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Vector Space Semi-Cayley Graphs

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Abstract. The vector space Cayley graph $\mathrm{Cay}(\mathcal{V},\mathrm{S})$ is a graph with the vertex set the whole vectors of the vector space $\mathcal{V}$ and two vectors v_1, v_2 join by an edge whenever $v_1 - v_2 \in S$ or $-S$, where S is a basis of $\mathcal{V}$. The vector space Cayley graph contains copies of the n-gons, where n is the cardinal number of the field that $\mathcal{V}$ is constructed over it. $\Gamma(\mathcal{V}, S)$ is another graph with vertex set $\mathcal{V}$, which is defined in this paper. It is a graph whose vertices v and w are adjacent whenever $c_1 v + c_2 \omega = \sum_{i=1}^{n} \alpha_i$, where $v, w \in \mathcal{V}$, $S = \{\alpha_1, \cdots, \alpha_n\}$ is an ordered basis for $\mathcal{V}$ and c_1, c_2 belong to the field that the vector space $\mathcal{V}$ is made of over. It is deduced that if S' is another basis for $\mathcal{V}$ which is constructed by special invertible matrix P, then $\Gamma(\mathcal{V}, S) \cong \Gamma(\mathcal{V}, S')$.

Keywords: Cayley graph, Vector space, Basis.

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1. Introduction

The Cayley graph is a mathematical term which is named after Arthur Cayley a British mathematician. It is a graph that encodes the abstract structure of a group. Suppose that G is a group and S is a generating set. The Cayley graph is a colored directed graph with the vertex set which is identified with G, for each generator s of S is assigned a color c_s. Moreover, the vertices corresponding to the elements g and gs are joined by a directed edge of color c_s, for

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any $g \in G$, $s \in S$. Thus the edge set consists of pairs of the form (g, gs), with $s \in S$ providing the color. The set S is usually assumed to be finite, symmetric $S = S^{-1}$ and the identity element of the group is excluded S. In this case, the uncolored Cayley graph is a simple undirected graph. We can consider S as a subset of non-identity elements G instead of being a generating set. A Cayley graph is connected if and only if $G = \langle S \rangle$. In general the Cayley graph over the group $\langle S \rangle$ is a component of the main Cayley graph over the group G. There are many researches about the Cayley graph have been done by some authors for instance see [3, 9].

Of course, there are some other ways to construct a graph associated to a given algebraic structure. We may refer to the works [1, 12].

Suppose $\mathcal{V}$ is a vector space over a field $\mathbb{F}$. If the dimension is n, then there is some basis of n elements for $\mathcal{V}$. When an order is chosen, the basis can be considered as an ordered basis. The elements of $\mathcal{V}$ are finite linear combinations of elements in the basis, which give rise to unique coordinate representations. Since a given vector v is a finite linear combination of basis elements, the only nonzero entries of the coordinate vector for v will be the nonzero coefficients of the linear combination representing v. Thus the coordinate vector for v is zero except in finitely many entries.

Mathematicians studied about vector spaces associated with a graph such as the vector spaces associated with the sets of cutsets, circuits, and subgraphs of graph. It is well known that the set of all subgraphs of a given graph G constitutes a linear vector space over the field of integers mod 2, where the addition of vectors is the ring-sum operation [5, 11].

In [10], the authors considered the following finite Euclidean graphs. Let $V = \mathbb{F}_q^n$ be the n-dimensional vector space over the finite field $\mathbb{F}_q$ where q is a power of a prime number. For $x, y \in V$, the Euclidean distance $d(x, y) \in \mathbb{F}_q$ is defined by $d(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2$. The Euclidean graph $E_q(n, a)$ was defined as the graph with vertex set V and edge set $E = \{(x, y) \in V \times V | x \neq y, d(x, y) = a\}$.

The original aim of this paper is to construct a graph associated to a vector space. In this paper, several type of semi-Cayley graphs are associated to a vector space. In the next section, by inspiration of the classical definition for the Cayley graph related to a group we define Cayley graph of a vector space. The general properties of vector space Cayley graph $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$ are discussed. We observe that $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$ is a graph which contains an induced subgraph which are cycles with $\mathrm{Card}(\mathbb{F})$ vertices. Finally, the graph $\Gamma(\mathcal{V}, S)$ is presented. We observe that $\Gamma(\mathcal{V}, S) \cong \Gamma(\mathcal{V}, S')$, if S' is constructed by special invertible matrix P.

Throughout the paper, all the notations and terminologies about the graphs are found in [2, 4].

2. The Vector Space Cayley Graph

For a vector space $\mathcal{V}$ with an ordered basis S we define a graph which its behavior is similar to the Cayley graph of a group and its generator set. Let us call it vector space Cayley graph and denote it by $\overrightarrow{Cay}(\mathcal{V},\mathrm{S})$. Clearly by definition of the ordinary Cayley digraph two vectors v_1, v_2 join by an arc whenever $v_1 - v_2 \in S$. It is a digraph without any loops and multiple arcs, so it is a simple digraph.

For a node, the number of head endpoints adjacent to a node is called the indegree of the node and the number of tail endpoints adjacent to a node is its outdegree. The indegree is denoted by $\deg^-(v)$ and the outdegree as $\deg^+(v)$. A vertex with $\deg^-(v)$=0 is called a source, as it is the origin of each of its incident edges. Similarly, a vertex with $\deg^+(v) = 0$ is called a sink.

The degree sum formula states that, for a directed graph, $\sum_{v\in V} \deg^+(v) = \sum_{v\in V} \deg^-(v) = |A|$, where A is the set of arcs. If for every node $v \in V$, $\deg^+(v) = \deg^-(v)$, the graph is called a balanced digraph.

It is clear that the zero vector is a sink node and is adjacent to all elements of the basis. Every two elements of the basis does not join. Moreover $\deg^-(v) = |\{v - s_i : s_i \in S\}| = \deg^+(v) = |\{s_i + v : s_i \in S\}|$, where S is the basis of the vector space and v an arbitrary vertex. Thus $\overrightarrow{Cay}(\mathcal{V},\mathrm{S})$ is a balanced digraph.

If we consider the adjacency of two vertices v_1 and v_2 as $v_1 - v_2 \in S$ or $-S$, then we have a simple graph without orientation, let us denote it by $\mathrm{Cay}(\mathcal{V},\mathrm{S})$, where $-S$ is the set of all additive inverse vectors of S. In the sequel by vector space Cayley graph we mean this undirected graph.

It is obvious that this graph is regular. If $\mathcal{V}$ is a finite dimensional vector space over the field of real numbers, then the degree of each vertex is equal to the $2\mathrm{Card}(S)$. Since there is no element in S which is equal to its inverse as S is a linear independent set. If $\mathcal{V}$ is a finite dimensional vector space with a basis which does not contain any element that is equal to its inverse, then the degree of each vertex is equal to the $2\mathrm{Card}(S)$.

Let $k > 0$ be an integer. A vertex k-coloring of a graph Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(\Gamma)$ of a graph Γ, is the minimum k for which Γ has a vertex k-coloring.

If we consider the vector space Cayley graph which is constructed over the field $\mathbb{F}$ such that $\mathrm{Card}(\mathbb{F}) > 2$, then $\mathrm{Cay}(\mathbb{F},\{1\})$ is a cycle graph. Thus $\mathrm{girth}(\mathrm{Cay}(\mathbb{F},\{1\})) = \mathrm{Card}(\mathbb{F})$ and $\chi(\mathrm{Cay}(\mathbb{F},\{1\})) = 2$ or 3 (depends on $\mathrm{Card}(\mathbb{F})$ is even or odd number), where the notation $\mathrm{Card}(\mathbb{F})$ is used to denote the cardinal number of the underlying set $\mathbb{F}$.

Proposition 2.1. *If* $\mathrm{Cay}(\mathcal{V},\mathrm{S})$ *is a vector space Cayley graph of the vector space* $\mathcal{V}$ *of dimension greater than or equal to* 2, *then* $\mathrm{girth}(\mathrm{Cay}(\mathcal{V},\mathrm{S})) \le 4$.

Proof. Since S contains at least two elements s_1, s_2, we have the cycle which is made by the vertices $0, s_1, s_2$ and $s_1 + s_2$. $\square$

Suppose $\mathcal{V}$ is a finite dimensional vector space with a basis which does not contain any element that is equal to its inverse. For every vertex v of the graph $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$, we can observe that v joins to $v - s_i$ and $v + s_i$, where $s_i \in S$. Thus, its neighborhood can be folded into half so that the two halves match exactly. By this fact the Figure (1) shows the adjacency condition for a vertex v, for which it draws in half. Actually, the vertex v join $v + s_i$ $(1 \le i \le n = \mathrm{Card}(S))$ vertices, that has not been displayed in the Figure (1). For a graph Γ and a

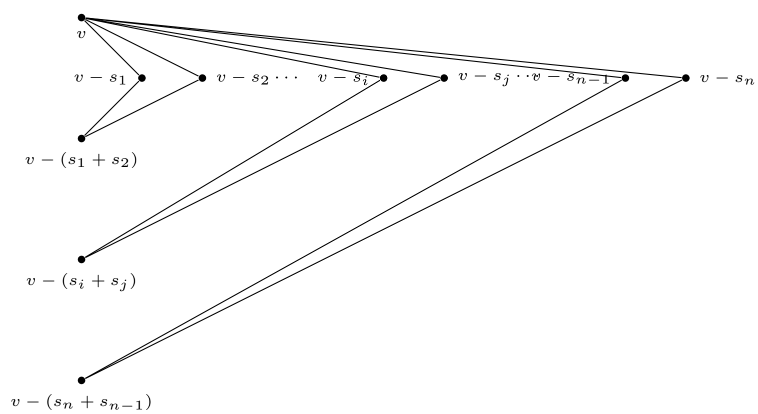

Figure 1

subset T of the vertex set $V(\Gamma)$, the closed neighborhood of T is denote by $N_\Gamma[T]$. It is the set of all neighborhoods of vertices belong to T, in which included the vertices of T too. If $N_\Gamma[T] = V(\Gamma)$, then T is said to be a dominating set of vertices in Γ. The domination number of a graph Γ, denoted by $\gamma(\Gamma)$, is the minimum size of a dominating set of the vertices in Γ.

Proposition 2.2. *If $\mathcal{V}$ is a finite dimensional vector space with a basis which does not contain any element that is equal to its inverse, then* $\gamma(\mathrm{Cay}(\mathcal{V}, \mathrm{S})) \ge \lceil \mathrm{Card}(\mathcal{V})/2\mathrm{Card}(S) \rceil$.

Proof. As discussed before the proposition, we observe that every vertex v dominate $2\mathrm{Card}(S)$ vertices of the graph $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$. This means, at least there is a certain vertex v for every $2\mathrm{Card}(S)$ vertices. Hence, we can form a dominating set with $\lceil \mathrm{Card}(\mathcal{V})/2\mathrm{Card}(S) \rceil$ vertices and clearly the smallest dominating set has at most $\lceil \mathrm{Card}(\mathcal{V})/2\mathrm{Card}(S) \rceil$ vertices. $\square$

Let T be a dominating set for $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$. The vertices of the graph are divided to two sets. The vertices which belong to T and the vertices which do not belong to T. The components of the coordinate of $v \in V \setminus T$ are the same as all components of the coordinate of at least one vector in T and the difference is just in one component.

It is clear that, if $\mathcal{V}$ is a trivial group, a field $\mathbb{F}$, a 2-dimensional vector space or a 3-dimensional vector space over the field $\mathbb{F}_2$, then $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$ is a planar graph.

Suppose R is a ring with identity, then an R-module $\mathcal{V}$ has a basis if and only if it is isomorphic to a direct sum of copies of left R-module R (see [8, Theorem 2.1]). If R is a division ring, then a unitary R-module is called a vector space.

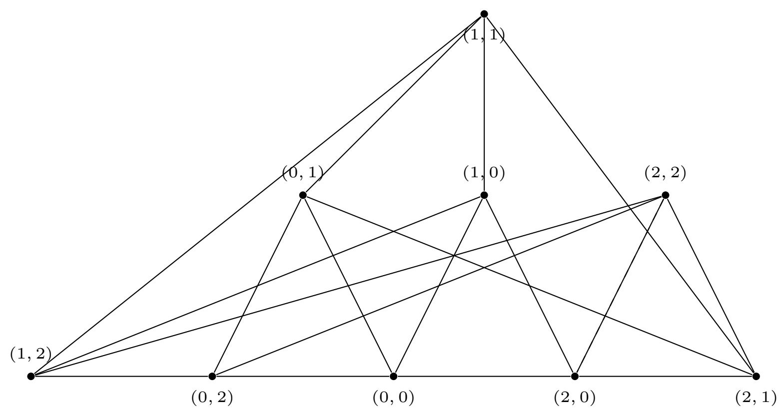

Figure 2

Proposition 2.3. *Let $\mathcal{V}$ be a vector space over the finite field $\mathbb{F} \neq \mathbb{F}_2$ with the basis S. Then $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$ is a graph which contains copies of induced subgraphs isomorphic to n-gons, where $n = \mathrm{Card}(\mathbb{F})$.*

Proof. By Theorem 2.1 in [8] mentioned above $\mathcal{V} \cong \bigoplus \sum \mathbb{F}$ and copies of $\mathbb{F}$ are indexed by $\mathrm{Card}(S)$. It is clear that $\mathrm{Cay}(\mathbb{F}, \{1\})$ is the induced subgraph of $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$. □

Example 2.4. Consider the vector space $\mathcal{V}$ with basis $S = \{(1,0), (0,1)\}$ over the field $\mathbb{Z}_3$. Therefore $\mathcal{V}$ has 9 vector

$$\{(0,0), (1,0), (0,1), (1,1), (1,2), (2,1), (2,2), (2,0), (0,2)\}.$$

The vector space Cayley graph associated to $\mathcal{V}$ has an induced subgraph made of triangles (See Figure (2)).

3. The Convex Vector Space Graph

In graph theory, the interval between a pair u, v of vertices in a graph G is the collection of all vertices that lie on some shortest path between u, v in G.

A subset C of vertices of a graph is said to be convex if it contains the interval between every pair of vertices in C.

Definition 3.1. Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. The convex vector space graph is a graph with the vertex set whole vectors of the vector space $\mathcal{V}$ and two vectors υ and ω join by an edge if there exist $c_1, c_2 \in \mathbb{F}$ such that $c_1\upsilon + c_2\omega \in S$, where S is an ordered basis for $\mathcal{V}$ and c_1, c_2 are not zero in the same time. We denote this graph by $\mathrm{Cay}^g(\mathcal{V}, S)$.

The vector space Cayley graph $\mathrm{Cay}(\mathcal{V}, \mathrm{S})$ can be considered as a subgraph of $\mathrm{Cay}^g(\mathcal{V}, S)$. It is clear that all the vertices which belong to the ordered basis S are adjacent and S is a convex set.

Lemma 3.2. *Let $\mathcal{V}$ be a vector space of finite dimension n over the field $\mathbb{F}$. The neighborhoods of the vertex υ with the coordinate $(x_1, \cdots, x_n)$ with respect to the ordered basis S, are all vectors with the coordinate $(-\frac{c_1}{c_2}x_1, \cdots, \frac{1}{c_2} - \frac{c_1}{c_2}x_k, \cdots, -\frac{c_1}{c_2}x_n)$, where $x_i, c_1, 0 \neq c_2 \in F$, and $1 \leq i, k \leq n$.*

Proof. Suppose ω is a vertex which is adjacent to the vector υ. It means there are $c_1, c_2 \in F$ such that $c_1\upsilon + c_2\omega = s_k \in S$. A computation implies the result. □

By Lemma 3.2 the zero vector is adjacent to all vectors with a coordinate of the form $(0, \ldots, \frac{1}{c_2}, \ldots, 0)$ which has just one non-zero component $(0 \neq c_2 \in \mathbb{F})$. This means the convex vector space graph is not complete. Moreover, if we omit the zero vector from the vector set of the convex vector space graph, then it is still not complete. For instance, the vertex $(1, 1)$ of the vector space $\mathcal{V}$ with basis $S = \{(1, 0), (0, 1)\}$ over the field $\mathbb{Z}_3$ does not join to $(2, 2)$.

Theorem 3.3. *Let $\mathcal{V}$ be a vector space. Then*

(i) *The diameter and girth of the convex vector space graph is 2 and 3, respectively.*
(ii) $\gamma(\Gamma) = 1$.

Proof. The assertion follows by the fact that all vertices join the elements of the basis of the vector space. □

One can easily deduce that the convex vector space graph is connected.

4. A graph $\Gamma(\mathcal{V}, S)$ associated to a vector space

Replace the term of adjacency of two vertices $\upsilon, \omega \in \mathcal{V}$ in Definition 3.1 by $c_1\upsilon + c_2\omega = \sum_{i=1}^{n} \alpha_i$, where $S = \{\alpha_1, \cdots, \alpha_n\}$ is an ordered basis for $\mathcal{V}$. Therefore by similar computation in Lemma 3.2 we deduce that the neighborhoods of the vertex υ with the coordinate $(x_1, \cdots, x_n)$ with respect to the ordered basis S, are all vectors with the coordinate $(\frac{1}{c_2} - \frac{c_1}{c_2}x_1, \cdots, \frac{1}{c_2} - \frac{c_1}{c_2}x_k, \cdots, \frac{1}{c_2} - \frac{c_1}{c_2}x_n)$, where $x_i, c_j \in F$ $j = 1, 2$ and $1 \leq i \leq n$. Let us denote this graph by $\Gamma(\mathcal{V}, S)$.

If $\mathcal{V}$ is a vector space over a finite field, then $\Gamma(\mathcal{V}, S)$ is not a regular graph, since the degree of zero vector is $\mathrm{Card}(\mathbb{F})-1$ and $\deg(\sum_{i=1}^{n}\alpha_i) = \mathrm{Card}(\mathcal{V})-1$.

If $\dim(\mathcal{V}) = 2$ with the ordered basis $S = \{\alpha_1, \alpha_2\}$, then clearly α_1, α_2 are adjacent. Moreover, $\upsilon \in \mathcal{V}$ is adjacent to α_i whenever the j-th component of its coordinate is not zero, where $i, j = 1, 2$ and $i \neq j$. By these facts we conclude the first and third part of the following result.

Proposition 4.1. *Suppose $\mathcal{V}$ is a vector space of dimension 2 over the field F. Then*

(i) *Two non-zero vertices υ_1 and υ_2 are adjacent, whenever*

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \neq 0,$$

where (x_i, y_i) is the coordinate of υ_i with respect to the ordered basis S.

(ii) $\gamma(\Gamma(\mathcal{V}, S)) = 1$.

Proof. (i) It is enough to verify the condition under which the system

$$\begin{cases} c_1x_1 + c_2x_2 = 1 \\ c_1y_1 + c_2y_2 = 1 \end{cases}$$

has a solution in $\mathbb{F}$.
(ii) Clearly $(1, 1)$ is adjacent to all other vertices. □

Assume $\dim(\mathcal{V}) = n > 2$. The vertices in the ordered basis S are not adjacent. Furthermore, the vertices υ is adjacent to the i-th vector s_i of the ordered basis S if all the component of its coordinate is a non-zero scaler x while its i-th component is $(1-c)x$. The adjacency of two arbitrary vertices υ, ω depends to the exitance of the scales c_1, c_2 for the system $c_1x_i + c_2y_i = 1$, where $1 \leq i \leq n$, x_i and y_i are the i-th term of the coordinate of υ and ω. It is obvious that the existence of the solution for this system tends to impossible when n tends to large numbers.
Since all the vertices of $\Gamma(\mathcal{V}, S)$ and the vector $\sum_{i=1}^{n}\alpha_i$ are adjacent, we conclude $\mathrm{diam}(\Gamma(\mathcal{V}, \mathrm{S})) = 2$ and $\mathrm{girth}(\Gamma(\mathcal{V}, \mathrm{S})) = 3$. Therefore $\Gamma(\mathcal{V}, S))$ has a cycle of order 3, it is not a bipartite graph. Let P be an invertible matrix such that the sum of entries in each row is 1. If $\mathcal{V}$ is a vector space of dimension n with ordered basis S, then there is a unique ordered basis S' for $\mathcal{V}$ such that the coordinate of every vertex with respect to S' are obtained by the coordinate of that vertex with respect to S and the matrix P (see [7, Theorem 8, Sec 2.4]).

Now we are able to present the following Theorem 4.2.

Theorem 4.2. *Let $\mathcal{V}$ be a vector space of dimension n, with two ordered basis S and S', such that S' is obtained by use of invertible matrix P defined in the above argument. Then $\Gamma(\mathcal{V}, S) \cong \Gamma(\mathcal{V}, S')$.*

Proof. It is enough to prove that the identity map $I : V(\Gamma(\mathcal{V}, S)) \to V(\Gamma(\mathcal{V}, S'))$ between the vertex set of two graphs, preserves the edges. Suppose υ, ω are two adjacent vertices in $\Gamma(\mathcal{V}, S)$. If the coordinate of υ is $(x_1, \cdots, x_n)$ with respect to the ordered basis S, then the coordinate of ω is $(\frac{1}{c_2} - \frac{c_1}{c_2}x_1, \cdots, \frac{1}{c_2} - \frac{c_1}{c_2}x_k, \cdots, \frac{1}{c_2} - \frac{c_1}{c_2}x_n)$, where $x_i, c_j \in F$, $j = 1, 2$ and $1 \leq i \leq n$, by the argument before the theorem. Now the question is whether υ and ω are adjacent in $\Gamma(V, S')$. In the other words are they adjacent with respect to S'? Assume p_{kl} are entries of the matrix P, $1 \leq k, l \leq n$. The coordinate of υ and ω with respect to S' is $(\Sigma_{r=1}^n p_{1r}x_r, \cdots, \Sigma_{r=1}^n p_{nr}x_r)$ and $(\frac{1}{c_2}\Sigma_{r=1}^n p_{1r} + (\frac{-c_1}{c_2})\Sigma_{r=1}^n p_{1r}x_r, \cdots, \frac{1}{c_2}\Sigma_{r=1}^n p_{1r} + (\frac{-c_1}{c_2})\Sigma_{r=1}^n p_{nr}x_r)$, respectively. Since P is an invertible matrix such that the sum of entries in each row is 1, we conclude υ, ω join by an edge in $\Gamma(\mathcal{V}, S')$. □

In order to have a sufficient apperception of the above theorem we take a look at the following example.

Example 4.3. Consider P as an invertible 2×2 matrix over the field $\mathbb{R}$. The sum of entries of each row is 1.

$$P = \begin{bmatrix} \frac{-1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

Suppose $S = \{(1, 0), (0, 1)\}$ is the basis for the vector space $\mathbb{R}^2$ and let $S' = \{(-2, 0), (3, 1)\}$ is a basis made by P as in [7, Theorem 8, Sec 2.4]. Moreover, φ is the graph isomorphism between two graphs $\Gamma(\mathbb{R}^2, S)$ and $\Gamma(\mathbb{R}^2, S')$. If $[v_1]_S = (x_1, y_1), [v_2]_S = (x_2, y_2)$ are two adjacent vertices in $\Gamma(\mathbb{R}^2, S)$, then $[\varphi(v_1)]_{S'} = (\frac{-1}{2}x_1 + \frac{3}{2}y_1, y_1)$ and $[\varphi(v_2)]_{S'} = (\frac{-1}{2}x_2 + \frac{3}{2}y_2, y_2)$ are adjacent. Since the adjacency of v_1, v_2 implies that $x_2 = \frac{1}{c_1} - \frac{c_1}{c_2}x_1$ and $y_2 = \frac{1}{c_1} - \frac{c_1}{c_2}y_1$, for $c_1, c_2 \in \mathbb{R}$. An easy computation ensure the adjacency of $\varphi(v_1)$ and $\varphi(v_2)$.

Similar result as Theorem 4.2 can be obtained for $\mathrm{Cay}^g(\mathcal{V}, S)$.

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