Iranian Journal of Mathematical Sciences and Informatics
Vol. 13, No. 2 (2018), pp 59-70
DOI: 10.7508/ijmsi.2018.13.005

L_1-Biharmonic Hypersurfaces in Euclidean Spaces with Three Distinct Principal Curvatures

Akram Mohammadpouri[*,a], Firooz Pashaie[b], Sepide Tajbakhsh[a]

[a]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.
[b]Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P.O.Box 55181-83111, Maragheh, Iran.

E-mail: pouri@tabrizu.ac.ir
E-mail: f_pashaie@maragheh.ac.ir
E-mail: st_sepide@yahoo.com

Abstract. A submanifold M^n of the Euclidean space $\mathbb{E}^{n+m}$ is said to be biharmonic if its position map $x : M^n \to \mathbb{E}^{n+m}$ satisfies the condition $\Delta^2 x = 0$, where Δ stands for the Laplace operator. A well-known conjecture of Bang-Yen Chen says that, the only biharmonic submanifolds of Euclidean spaces are the minimal ones. In this paper, we consider a modified version of the conjecture, replacing Δ by its extension, L_1-operator (namely, L_1-conjecture). The L_1-conjecture states that any L_1-biharmonic Euclidean hypersurface is 1-minimal. We prove that the L_1-conjecture is true for L_1-biharmonic hypersurfaces with three distinct principal curvatures and constant mean curvature of a Euclidean space of arbitrary dimension.

Keywords: Linearized operators L_1, L_1-Biharmonic hypersurfaces, 1-Minimal.

2000 Mathematics subject classification: 53D12, 53C40, 53C42.

*Corresponding Author

Received 14 December 2015; Accepted 27 October 2016

1. Introduction

The concept of harmonic maps plays important roles in differential geometry, computational geometry and physical theories of elastics and fluid mechanics. In applied mathematics, some partial differential equations have analytical solutions in terms of harmonic functions (see for instance [13, 14]). Sometimes, it becomes very difficult to find harmonic functions whereas biharmonic ones make help us to solve related differential equations. As a geometric example, there exists no harmonic map as $\mathbb{T}^2 \to \mathbb{S}^2$ (whatever the metrics chosen) in the homotopy class of Brower degree ± 1 and hence, it is useful to find a biharmonic map from $\mathbb{T}^2$ into $\mathbb{S}^2$ ([9]). Obviously, harmonic maps are biharmonic but not vis versa. If a map biharmonic is non-harmonic, then it is said to be *proper-biharmonic*. Proper-biharmonic maps facilitate the study of pseudo-umbilical and parallel submanifolds.

A well-known conjecture of Bang-Yen Chen (in 1987) says that every biharmonic Riemannian submanifold of Euclidean m-space (of arbitrary dimension m), $\mathbb{E}^m$, is minimal. Chen himself has proven his conjecture for Euclidean surfaces in 3-space, $\mathbb{E}^3$. The conjecture has been affirmed in some extended cases. In 1992, I. Dimitrić proved that any biharmonic hypersurface in $\mathbb{E}^m$ (of arbitrary dimension m) with at most two distinct principal curvatures is minimal ([7]). Also, in 1995, T. Hasanis and T. Vlachos have proven Chen conjecture on the 3-dimensional Euclidean hypersurfaces ([12]). K. Akutagawa and S. Maeta ([2]) have studied on a general version of the conjecture on compelet biharmonic submanifolds of the Euclidean spaces. In 2013, B.Y. Chen and M.I. Munteanu ([6]) affirmed the conjecture for every $\delta(2)$-ideal or $\delta(3)$-ideal hypersurface of the Euclidean space of arbitrary dimension. Recently, R. Gupta ([11]) has proven that every biharmonic hypersurface with three distinct principal curvatures in $\mathbb{E}^m$ is minimal. On the other hand, there exists a nice relation between the finite type hypersurfaces and biharmonic ones (see [7]). The theory of finite type hypersurfaces has been interested by B.Y. Chen and followed by L.J. Alias, S.M.B. Kashani and others (see [3, 5, 15]). One can see main results in the last chapter of Chen's book ([5]). In [15], S.M.B. Kashani has introduced the notion of L_r-finite type hypersurface as an extension of finite type hypersurface in the Euclidean space, followed by the first author in her doctoral thesis.

The map L_r, as a natural extension of the Laplace operator $L_0 = \Delta$, stands for the linearized operator of $(r+1)$th mean curvature of a hypersurface M^n in $\mathbb{E}^{n+1}$, for $r = 0, \cdots, n-1$ (see [3, 17]). The L_r-operator is given by $L_r(f) = tr(P_r \circ \nabla^2 f)$ for any smooth function f on M^n, where P_r is the r-th Newton transformation associated to the second fundamental from of M^n and $\nabla^2 f$ is the hessian of f.

It seems interesting to generalize the definition of biharmonic hypersurface by replacing Δ by L_r. We call these hypersurfaces L_r-*biharmonic*. Recently, M. Aminian and S.M.B. kashani ([4]) have stated a general version of Chen conjecture, which says that, every Euclidean hypersurface $x : M^n \to \mathbb{E}^{n+1}$ satisfying the condition $L_r^2 x = 0$ for some r, $(0 \leq r \leq n-1)$ is r-minimal. They proved that the L_r-conjecture is true for Euclidean hypersufaces with at most two principal curvatures and L_r-finite type hypersurfaces. In this paper, we prove that the L_1-conjecture is true for Euclidean hypersufaces with three distinct principal curvature and constant mean curvature. Here is our main result:

Theorem 1.1. *Every L_1-biharmonic hypersurface in $\mathbb{E}^{n+1}$ with constant mean curvature and three distinct principal curvatures is 1-minimal.*

2. Preliminaries

In this section, we recall preliminaries from [3, 10]. Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from a Riemannian manifold M^n of dimension n into the Euclidean space $\mathbb{E}^{n+1}$, with a unite normal vector field (Gauss map) N. The symbols ∇^0 and ∇ stands for Levi-Civita connections on $\mathbb{E}^{n+1}$ and M^n, respectively. The Gauss formula on M^n is given by $\nabla^0_X Y = \nabla_X Y + < SX, Y > N$, where $S : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the shape operator (or Weingarten endomorphism) of M^n defined by $SX = -\nabla^0_X N$, for every tangent vector fields X and Y on M^n. As it is known, at each point $p \in M$, S_p is a self-adjoint linear endomorphism on the tangent space, T_pM, and its eigenvalues $\lambda_1(p), \cdots, \lambda_n(p)$ are defined as the principal curvatures and the corresponding orthonormal vectors (local basis) $\{e_1, \cdots, e_n\}$ are called the principal directions on M^n. The characteristic polynomial of S is defined by

$$Q_S(t) = \det(tI - S) = \sum_{k=0}^{n} (-1)^k a_k t^{n-k},$$

where $a_0 = 1$ and for $k = 1, \cdots, n$, a_k is given by

$$a_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k}. \tag{2.1}$$

The *r-th mean curvature* H_r of M^n is defined by $\binom{n}{r} H_r = a_r$ for $1 \leq r \leq n$ and $H_0 = 1$. If $H_{r+1} \equiv 0$, the hypersurface M^n is said to be *r-minimal*. The *r-th Newton transformation* of M^n is the operator $P_r : \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ defined by

$$P_r = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j} S^j = \sum_{j=0}^{r} (-1)^j a_{r-j} S^j.$$

Equivalently,

$$P_0 = I, \ P_r = \binom{n}{r} H_r I - S \circ P_{r-1}.$$

At each point $p \in M$, the restricted map $P_r : T_pM \to T_pM$ is a self-adjoint linear operators that commutes with S and its eigenvalues with respect to the orthonormal (local) basis $\{e_1, \cdots, e_n\}$ of principal directions on M^n are given by $P_re_i = \mu_{i,r}e_i$ (for $i = 1, \cdots, n$), where

$$\mu_{i,r} = \sum_{\substack{1\le i_1<\cdots<i_r\le n \\ i_j\neq i}} \lambda_{i_1}\cdots\lambda_{i_r}.$$

We will use a helpful formula from [3] as:

$$tr(S^2 \circ P_1) = \frac{n(n-1)(n-2)}{2}(2HH_2 - H_3). \tag{2.2}$$

Associated to the Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ given by $L_r(f) = tr(P_r \circ \nabla^2 f)$, where $\nabla^2 f : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes a self-adjoint linear operator metrically equivalent to the Hessian of f, given by $< \nabla^2 f(X), Y >=< \nabla_X(\nabla f), Y >$ for every vector fields X and Y on M. In terms of the local orthonormal basis $\{e_1, \ldots, e_n\}$, $L_r(f)$ is given by

$$L_r(f) = \sum_{i=1}^{n} \mu_{i,r}(e_ie_if - \nabla_{e_i}e_if). \tag{2.3}$$

3. L_r-Biharmonic Hypersurfeces in $\mathbb{E}^{n+1}$

Let $x : M^n \to \mathbb{E}^{n+1}$ be a connected orientable hypersurface immersed into the Euclidean space, with Gauss map N. By definition, M^n is called a L_r-biharmonic hypersurface if its position vector field satiesfies the condition $L_r^2x = 0$. By the equality $L_rx = c_rH_{r+1}N$ from [3], the condition $L_r^2x = 0$ has another equivalent expression as $L_r(H_{r+1}N) = 0$. It is clear that the r-minimal hypersurface is L_r-biharmonic. By formulae in [3] page 122, we have

$$L_r^2x = -2c_r(S\circ P_r)\nabla H_{r+1} - c_r\binom{n}{r+1}H_{r+1}\nabla H_{r+1} - c_r(tr(S^2\circ P_r)H_{r+1} - L_rH_{r+1})N, \tag{3.1}$$

where $c_r = (r+1)\binom{n}{r+1}$.

By using this formula for L_r^2x and the identifying normal and tangent parts of the L_r-biharmonic condition $L_r^2x = 0$, one obtains necessary and sufficient conditions for M^n to be L_r-biharmonic in $\mathbb{E}^{n+1}$, namely

$$L_rH_{r+1} = tr(S^2 \circ P_r)H_{r+1} \tag{3.2}$$

and

$$(S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2}\binom{n}{r+1}H_{r+1}\nabla H_{r+1}. \tag{3.3}$$

3.1. **Proof of theorem 1.1.** From now on, we concentrate on L_1-biharmonic hypersurfaces M^n in a Euclidean space $\mathbb{E}^{n+1}$ with three distinct principal curvatures and constant mean curvature H. We assume that the 2th mean curvature H_2 is not constant, so there exists of an open connected subset $\mathcal{U}$ of M, with $\nabla H_2(p) \neq 0$ for all $p \in \mathcal{U}$. We shall contradict the condition $\nabla H_2(p) \neq 0,\ \forall p \in \mathcal{U}$.

We assume that $\{e_1, e_2, \ldots, e_{n-1}, e_n\}$ be a local orthonormal frame of principal directions of the shape opreator, S, on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ $(1 \leq i \leq n)$. Then we have $P_2 e_i = \mu_{i,2} e_i$, for every i. We have

$$H_2 = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j. \tag{3.4}$$

From (3.3) (using the inductive definition of P_2) we get

$$P_2(\nabla H_2) = \frac{3}{4} n(n-1) H_2 \nabla H_2 \ on\ \mathcal{U}. \tag{3.5}$$

Observe from (3.5) that ∇H_2 is an eigenvector of P_2 with the corresponding eigenvalue $\frac{3}{4}n(n-1)H_2$. Without loss of generality, we can choose e_1 such that e_1 is parallel to ∇H_2. Since the shape operator S and P_2 can be simultaneously diagonalized, therefore the shape operator S of M^n takes the form with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_{n-1}, e_n\}$

$$\begin{pmatrix} .\lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \lambda_n \end{pmatrix}. \tag{3.6}$$

Then we have

$$\mu_{1,2} = \frac{3}{4} n(n-1) H_2. \tag{3.7}$$

We can decompose $\nabla H_2 = \sum_{i=1}^n e_i(H_2) e_i$. Since e_1 is parallel to ∇H_2, it follows that

$$e_1(H_2) \neq 0,\ e_i(H_2) = 0\ i = 2, \ldots, n. \tag{3.8}$$

We write

$$\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k,\ i,j = 1, 2, \ldots, n. \tag{3.9}$$

The compatibility conditions $\nabla_{e_k} < e_i, e_i >= 0$ and $\nabla_{e_k} < e_i, e_j >= 0$ imply respectively that

$$\omega_{ki}^i = 0,\ \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.10}$$

for $i \neq j$ and $i, j, k = 1, 2, \ldots, n$. Furthermore, it follows from the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.11}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{3.12}$$

for distinct $i, j, k = 1, 2, \ldots, n$.

Since $\mu_{1,2} = \dfrac{3}{4}n(n-1)H_2$, from (3.4) we have

$$H_2 = \frac{4}{n(n-1)}\lambda_1(\lambda_1 - nH), \tag{3.13}$$

therefore, we get

$$e_1(\lambda_1) \neq 0,\ e_i(\lambda_1) = 0\ i = 2, \ldots, n. \tag{3.14}$$

One can compute that

$$[e_i, e_j](\lambda_1) = 0,\ i, j = 2, \ldots, n,$$

which yields directly

$$\omega_{ij}^1 = \omega_{ji}^1, \tag{3.15}$$

for $i \neq j$ and $i, j = 2, \ldots, n$.

Now we show that $\lambda_j \neq \lambda_1$ for $j = 2, \ldots, n$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, by putting $i = 1$ in (3.11) we have that

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts the first expression of (3.14).

Since M^n has three distinct principal curvatures, we can assume that $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = \lambda$ and $\lambda_n \neq \lambda$, hence $\lambda_n = nH - \lambda_1 - (n-2)\lambda$.

Consider Eqs. (3.11) and (3.12).

For $i, j = 2, 3, \ldots, n-1$, and $i \neq j$ in (3.11). One has

$$e_j(\lambda) = 0,\ for\ j = 2, \ldots, n-1. \tag{3.16}$$

For $j = 1$ and $i \neq 1$ in (3.11), by (3.14) we have $\omega_{1i}^1 = 0$ $(i \neq 1)$. Moreover, by the first expression of (3.10) we have

$$\omega_{1i}^1 = 0,\ i = 1, 2, \ldots, n.$$

For $i = 2, \ldots, n-1, j = n$ in (3.11), by (3.16) we have

$$\omega_{ni}^n = 0,\ i = 2, 3, \ldots, n-1.$$

For $i = 1, j = 2, 3, \ldots, n$ in (3.11), we obtain

$$\omega_{n1}^n = -\frac{e_1(\lambda_1 + (n-2)\lambda)}{2\lambda_1 + (n-2)\lambda - nH},\ \omega_{j1}^j = \frac{e_1(\lambda)}{\lambda_1 - \lambda},\ j = 2, 3, \ldots, n-1. \tag{3.17}$$

For $i = n, j = 2, 3, \ldots, n-1$ in (3.11), we obtain

$$\omega_{jn}^j = \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda},\ j = 2, 3, \ldots, n-1.$$

For $i = 1$, $j \neq k$ and $j, k = 2, 3, \ldots, n-1$ in (3.12), we have

$$\omega_{k1}^{j} = 0, \ j \neq k, \ and \ j, k = 2, 3, \ldots, n-1.$$

For $i = n$, $j \neq k$ and $j, k = 2, 3, \ldots, n-1$, in (3.12), we get

$$\omega_{kn}^{j} = 0, \ j \neq k, \ and \ j, k = 2, 3, \ldots, n-1.$$

For $i = n$ and $j = 1, k = 2, 3, \ldots, n-1$ in (3.12), and using (3.15) we get

$$\omega_{kn}^{1} = \omega_{nk}^{1} = 0, \ k = 2, 3, \ldots, n-1.$$

Similarly, we can also obtain

$$\omega_{1k}^{n} = \omega_{k1}^{n} = 0, \ k = 2, 3, \ldots, n-1.$$

Let us introduce two smooth functions α and β as followes:

$$\alpha = \frac{e_1(\lambda)}{\lambda_1 - \lambda}, \ \beta = \frac{e_1(\lambda_1 + (n-2)\lambda)}{2\lambda_1 + (n-2)\lambda - nH}. \tag{3.18}$$

We have the following:

Lemma 3.1. *Let M^n be an n-dimensional biharmonic hypersurface with three distnct principle curvature in Euclidean space, having the shape operator given by (3.6) with respect to suitable orthonormal farame $\{e_1, e_2, \ldots, e_{n-1}, e_n\}$. Then we obtain*

$$\nabla_{e_1} e_1 = 0, \ \nabla_{e_i} e_1 = \alpha e_i, \ i = 2, \ldots, n-1, \ \nabla_{e_n} e_1 = -\beta e_n,$$

$$\nabla_{e_i} e_i = -\alpha e_1 + \sum_{i \neq j, j=2}^{n-1} \omega_{ii}^{j} e_j - \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} e_n, \ i = 2, 3, \ldots, n-1,$$

$$\nabla_{e_i} e_j = \sum_{i \neq j, k=2}^{n-1} \omega_{ij}^{k} e_k, \ i, j = 2, 3, \ldots, n-1,$$

$$\nabla_{e_1} e_n = 0, \ \nabla_{e_n} e_n = \beta e_1, \ \nabla_{e_i} e_n = \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} e_i, \ i = 2, 3, \ldots, n-1. \tag{3.19}$$

where ω_{ij}^{k} satisfies (3.10) for $i, j, k = 1, 2, \ldots, n-1, n$.

Recall the definition of the Gauss curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Using Lemma 3.1, Gauss equation and comparing the coefficients with respect to a orthonormal basis $\{e_1, e_2, \ldots, e_{n-1}, e_n\}$, we find the following:

- $X = e_1, \ Y = e_2, \ Z = e_1$,

$$e_1(\alpha) + \alpha^2 = -\lambda_1 \lambda; \tag{3.20}$$

- $X = e_1, \ Y = e_2, \ Z = e_n$,

$$e_1 \left(\frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} \right) + \alpha \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} = 0; \tag{3.21}$$

- $X = e_1,\ Y = e_n,\ Z = e_1,$

$$-e_1(\beta) + \beta^2 = -\lambda_1(nH - \lambda_1 - (n-2)\lambda); \tag{3.22}$$

- $X = e_3,\ Y = e_n,\ Z = e_1,$

$$e_n(\alpha) + (\alpha+\beta)\frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} = 0; \tag{3.23}$$

- $X = e_n,\ Y = e_2,\ Z = e_n,$

$$-e_n\left(\frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda}\right) + \alpha\beta - \left(\frac{e_4(\lambda)}{nH - \lambda_1 - (n-1)\lambda}\right)^2 = \lambda(nH - \lambda_1 - (n-2)\lambda). \tag{3.24}$$

Now, we consider the L_1-biharmonic equation (3.2). It follows from (2.3) and (3.19) that

$$\begin{aligned}(\lambda_1 - nH)e_1e_1(H_2) + [(n-2)(\lambda - nH)\alpha + (\lambda_1 + (n-2)\lambda)\beta]\, e_1(H_2)\\ - \frac{n(n-1)(n-2)}{2}H_2(2HH_2 - H_3) = 0.\end{aligned} \tag{3.25}$$

From (3.8) and (3.19), we obtain

$$e_ie_1(H_2) = 0,\ i = 2, 3, \dots, n-1, n. \tag{3.26}$$

Differentiating α and β along e_n, we get equations

$$(\lambda_1 - \lambda)e_n(\alpha) - \alpha e_n(\lambda) = e_ne_1(\lambda),$$

$$(2\lambda_1 + (n-2)\lambda - nH)e_n(\beta) + (n-2)\beta e_n(\lambda) = (n-2)e_ne_1(\lambda),$$

respectively and eliminating $e_ne_1(\lambda)$, we have

$$(2\lambda_1 + (n-2)\lambda - nH)e_n(\beta) = (n-2)\left[(\lambda_1 - \lambda)e_n(\alpha) - (\alpha+\beta)e_n(\lambda)\right].$$

Putting the value of $e_n(\alpha)$ from (3.23) in the above equation, we find

$$e_n(\beta) = \frac{e_n(\lambda)(n-2)(\alpha+\beta)(n\lambda - nH)}{(2\lambda_1 + (n-2)\lambda - nH)(nH - \lambda_1 - (n-1)\lambda)}.$$

Differentiating (3.25) along e_n and using (3.26), (3.23) and $e_n(\beta)$, we get

$$(n-2)e_n(\lambda)\left[\frac{(\alpha+\beta)A}{2\lambda_1 + (n-2)\lambda - nH}e_1(H_2) - H_2(nH - \lambda_1 - (n-1)\lambda)B\right] = 0. \tag{3.27}$$

where $A := 4nH\lambda_1 - 2{\lambda_1}^2 - 2(n-1)\lambda\lambda_1 + 2n(n-1)H\lambda - 2n^2H^2$ and $B := n^2H^2 + 3\lambda_1^2 + (3(n-2)^2 - 3)\lambda^2 + (2n - 4n(n-2))H\lambda - 4nH\lambda_1 + 6(n-2)\lambda\lambda_1$. We claim that $e_n(\lambda) = 0$. Indeed, if $e_n(\lambda) \neq 0$, then

$$\frac{(\alpha+\beta)A}{2\lambda_1 + (n-2)\lambda - nH}e_1(H_2) - H_2(nH - \lambda_1 - (n-1)\lambda)B = 0, \tag{3.28}$$

Now, differentiating (3.28) along e_n, we have

$$\frac{(\alpha+\beta)\left[A((n-4)\lambda_1+2(n-2)^2\lambda+(n-2n(n-2))H)+C\right]e_1(H_2)}{(2\lambda_1+(n-2)\lambda-nH)^2}+H_2D=0, \tag{3.29}$$

where $C:=(2n(n-1)H-2(n-1)\lambda_1)(nH-\lambda_1-(n-1)\lambda)(2\lambda_1+(n-2)\lambda-nH)$ and
$D=:-(nH-\lambda_1-(n-1)\lambda)^2[(6(n-2)^2-6)\lambda+(2n-4n(n-2))H+6(n-2)\lambda_1]\ +(n-1)(nH-\lambda_1-(n-1)\lambda)B$.

Eliminating $e_1(H_2)$ from (3.28) and (3.29), we obtain

$$\begin{aligned} &-AD(2\lambda_1+(n-2)\lambda-nH)=(nH-\lambda_1-(n-1)\lambda)B[A((n-4)\lambda_1\\ &+2(n-2)^2\lambda+(n-2n(n-2))H)+C] \end{aligned} \tag{3.30}$$

After four times differentiating (3.30) along e_n, we get that $nH=\lambda_1$, which is not possible since λ_1 is not constant. Consequently, $e_n(\lambda)=0$. Therefore, (3.24) reduces to

$$\alpha\beta=\lambda(nH-\lambda_1-(n-2)\lambda). \tag{3.31}$$

Note that (3.13) yields

$$e_1(H_2)=-\frac{4(n-2)}{n(n-1)}(2\lambda_1-nH)e_1(\lambda)+\frac{4}{n(n-1)}(2\lambda_1+(n-2)\lambda-nH)(2\lambda_1-nH)\beta. \tag{3.32}$$

By using (3.32), (3.31), (3.22) and (3.20), we obtain

$$\begin{aligned} e_1e_1(H_2)=&\frac{4(n-2)}{n(n-1)}\lambda_1\lambda(\lambda_1-\lambda)(2\lambda_1-nH)\\ &+\frac{4}{n(n-1)}(nH-\lambda_1-(n-2)\lambda)(2\lambda_1-nH)((3n-2)\lambda_1\lambda+2{\lambda_1}^2-2nH\lambda-nH\lambda_1)\\ &+\left[-n\alpha+3\beta+2\frac{(2\lambda_1+(n-2)\lambda-nH)\beta-(n-2)(\lambda_1-\lambda)\alpha}{2\lambda_1-nH}\right]e_1(H_2). \end{aligned} \tag{3.33}$$

Combining (3.25) with (3.33) gives

$$(P_{1,2}\alpha+P_{2,2}\beta)e_1(H_2)=P_{3,6}, \tag{3.34}$$

where $P_{1,2},\ P_{2,2}$ and $P_{3,6}$ are polynomials in terms of λ and λ_1 of degrees 2, 2 and 6 respectively.

Differentiating (3.34) along e_1 and using (3.31), (3.22), (3.20) and (3.34), we get following relation

$$P_{4,8}\alpha+P_{5,8}\beta=P_{6,5}e_1(H_2), \tag{3.35}$$

where $P_{4,8},\ P_{5,8}$ and $P_{6,5}$ are polynomials in terms of λ and λ_1 of degrees 8, 8 and 5 respectively.

Also, we have

$$e_1(H_2) = \frac{4}{n(n-1)}(2\lambda_1 - nH)\left(\beta(2\lambda_1 + (n-2)\lambda - nH) - (n-2)\alpha(\lambda_1 - \lambda)\right). \tag{3.36}$$

Combining (3.35) and (3.36), we obtain

$$\begin{aligned} &\left(P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1 - nH)\right)\alpha \\ &+\left(P_{5,8} - \frac{4}{n(n-1)}P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)\right)\beta = 0. \end{aligned} \tag{3.37}$$

On the other hand, combining (3.36) with (3.34) and using (3.31), we find

$$P_{2,2}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)\beta^2 - P_{1,2}(n-2)(\lambda_1-\lambda)(2\lambda_1 - nH)\alpha^2 = L, \tag{3.38}$$

where L is given by

$$\begin{aligned} L &= \lambda(nH - \lambda_1 - (n-2)\lambda)(2\lambda_1 - nH) \\ &(P_{2,2}(n-2)(\lambda_1-\lambda) - P_{1,2}(2\lambda_1 + (n-2)\lambda - nH)) \\ &\quad + \frac{n(n-1)}{4}P_{3,6}. \end{aligned} \tag{3.39}$$

Using (3.37) and (3.31), we get

$$\alpha^2 = \frac{\frac{4}{n(n-1)}P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH) - P_{5,8}}{P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1 - nH)}\lambda(nH - \lambda_1 - (n-2)\lambda),$$

$$\beta^2 = \frac{-\frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1 - nH) - P_{4,8}}{P_{5,8} - \frac{4}{n(n-1)}P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)}\lambda(nH - \lambda_1 - (n-2)\lambda). \tag{3.40}$$

Eliminating α^2 and β^2 from (3.38), we obtain

$$\begin{aligned} &\lambda(nH - \lambda_1 - (n-2)\lambda)(2\lambda_1 - nH) \\ &[P_{1,2}(n-2)(\lambda_1-\lambda)(P_{5,8} - \frac{4}{n(n-1)}P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH))^2 \\ &- P_{2,2}(2\lambda_1 + (n-2)\lambda - nH)(P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1 - nH))^2] \\ &= L(P_{5,8} - \frac{4}{n(n-1)}P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)) \\ &(P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1 - nH)), \end{aligned} \tag{3.41}$$

which is a polynomial equation of degree 22 in terms of λ and λ_1.

Now consider an integral curve of e_1 passing through $p = \gamma(t_0)$ as $\gamma(t),\ t \in I$.

Since $e_i(\lambda_1) = e_i(\lambda) = 0$ for $i = 2, \dots, n$ and $e_1(\lambda_1),\ e_1(\lambda) \neq 0$, we can assume $t = t(\lambda)$ and $\lambda_1 = \lambda_1(\lambda)$ in some neighborhood of $\lambda_0 = \lambda(t_0)$. Using (3.37), we have

$$\begin{aligned}\frac{d\lambda_1}{d\lambda} &= \frac{d\lambda_1}{dt}\frac{dt}{d\lambda} = \frac{e_1(\lambda_1)}{e_1(\lambda)}\\ &= \frac{(2\lambda_1 + (n-2)\lambda - nH)\beta - (n-2)(\lambda_1 - \lambda)\alpha}{(\lambda_1 - \lambda)\alpha}\\ &= \frac{\left(P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1 - \lambda)(2\lambda_1 - nH)\right)(2\lambda_1 + (n-2)\lambda - nH)}{\left(\frac{4}{n(n-1)}P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH) - P_{5,8}\right)(\lambda_1 - \lambda)} - (n-2)\end{aligned} \tag{3.42}$$

Differentiating (3.41) with respect to λ and substituting $\frac{d\lambda_1}{d\lambda}$ from (3.42), we get

$$f(\lambda_1, \lambda) = 0, \tag{3.43}$$

another algebraic equation of degree 30 in terms of λ_1 and λ.

We rewrite (3.41) and (3.43) respectively in the following forms

$$\sum_{i=0}^{22} f_i(\lambda_1)\lambda^i, \qquad \sum_{i=0}^{30} g_i(\lambda_1)\lambda^i, \tag{3.44}$$

where $f_i(\lambda_1)$ and $g_j(\lambda_1)$ are polynomial functions of λ_1. We eliminate λ^{30} between these two polynomials of (3.44) by multiplying $g_{30}\lambda^8$ and f_{22} respectively on the first and second equations of (3.44), we obtain a new polynomial equation in λ of degree 29. Combining this equation with the first equation of (3.44), we successively obtain a polynomial equation in λ of degree 28. In a similar way, by using the first equation of (3.44) and its consequences we are able to gradually eliminate λ. At last, we obtain a non-trivial algebraic polynomial equation in λ_1 with constant coefficients. Therefore, we conclude that the real function λ_1 must be a constant, which is a contradiction. Hence H_2 is constant on M^n. If $H_2 \neq 0$, by using (3.2) and (2.2) we obtain that H_3 is constant. Therefore all the mean curvatures H_r are constant functions, this is equivalent to M^n is isoparametric. An isoparametric hypersurface of Euclidean space can have at most two distinct principal curvatures ([18]), which is a contradiction. So $H_2 = 0$.

In conclusion, we get Theorem 1.1.

Acknowledgments

The first and second authors wish to thank their supervisor Prof. S.M.B. Kashani, for introducing them the generalizations of Laplace operator and finite type hypersurfaces.

References

1. G.B. Airy, On the Strains in the Interior of Beams, *Philosophical Transactions of the Royal Society*, **153**, (1863), 49-79.
2. K. Akutagawa, S. Maeta, Biharmonic Properly Immersed Submanifolds in Euclidean Spaces, *Geometriae Dedicata*, **164**(1), (2013), 351-355.
3. L. J. Alías, N. Gürbüz, An Extension of Takahashi Theorem for the Linearized Operators of the Higher Order Mean Curvatures, *Geometriae Dedicata*, **121**(1), (2006), 113-127.
4. M. Aminian, S. M. B. kashani, L_k-Biharmonic Hypersurfaces in Euclidean Space, *Taiwanese Journal of Mathematics*, **19**(3), (2015), 659-976.
5. B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, Series in Pure Mathematics, 2. World Scientific Publishing Co, Singapore, 2014.
6. B. Y. Chen, M. I. Munteanu, Biharmonic Ideal hypersurfaces in Euclidean Spaces, *Differential Geometry and its Applications*, **31**, (2013), 1-16.
7. I. Dimitrić, Submanifolds of $\mathbb{E}^m$ with Harmonic Mean Curvature Vector, *Bulletin of the Institute of Mathematics Academia Sinica*, **20**(1), (1992), 53-65.
8. J. Eells, J.H. Sampson, Harmonic Mappings of Riemannian Manifolds, *American Journal of Mathematics*, **86**(1), (1964), 109-160.
9. J. Eells, J. C. Wood, Restrictions on Harmonic Maps of Surfaces, *Topology* **15**(3), (1976), 263-266.
10. Y. Fu, Biharmonic Hypersurfaces with Three Distinct Principal Curvatures in Euclidean 5-Space, *Journal of Geometry and Physics*, **75**, (2014), 113-119.
11. R. Gupta, On Biharmonic Hypersurfaces in Euclidean Space of Arbitary Dimension, *Glasgow Mathematical Journal*, **57**, (2015), 633-642.
12. T. Hasanis, T. Vlachos, Hypersurfaces in $\mathbb{E}^4$ with Harmonic Mean Curvature Vector Field, *Mathematische Nachrichten*, **172**(1), (1995), 145-169.
13. W. Hussain, The Effect of Pure Shear on the Reflection of Plane Waves at the Boundary of an Elastic Half-Space, *Iranian Journal of Mathematical Sciences and Informatics*, **2**(2), (2007), 1-16.
14. J. A. Jerline , L. B. Michaelraj, On Harmonic Index and Diameter of Unicyclic Graphs, *Iranian Journal of Mathematical Sciences and Informatics*, **11**(1), (2016), 115-122.
15. S. M. B. Kashani, On Some L_1-Finite Type (Hyper)surfaces in $\mathbb{R}^{n+1}$, *Bulletin of the Korean Mathematical Society*, **46**(1), (2009), 35-43.
16. J. C. Maxwell, *On Reciprocal Diagrams in Space, and Their Relation to Airys Function of Stress*, Proc. London. Math. Soc. **102**(2), 1868.
17. R. C. Reilly, Variational Properties of Functions of the Mean Curvatures for Hypersurfaces in Space Forms, *Journal of Differential Geometry*, **8**(3), (1973), 465-477.
18. B. Segre, Famiglie di Ipersuperficie Isoparametrische Negli Spazi Euclidei ad un Qualunque Numero di Dimensioni, *Atti della Accademia d'Italia, Rendiconti della Classe di Scienze Fisiche, Matematiche e Naturali*, **27**, (1938), 203-207.