# Balanced Degree-Magic Labelings of Complete Bipartite Graphs under Binary Operations 

Phaisatcha Inpoonjai*, Thiradet Jiarasuksakun<br>Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand.

> E-mail: phaisatcha_in@outlook.com E-mail: thiradet.jia@mail.kmutt.ac.th


#### Abstract

A graph is called supermagic if there is a labeling of edges where the edges are labeled with consecutive distinct positive integers such that the sum of the labels of all edges incident with any vertex is constant. A graph $G$ is called degree-magic if there is a labeling of the edges by integers $1,2, \ldots,|E(G)|$ such that the sum of the labels of the edges incident with any vertex $v$ is equal to $(1+|E(G)|) \operatorname{deg}(v) / 2$. Degree-magic graphs extend supermagic regular graphs. In this paper we find the necessary and sufficient conditions for the existence of balanced degree-magic labelings of graphs obtained by taking the join, composition, Cartesian product, tensor product and strong product of complete bipartite graphs.


Keywords: Complete bipartite graphs, Supermagic graphs, Degree-magic graphs, Balanced degree-magic graphs.

2000 Mathematics subject classification: 05C78.

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## 1. Introduction

We consider simple graphs without isolated vertices. If $G$ is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of $G$, respectively. Cardinalities of these sets are called the order and size of $G$.

Let a graph $G$ and a mapping $f$ from $E(G)$ into positive integers be given. The index mapping of $f$ is the mapping $f^{*}$ from $V(G)$ into positive integers defined by

$$
f^{*}(v)=\sum_{e \in E(G)} \eta(v, e) f(e) \quad \text { for every } \quad v \in V(G)
$$

where $\eta(v, e)$ is equal to 1 when $e$ is an edge incident with a vertex $v$, and 0 otherwise. An injective mapping $f$ from $E(G)$ into positive integers is called a magic labeling of $G$ for an index $\lambda$ if its index mapping $f^{*}$ satisfies

$$
f^{*}(v)=\lambda \quad \text { for all } \quad v \in V(G)
$$

A magic labeling $f$ of a graph $G$ is called a supermagic labeling if the set $\{f(e): e \in E(G)\}$ consists of consecutive positive integers. We say that a graph $G$ is supermagic (magic) whenever a supermagic (magic) labeling of $G$ exists.

A bijective mapping $f$ from $E(G)$ into $\{1,2, \ldots,|E(G)|\}$ is called a degreemagic labeling (or only d-magic labeling) of a graph $G$ if its index mapping $f^{*}$ satisfies

$$
f^{*}(v)=\frac{1+|E(G)|}{2} \operatorname{deg}(v) \quad \text { for all } \quad v \in V(G)
$$

A d-magic labeling $f$ of a graph $G$ is called balanced if for all $v \in V(G)$, the following equation is satisfied

$$
\begin{aligned}
\mid\{e \in E(G) & : \eta(v, e)=1, f(e) \leq\lfloor|E(G)| / 2\rfloor\} \mid \\
& =|\{e \in E(G): \eta(v, e)=1, f(e)>\lfloor|E(G)| / 2\rfloor\}|
\end{aligned}
$$

We say that a graph $G$ is degree-magic (balanced degree-magic) or only d-magic when a d-magic (balanced d-magic) labeling of $G$ exists.

The concept of magic graphs was introduced by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. There are now many papers published on magic and supermagic graphs; see $[6,7,10]$ for more comprehensive references. The concept of degree-magic graphs was then introduced by Bezegová and Ivančo [2] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degreemagic and balanced degree-magic complete bipartite graphs in [2]. They also characterized degree-magic complete tripartite graphs in [4]. Some of these concepts are investigated in $[1,3,5]$. We will hereinafter use the auxiliary results from these studies.

Theorem 1.1. [2] Let $G$ be a regular graph. Then $G$ is supermagic if and only if it is $d$-magic.

Theorem 1.2. [2] Let $G$ be a d-magic graph of even size. Then every vertex of $G$ has an even degree and every component of $G$ has an even size.
Theorem 1.3. [2] Let $G$ be a balanced d-magic graph. Then $G$ has an even number of edges and every vertex has an even degree.

Theorem 1.4. [2] Let $G$ be a d-magic graph having a half-factor. Then $2 G$ is a balanced d-magic graph.

Theorem 1.5. [2] Let $H_{1}$ and $H_{2}$ be edge-disjoint subgraphs of a graph $G$ which form its decomposition. If $H_{1}$ is d-magic and $H_{2}$ is balanced d-magic, then $G$ is a d-magic graph. Moreover, if $H_{1}$ and $H_{2}$ are both balanced d-magic, then $G$ is a balanced d-magic graph.

Proposition 1.6. [2] For $p, q>1$, the complete bipartite graph $K_{p, q}$ is d-magic if and only if $p \equiv q(\bmod 2)$ and $(p, q) \neq(2,2)$.

Theorem 1.7. [2] The complete bipartite graph $K_{p, q}$ is balanced d-magic if and only if the following statements hold:
(i) $p \equiv q \equiv 0(\bmod 2)$;
(ii) if $p \equiv q \equiv 2(\bmod 4)$, then $\min \{p, q\} \geq 6$.

Lemma 1.8. [4] Let m,n and o be even positive integers. Then the complete tripartite graph $K_{m, n, o}$ is balanced d-magic.

## 2. Balanced Degree-Magic Labelings in the Join of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the join of graphs $G$ and $H$, denoted by $G+H$, consists of $G \cup H$ and all edges joining a vertex of $G$ and a vertex of $H$. For any positive integers $p$ and $q$, we consider the join $K_{p, q}+K_{p, q}$ of complete bipartite graphs. Let $K_{p, q}+K_{p, q}$ be a d-magic graph. Since $\operatorname{deg}(v)$ is $p+2 q$ or $2 p+q$ and $f^{*}(v)=\left(2 p q+(p+q)^{2}+1\right) \operatorname{deg}(v) / 2$ for any $v \in V\left(K_{p, q}+K_{p, q}\right)$, we have

Proposition 2.1. Let $K_{p, q}+K_{p, q}$ be a d-magic graph. Then $p$ or $q$ is even.
Proposition 2.2. Let $K_{p, q}+K_{p, q}$ be a balanced d-magic graph. Then both $p$ and $q$ are even.

Proposition 2.3. Let $p$ and $q$ be even positive integers. Then $K_{p+q, p+q}$ is a balanced d-magic graph.

Proof. Applying Theorem 1.7, $K_{p+q, p+q}$ is a balanced d-magic graph.


Figure 1. A balanced d-magic graph $K_{2,6}+K_{2,6}$ with 16 vertices and 88 edges.

Theorem 2.4. Let $p$ and $q$ be even positive integers. Then $K_{p, q}+K_{p, q}$ is a balanced d-magic graph.

Proof. Let $p$ and $q$ be even positive integers. We consider the following two cases:
Case I. If $(p, q)=(2,2)$, the graph $K_{2,2}+K_{2,2}$ is decomposable into three balanced d-magic subgraphs isomorphic to $K_{2,4}$. According to Theorem 1.5, $K_{2,2}+K_{2,2}$ is a balanced d-magic graph.
Case II. If $(p, q) \neq(2,2)$, then $K_{p+q, p+q}$ is balanced d-magic by Proposition 2.3 , and $2 K_{p, q}$ is balanced d-magic by Theorem 1.4. Since $K_{p, q}+K_{p, q}$ is the graph such that $K_{p+q, p+q}$ and $2 K_{p, q}$ form its decomposition, by Theorem 1.5, $K_{p, q}+K_{p, q}$ is a balanced d-magic graph.

We know that $K_{2,6}$ is d-magic, but it is not balanced d-magic. Applying Theorem 2.4, we can construct a balanced d-magic graph $K_{2,6}+K_{2,6}$ (see Figure 1) with the labels on edges of $K_{2,6}+K_{2,6}$ in Table 2.

We will now generalize to find the necessary and sufficient conditions for the existence of balanced d-magic labelings of the join of complete bipartite graphs in a general form. For any positive integers $p, q, r$ and $s$, we consider the join $K_{p, q}+K_{r, s}$ of complete bipartite graphs. Let $K_{p, q}+K_{r, s}$ be a dmagic graph. Since $\operatorname{deg}(v)$ is $p+r+s, q+r+s, p+q+r$ or $p+q+s$ and $f^{*}(v)=(p q+(p+q)(r+s)+r s+1) \operatorname{deg}(v) / 2$ for any $v \in V\left(K_{p, q}+K_{r, s}\right)$, we have

Proposition 2.5. Let $K_{p, q}+K_{r, s}$ be a d-magic graph. Then the following conditions hold:
(i) only one of $p, q, r$ and $s$ is even or
(ii) only two of $p, q, r$ and $s$ are even or
(iii) all of $p, q, r$ and $s$ are even.

| Vertices | $a_{1}$ | $a_{2}$ | $a_{3}$ | $a_{4}$ | $a_{5}$ | $a_{6}$ | $c_{1}$ | $c_{2}$ | $d_{1}$ | $d_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{1}$ | 15 | 70 | 75 | 26 | 23 | 62 | 18 | 67 | 1 | 88 |
| $b_{2}$ | 74 | 16 | 17 | 63 | 66 | 24 | 71 | 25 | 11 | 78 |
| $b_{3}$ | 69 | 19 | 14 | 68 | 61 | 27 | 76 | 22 | 3 | 86 |
| $b_{4}$ | 36 | 57 | 56 | 37 | 44 | 49 | 29 | 48 | 85 | 4 |
| $b_{5}$ | 31 | 54 | 59 | 42 | 39 | 46 | 34 | 51 | 84 | 5 |
| $b_{6}$ | 58 | 32 | 33 | 47 | 50 | 40 | 55 | 41 | 83 | 6 |
| $d_{1}$ | 20 | 73 | 72 | 21 | 28 | 65 | 13 | 64 | - | - |
| $d_{2}$ | 53 | 35 | 30 | 52 | 45 | 43 | 60 | 38 | - | - |
| $c_{1}$ | 77 | 87 | 79 | 9 | 8 | 7 | - | - | - | - |
| $c_{2}$ | 12 | 2 | 10 | 80 | 81 | 82 | - | - | - | - |

Table 1. The labels on edges of balanced d-magic graph $K_{2,6}+K_{2,6}$.

Proposition 2.6. Let $K_{p, q}+K_{r, s}$ be a balanced d-magic graph. Then $p, q, r$ and $s$ are even.

Now we are able to show a sufficient condition for the existence of balanced d-magic labelings of the join of complete bipartite graphs $K_{p, q}+K_{r, s}$.

Theorem 2.7. Let $p, q, r$ and $s$ be even positive integers. Then $K_{p, q}+K_{r, s}$ is a balanced d-magic graph.

Proof. Let $p, q, r$ and $s$ be even positive integers. We consider the following two cases:
Case I. If at least one of $p, q, r$ and $s$ is not congruent to 2 modulo 4 . Suppose that $p$ is not congruent to 2 modulo 4 . Thus, $K_{p, q}$ is balanced d-magic by Theorem 1.7. Since $r, s$ and $p+q$ are even, $K_{r, s, p+q}$ is balanced d-magic by Lemma 1.8. The graph $K_{p, q}+K_{r, s}$ is decomposable into two balanced dmagic subgraphs isomorphic to $K_{p, q}$ and $K_{r, s, p+q}$. According to Theorem 1.5, $K_{p, q}+K_{r, s}$ is a balanced d-magic graph.
Case II. If $p, q, r$ and $s$ are congruent to 2 modulo 4 . Thus $q+r, q+s$ and $p+q$ are not congruent to 2 modulo 4. By Theorem 1.7, $K_{p, q+r}, K_{r, q+s}$ and $K_{s, p+q}$ are balanced d-magic. The graph $K_{p, q}+K_{r, s}$ is decomposable into three balanced d-magic subgraphs isomorphic to $K_{p, q+r}, K_{r, q+s}$ and $K_{s, p+q}$. According to Theorem 1.5, $K_{p, q}+K_{r, s}$ is a balanced d-magic graph.

Corollary 2.8. Let $p, q, r$ and $s$ be even positive integers. If $p=q=r=s$, then $K_{p, q}+K_{r, s}$ is a supermagic graph.

Proof. Applying Theorems 1.1 and 2.7.


Figure 2. A balanced d-magic graph $K_{2,4}+K_{2,10}$ with 18 vertices and 100 edges.

| Vertices | $a_{1}$ | $a_{2}$ | $a_{3}$ | $a_{4}$ | $a_{5}$ | $a_{6}$ | $a_{7}$ | $a_{8}$ | $a_{9}$ | $a_{10}$ | $c_{1}$ | $c_{2}$ | $d_{1}$ | $d_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{1}$ | 31 | 70 | 79 | 22 | 57 | 61 | 42 | 41 | 58 | 44 | 85 | 16 | 100 | 1 |
| $b_{2}$ | 23 | 78 | 71 | 30 | 45 | 52 | 54 | 53 | 49 | 50 | 84 | 17 | 2 | 99 |
| $b_{3}$ | 77 | 24 | 29 | 72 | 56 | 46 | 48 | 47 | 55 | 51 | 18 | 83 | 3 | 98 |
| $b_{4}$ | 76 | 25 | 28 | 73 | 39 | 43 | 59 | 60 | 40 | 62 | 19 | 82 | 97 | 4 |
| $d_{1}$ | 75 | 26 | 27 | 74 | 38 | 64 | 65 | 36 | 67 | 33 | 81 | 20 | - | - |
| $d_{2}$ | 21 | 80 | 69 | 32 | 68 | 37 | 35 | 66 | 34 | 63 | 15 | 86 | - | - |
| $c_{1}$ | 96 | 6 | 7 | 93 | 92 | 10 | 11 | 89 | 88 | 14 | - | - | - | - |
| $c_{2}$ | 5 | 95 | 94 | 8 | 9 | 91 | 90 | 12 | 13 | 87 | - | - | - | - |

Table 2. The labels on edges of balanced d-magic graph
$K_{2,4}+K_{2,10}$.

Since 4 is not congruent to 2 modulo 4, applying Theorem 2.7, a balanced d-magic graph $K_{2,4}+K_{2,10}$ is constructed (see Figure 2), and the labels on edges of $K_{2,4}+K_{2,10}$ are shown in Table 2.

## 3. Balanced Degree-Magic Labelings in the Composition of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the composition of graphs $G$ and $H$, denoted by $G \cdot H$, is a graph such that the vertex set of $G \cdot H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \cdot H$ if and only if either $u$ is adjacent with $x$ in $G$ or $u=x$ and $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the composition $K_{p, q} \cdot K_{r, s}$ of complete bipartite graphs. Let $K_{p, q} \cdot K_{r, s}$ be a d-magic graph. Since $\operatorname{deg}(v)$ is $(r+s) p+r,(r+s) p+s,(r+s) q+r$ or $(r+s) q+s$ and
$f^{*}(v)=\left(p q(r+s)^{2}+r s(p+q)+1\right) \operatorname{deg}(v) / 2$ for any $v \in V\left(K_{p, q} \cdot K_{r, s}\right)$, we have

Proposition 3.1. Let $K_{p, q} \cdot K_{r, s}$ be a d-magic graph. Then the following conditions hold:
(i) only one of $p, q, r$ and $s$ is even or
(ii) at least both $r$ and $s$ are even.

Proposition 3.2. Let $K_{p, q} \cdot K_{r, s}$ be a balanced d-magic graph. Then at least both $r$ and $s$ are even.

In the next result we find a sufficient condition for the existence of balanced d-magic labelings of the composition of complete bipartite graphs $K_{p, q} \cdot K_{r, s}$.
Theorem 3.3. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers. Then $K_{p, q} \cdot K_{r, s}$ is a balanced d-magic graph.
Proof. Let $p$ and $q$ be positive integers, and let $k=\min \{p, q\}$ and $h=$ $\max \{p, q\}$. It is clear that $K_{r+s, r+s}, K_{r, s}+K_{r, s}$ and $K_{r, s, r+s}$ are balanced d-magic by Proposition 2.3, Theorem 2.4 and Lemma1.8, respectively. The graph $K_{p, q} \cdot K_{r, s}$ is decomposable into $k$ balanced d-magic subgraphs isomorphic to $K_{r, s}+K_{r, s}, h(k-1)$ balanced d-magic subgraphs isomorphic to $K_{r+s, r+s}$ and $h-k$ balanced d-magic subgraphs isomorphic to $K_{r, s, r+s}$. According to Theorem 1.5, $K_{p, q} \cdot K_{r, s}$ is a balanced d-magic graph.

Notice that the graph composition $K_{p, q} \cdot K_{r, s}$ is naturally nonisomorphic to $K_{r, s} \cdot K_{p, q}$ except for the case $(p, q)=(r, s)$.
Corollary 3.4. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers. If $p=q$ and $r=s$, then $K_{p, q} \cdot K_{r, s}$ is a supermagic graph.

Proof. Applying Theorems 1.1 and 3.3.
The following example is a balanced d-magic graph $K_{1,2} \cdot K_{2,2}$ (see Figure 3) with the labels on edges of $K_{1,2} \cdot K_{2,2}$ in Table 3.

## 4. Balanced Degree-Magic Labelings in the Cartesian Product of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the Cartesian product of graphs $G$ and $H$, denoted by $G \times H$, is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \times H$ if and only if either $u=x$ and $v$ is adjacent with $y$ in $H$ or $v=y$ and $u$ is adjacent with $x$ in $G$. For any positive integers $p, q, r$ and $s$, we consider the Cartesian product $K_{p, q} \times K_{r, s}$ of complete bipartite graphs. Let $K_{p, q} \times K_{r, s}$ be a d-magic graph. Since $\operatorname{deg}(v)$ is $p+r, p+s, q+r$ or $q+s$ and $f^{*}(v)=(p q(r+s)+r s(p+q)+1) \operatorname{deg}(v) / 2$ for any $v \in V\left(K_{p, q} \times K_{r, s}\right)$, we have


Figure 3. A balanced d-magic graph $K_{1,2} \cdot K_{2,2}$ with 12 vertices and 44 edges.

| Vertices | $c_{1}$ | $c_{2}$ | $d_{1}$ | $d_{2}$ | $e_{1}$ | $e_{2}$ | $f_{1}$ | $f_{2}$ | $b_{1}$ | $b_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{1}$ | 12 | 34 | 43 | 2 | 27 | 26 | 20 | 17 | 35 | 9 |
| $a_{2}$ | 33 | 11 | 1 | 44 | 19 | 18 | 25 | 28 | 10 | 36 |
| $b_{1}$ | 8 | 38 | 39 | 5 | 32 | 14 | 15 | 29 | - | - |
| $b_{2}$ | 37 | 7 | 6 | 40 | 13 | 31 | 30 | 16 | - | - |
| $d_{1}$ | 4 | 42 | - | - | - | - | - | - | - | - |
| $d_{2}$ | 41 | 3 | - | - | - | - | - | - | - | - |
| $f_{1}$ | - | - | - | - | 23 | 22 | - | - | - | - |
| $f_{2}$ | - | - | - | - | 21 | 24 | - | - | - | - |

Table 3. The labels on edges of balanced d-magic graph $K_{1,2} \cdot K_{2,2}$.

Proposition 4.1. Let $K_{p, q} \times K_{r, s}$ be a d-magic graph. Then the following conditions hold:
(i) only one of $p, q, r$ and $s$ is even or
(ii) all of $p, q, r$ and $s$ are either odd or even.

Proposition 4.2. Let $K_{p, q} \times K_{r, s}$ be a balanced d-magic graph. Then $p, q, r$ and $s$ are either odd or even.

In the next result we are able to find a sufficient condition for the existence of balanced d-magic labelings of the Cartesian product of complete bipartite graphs $K_{p, q} \times K_{r, s}$.

Theorem 4.3. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq(2,2)$ and $(r, s) \neq(2,2)$. Then $K_{p, q} \times K_{r, s}$ is a balanced d-magic graph.

Proof. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq(2,2)$ and $(r, s) \neq$ $(2,2)$. Since $K_{p, q}$ and $K_{r, s}$ are d-magic by Proposition $1.6,2 K_{p, q}$ and $2 K_{r, s}$ are balanced d-magic by Theorem 1.4. The graph $K_{p, q} \times K_{r, s}$ is decomposable into $(r+s) / 2$ balanced d-magic subgraphs isomorphic to $2 K_{p, q}$ and $(p+q) / 2$


Figure 4. A balanced d-magic graph $K_{2,4} \times K_{2,4}$ with 36 vertices and 96 edges.
balanced d-magic subgraphs isomorphic to $2 K_{r, s}$. According to Theorem 1.5, $K_{p, q} \times K_{r, s}$ is a balanced d-magic graph.

Observe that the Cartesian product graph $K_{p, q} \times K_{r, s}$ is naturally isomorphic to $K_{r, s} \times K_{p, q}$.

Corollary 4.4. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq(2,2)$ and $(r, s) \neq(2,2)$. If $p=q$ and $r=s$, then $K_{p, q} \times K_{r, s}$ is a supermagic graph.
Proof. Applying Theorems 1.1 and 4.3.
The following example is a balanced d-magic graph $K_{2,4} \times K_{2,4}$ (see Figure $4)$, and the labels on edges of $K_{2,4} \times K_{2,4}$ are shown in Table 4.

| Vertices | $b_{1}$ | $b_{2}$ | $b_{3}$ | $b_{4}$ | $e_{1}$ | $e_{2}$ | $e_{3}$ | $e_{4}$ | $a_{3}$ | $a_{4}$ | $a_{9}$ | $a_{10}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{1}$ | 96 | 2 | 3 | 93 | - | - | - | - | 72 | - | 25 | - |
| $a_{2}$ | 1 | 95 | 94 | 4 | - | - | - | - | - | 64 | - | 33 |
| $a_{7}$ | - | - | - | - | 8 | 90 | 91 | 5 | 27 | - | 70 | - |
| $a_{8}$ | - | - | - | - | 89 | 7 | 6 | 92 | - | 35 | - | 62 |
| $c_{1}$ | 48 | - | - | - | 51 | - | - | - | 88 | 9 | - | - |
| $c_{2}$ | - | 32 | - | - | - | 67 | - | - | 10 | 87 | - | - |
| $c_{3}$ | - | - | 40 | - | - | - | 59 | - | 11 | 86 | - | - |
| $c_{4}$ | - | - | - | 56 | - | - | - | 43 | 85 | 12 | - | - |
| $f_{1}$ | 49 | - | - | - | 46 | - | - | - | - | - | 16 | 81 |
| $f_{2}$ | - | 65 | - | - | - | 30 | - | - | - | - | 82 | 15 |
| $f_{3}$ | - | - | 57 | - | - | - | 38 | - | - | - | 83 | 14 |
| $f_{4}$ | - | - | - | 41 | - | - | - | 54 | - | - | 13 | 84 |


| Vertices | $d_{1}$ | $d_{2}$ | $d_{3}$ | $d_{4}$ | $g_{1}$ | $g_{2}$ | $g_{3}$ | $g_{4}$ | $a_{3}$ | $a_{4}$ | $a_{9}$ | $a_{10}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{5}$ | 24 | 74 | 75 | 21 | - | - | - | - | 26 | - | 71 | - |
| $a_{6}$ | 73 | 23 | 22 | 76 | - | - | - | - | - | 34 | - | 63 |
| $a_{11}$ | - | - | - | - | 80 | 18 | 19 | 77 | 69 | - | 28 | - |
| $a_{12}$ | - | - | - | - | 17 | 79 | 78 | 20 | - | 61 | - | 36 |
| $c_{1}$ | 50 | - | - | - | 45 | - | - | - | - | - | - | - |
| $c_{2}$ | - | 66 | - | - | - | 29 | - | - | - | - | - | - |
| $c_{3}$ | - | - | 58 | - | - | - | 37 | - | - | - | - | - |
| $c_{4}$ | - | - | - | 42 | - | - | - | 53 | - | - | - | - |
| $f_{1}$ | 47 | - | - | - | 52 | - | - | - | - | - | - | - |
| $f_{2}$ | - | 31 | - | - | - | 68 | - | - | - | - | - | - |
| $f_{3}$ | - | - | 39 | - | - | - | 60 | - | - | - | - | - |
| $f_{4}$ | - | - | - | 55 | - | - | - | 44 | - | - | - | - |

Table 4. The labels on edges of balanced d-magic graph $K_{2,4} \times K_{2,4}$.

## 5. Balanced Degree-Magic Labelings in the Tensor Product of Complete Bipartite Graphs

For two vertex-disjoint graphs $G$ and $H$, the tensor product of graphs $G$ and $H$, denoted by $G \oplus H$, is a graph such that the vertex set of $G \oplus H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \oplus H$ if and only if $u$ is adjacent with $x$ in $G$ and $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the tensor product $K_{p, q} \oplus K_{r, s}$ of complete bipartite graphs. Let $K_{p, q} \oplus K_{r, s}$ be a d-magic graph. Since $\operatorname{deg}(v)$ is $p r, p s, q r$ or $q s$ and $f^{*}(v)=(2 p q r s+1) \operatorname{deg}(v) / 2$ for any $v \in V\left(K_{p, q} \oplus K_{r, s}\right)$, we have

Proposition 5.1. Let $K_{p, q} \oplus K_{r, s}$ be a balanced d-magic graph. Then $p$ and $q$ are even or $r$ and $s$ are even.

Now we can prove a sufficient condition for the existence of balanced d-magic labelings of the tensor product of complete bipartite graphs $K_{p, q} \oplus K_{r, s}$.

Theorem 5.2. Let $p$ and $q$ be positive integers with $(p, q) \neq(1,1)$. Then $K_{p, q} \oplus K_{2,2}$ is a balanced d-magic graph.

Proof. Let $p$ and $q$ be positive integers with $(p, q) \neq(1,1)$. Let $k=\min \{p, q\}$ and $h=\max \{p, q\}$. Since $K_{2,2 h}$ is d-magic by Proposition 1.6, $2 K_{2,2 h}$ is balanced d-magic by Theorem 1.4. The graph $K_{p, q} \oplus K_{2,2}$ is decomposable into $k$ balanced d-magic subgraphs isomorphic to $2 K_{2,2 h}$. According to Theorem 1.5, $K_{p, q} \oplus K_{2,2}$ is a balanced d-magic graph.


Figure 5. A balanced d-magic graph $K_{1,3} \oplus K_{2,2}$ with 16 vertices and 24 edges.

| Vertices | $b_{1}$ | $b_{2}$ | $b_{3}$ | $b_{4}$ | $b_{5}$ | $b_{6}$ | $b_{7}$ | $b_{8}$ | $b_{9}$ | $b_{10}$ | $b_{11}$ | $b_{12}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{1}$ | - | - | 1 | 11 | - | - | 3 | 21 | - | - | 20 | 19 |
| $a_{2}$ | - | - | 24 | 14 | - | - | 22 | 4 | - | - | 5 | 6 |
| $a_{3}$ | 13 | 23 | - | - | 15 | 9 | - | - | 8 | 7 | - | - |
| $a_{4}$ | 12 | 2 | - | - | 10 | 16 | - | - | 17 | 18 | - | - |

Table 5. The labels on edges of balanced d-magic graph $K_{1,3} \oplus K_{2,2}$.

Theorem 5.3. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers with $(r, s) \neq(2,2)$. Then $K_{p, q} \oplus K_{r, s}$ is a balanced d-magic graph.
Proof. Let $p$ and $q$ be positive integers, and let $r$ and $s$ be even positive integers with $(r, s) \neq(2,2)$. Since $K_{r, s}$ is d-magic by Proposition $1.6,2 K_{r, s}$ is balanced d-magic by Theorem 1.4. The graph $K_{p, q} \oplus K_{r, s}$ is decomposable into $p q$ balanced d-magic subgraphs isomorphic to $2 K_{r, s}$. According to Theorem 1.5, $K_{p, q} \oplus K_{r, s}$ is a balanced d-magic graph.

It is clear that the tensor product graph $K_{p, q} \oplus K_{r, s}$ is isomorphic to $K_{r, s} \oplus$ $K_{p, q}$.
Corollary 5.4. Let $p, q$ be positive integers with $(p, q) \neq(1,1)$, and let $r, s$ be even positive integers. If $p=q$ and $r=s$, then $K_{p, q} \oplus K_{r, s}$ is a supermagic graph.

Proof. Applying Theorems 1.1, 5.2 and 5.3.
Below is an example of balanced d-magic graph $K_{1,3} \oplus K_{2,2}$ (see Figure 5), and the labels on edges of $K_{1,3} \oplus K_{2,2}$ are shown in Table 5.

## 6. Balanced Degree-Magic Labelings in the Strong Product of Complete Bipartite Graphs

the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \otimes H$ if and only if $u=x$ and $v$ is adjacent with $y$ in $H$, or $v=y$ and $u$ is adjacent with $x$ in $G$, or $u$ is adjacent with $x$ in $G$ and $v$ is adjacent with $y$ in $H$. For any positive integers $p, q, r$ and $s$, we consider the strong product $K_{p, q} \otimes K_{r, s}$ of complete bipartite graphs. Let $K_{p, q} \otimes K_{r, s}$ be a d-magic graph. Since $\operatorname{deg}(v)$ is $p+r+p r, p+s+p s, q+r+q r$ or $q+s+q s$ and $f^{*}(v)=(p q(r+s)+r s(p+q)+2 p q r s+1) \operatorname{deg}(v) / 2$ for any $v \in V\left(K_{p, q} \otimes K_{r, s}\right)$, we have

Proposition 6.1. Let $K_{p, q} \otimes K_{r, s}$ be a d-magic graph. Then the following conditions hold:
(i) only one of $p, q, r$ and $s$ is even or
(ii) all of $p, q, r$ and $s$ are even.

Proposition 6.2. Let $K_{p, q} \otimes K_{r, s}$ be a balanced d-magic graph. Then $p, q, r$ and $s$ are even.

We conclude this paper with an identification of the sufficient condition for the existence of balanced d-magic labelings of the strong product of complete bipartite graphs $K_{p, q} \otimes K_{r, s}$.

Theorem 6.3. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq(2,2)$ and $(r, s) \neq(2,2)$. Then $K_{p, q} \otimes K_{r, s}$ is a balanced d-magic graph.
Proof. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq(2,2)$ and $(r, s) \neq$ $(2,2)$. Thus, $K_{p, q} \times K_{r, s}$ is balanced d-magic by Theorem 4.3, and $K_{p, q} \oplus K_{r, s}$ is balanced d-magic by Theorem 5.3. Since $K_{p, q} \otimes K_{r, s}$ is the graph such that $K_{p, q} \times K_{r, s}$ and $K_{p, q} \oplus K_{r, s}$ form its decomposition, by Theorem 1.5, $K_{p, q} \otimes K_{r, s}$ is a balanced d-magic graph.

It is clear that the strong product graph $K_{p, q} \otimes K_{r, s}$ is isomorphic to $K_{r, s} \otimes$ $K_{p, q}$.

Corollary 6.4. Let $p, q, r$ and $s$ be even positive integers with $(p, q) \neq(2,2)$ and $(r, s) \neq(2,2)$. If $p=q$ and $r=s$, then $K_{p, q} \otimes K_{r, s}$ is a supermagic graph.
Proof. Applying Theorems 1.1 and 6.3.

## Acknowledgments

The authors would like to thank the anonymous referee for careful reading and the helpful comments improving this paper.

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[^0]:    * Corresponding Author

    Received 26 December 2015; Accepted 1 June 2016
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