_{r,s} \) is a supermagic graph.

Proof. Applying Theorems 1.1, 5.2 and 5.3. \( \square \)

Below is an example of balanced d-magic graph \( K_{1,3} \oplus K_{2,2} \) (see Figure 5), and the labels on edges of \( K_{1,3} \oplus K_{2,2} \) are shown in Table 5.

6. BALANCED DEGREE-MAGIC LABELINGS IN THE STRONG PRODUCT OF COMPLETE BIPARTITE GRAPHS

For two vertex-disjoint graphs \( G \) and \( H \), the strong product of graphs \( G \) and \( H \), denoted by \( G \otimes H \), is a graph such that the vertex set of \( G \otimes H \) is
the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \otimes H$ if and only if $u = x$ and $v$ is adjacent with $y$ in $H$, or $v = y$ and $u$ is adjacent with $x$ in $G$, or $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the strong product $K_{p,q} \otimes K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \otimes K_{r,s}$ be a d-magic graph. Since $\deg(v) = p + r + pr, p + s + ps, q + r + qr$ or $q + s + qs$ and $f^*(v) = (pq(r+s) + rs(p+q) + 2pqrs + 1) \deg(v)/2$ for any $v \in V(K_{p,q} \otimes K_{r,s})$, we have

**Proposition 6.1.** Let $K_{p,q} \otimes K_{r,s}$ be a d-magic graph. Then the following conditions hold:

(i) only one of $p, q, r$ and $s$ is even or

(ii) all of $p, q, r$ and $s$ are even.

**Proposition 6.2.** Let $K_{p,q} \otimes K_{r,s}$ be a balanced d-magic graph. Then $p, q, r$ and $s$ are even.

We conclude this paper with an identification of the sufficient condition for the existence of balanced d-magic labelings of the strong product of complete bipartite graphs $K_{p,q} \otimes K_{r,s}$.

**Theorem 6.3.** Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Then $K_{p,q} \otimes K_{r,s}$ is a balanced d-magic graph.

**Proof.** Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Thus, $K_{p,q} \times K_{r,s}$ is balanced d-magic by Theorem 4.3, and $K_{p,q} \otimes K_{r,s}$ is balanced d-magic by Theorem 5.3. Since $K_{p,q} \otimes K_{r,s}$ is the graph such that $K_{p,q} \times K_{r,s}$ and $K_{p,q} \oplus K_{r,s}$ form its decomposition, by Theorem 1.5, $K_{p,q} \otimes K_{r,s}$ is a balanced d-magic graph. □

It is clear that the strong product graph $K_{p,q} \otimes K_{r,s}$ is isomorphic to $K_{r,s} \otimes K_{p,q}$.

**Corollary 6.4.** Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. If $p = q$ and $r = s$, then $K_{p,q} \otimes K_{r,s}$ is a supermagic graph.

**Proof.** Applying Theorems 1.1 and 6.3. □

**Acknowledgments**

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**References**