Some Algebraic and Combinatorial Properties of the Complete $T$-Partite Graphs

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Abstract. In this paper, we characterize the shellable complete $t$-partite graphs. It is also shown that for these types of graphs the concepts vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent. Furthermore, we give a combinatorial condition for the Cohen-Macaulay complete $t$-partite graphs.

Keywords: Cohen-Macaulay, shellable, Vertex decomposable, Edge ideal.

2000 Mathematics subject classification: 13C14, 05E99, 13F55

1. Introduction

Let $G$ be a finite simple graph on $n$ vertices. Let $V_G$ and $E_G$ denote, respectively, the vertex set and the edge set of $G$. An independent set in $G$ is a subset of $V_G$ which none of elements are adjacent and the independence complex $\Delta_G$ of a graph $G$ is defined by

$$\Delta_G = \{ A \subseteq V_G : A \text{ is an independent set in } A \}.$$

We denote by $\text{MIS}(G)$ the set of all maximal independent sets in $G$ (the set of facets of $\Delta_G$); see [16]. In this paper, we obtain a lower bound for $|\text{MIS}(G)|$. Recently, researchers studied the algebraic properties of a commutative ring by...
its associated combinatorial structure, like for instance zero divisor graph; see [1, 8, 13]. Now let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field $k$ in $n$ variables $x_1, \ldots, x_n$. Identifying $i \in V_G$ with the variable $x_i$ in $R$, the edge ideal $I(G)$ of $G$ will be defined as the monomial ideal generated by all of monomials $x_ix_j$ such that $\{x_i, x_j\} \in E_G$.

In recent years, researchers tried to identify Cohen-Macaulay graphs in terms of their combinatorial properties. Estrada and Villarreal in [3] showed that Cohen-Macaulayness and shellability of a bipartite graph $G$ are the same. Herzog and Hibi in [6] proved that a bipartite graph $G$ is Cohen-Macaulay if and only if $|V_1| = |V_2|$ and there is an order on vertices of $V_1$ and $V_2$ as $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, respectively, such that:

i) $x_i \sim y_i$ for $i = 1, \ldots, n$,

ii) if $x_i \sim y_j$, then $i \leq j$,

iii) if $x_i \sim y_j$ and $x_j \sim y_k$, then $x_i \sim y_k$.

Here, we present a necessary and sufficient condition under which shellability of a complete $t$-partite graph is equivalent to Cohen-Macaulayness.

It is known that any vertex decomposable graph is shellable (hence sequentially Cohen-Macaulay), but the converse is not valid in general. So, it is interesting to know a family of graphs in which the property of shellability, vertex decomposability and sequentially Cohen-Macaulayness are the same. Francisco and Van Tuyl in [4] showed that $n$-cycles for $n = 3, 5$ belong to this family of graphs. F. Mohammadi and D. Kiani in [10] proved that in $\theta_{n_1, \ldots, n_k}$ for $\{n_1, \ldots, n_k\} \neq \{2, 5\}$, vertex decomposability, shellability and sequentially Cohen-Macaulayness are coincide. Van Tuyl in [14] showed that in bipartite graphs, three concepts are equivalent. In this paper, attempts have been particularly made to introduce another member of this family, that is, complete $t$-partite graphs.

Herzog and Hibi, in [6], showed that a bipartite graph without isolated vertices $G$ is unmixed if and only if there exists a bipartition $V_1 = \{x_1, \ldots, x_g\}$ and $V_2 = \{y_1, \ldots, y_g\}$ of $V_G$ such that:

i) $\{x_i, y_i\} \in E_G$ for all $i$,

ii) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in $E_G$ and $i, j, k$ are distinct, then $\{x_i, y_k\} \in E_G$.

In the current paper, among other results, we provide a condition for identifying all the unmixed complete $t$-partite graphs.

### 2. Main Results

A graph $G$ is $t$-partite if its vertex set can be partitioned into disjoint independent subsets $V_1, \ldots, V_t$. Moreover, in this paper, we consider $t$ as the smallest number that has this property. The graph $G$ is called complete $t$-partite graph if its vertex set can be partitioned into disjoint independent subsets $V_1, \ldots, V_t$ such that for all $u$ and $v$ in different partition sets, $uv \in E_G$. A
**k-coloring** of a graph $G$ is a labeling $f : V(G) \rightarrow S$ where $|S| = k$. The labels are considered as colors and the set of vertices of one given color form a color class. A $k$-coloring is said to be proper if adjacent vertices have different labels. A graph is **k-colorable** if it has a proper $k$-coloring. The chromatic number of graph $G$, $\chi(G)$, is the least $k$ such that $G$ is $k$-colorable [11].

The following proposition gives a lower bound for cardinality of the set of all maximal independent sets in $G$, $|\text{MIS}(G)|$, in terms of chromatic number.

**Proposition 2.1.** Let $G$ be a graph. If $\chi(G) = t$, then $|\text{MIS}(G)| \geq t$.

*Proof.* Since $\chi(G) = t$, there exist $t$ color classes for $G$. Hence, the graph $G$ can be considered as a $t$-partite graph. Suppose that $V_i$ is the set of elements of $i$-th color class. Then $V_i$ is an independent set of $G$ and there exists a maximal independent set $F_i$ in $G$ such that $V_i \subseteq F_i$ for all $1 \leq i \leq t$. Thus, there exists at least $t$ maximal independent set for $G$. $\square$

**Remark 2.2.** Using the definition of a complete $t$-partite graph, it follows that $|\text{MIS}(G)| = t$ for any complete $t$-partite graph $G$.

**Definition 2.3.** A simplicial complex $\Delta$ is called **shellable** if the facets (maximal faces) of $\Delta$ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_i = \{v\}$. We call $F_1, \ldots, F_s$ to be a **shelling** of a shellable complex, $\Delta$, when the facets are ordered as in the definition.

Now by the next theorem, all shellable complete $t$-partite graphs can be classified.

**Theorem 2.4.** Let $G$ be a complete $t$-partite graph. $G$ is shellable if and only if $G$ is $t$-colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.

*Proof.* $\Leftarrow$) Assume that $V_G = \{x_1, \ldots, x_n\}$ is the set of vertices. To prove that $G$ is shellable, we have to find a shelling $F_1, \ldots, F_t$ for $\Delta_G$. Since proper $t$-vertex coloring gives a partition of $V_G$ into $t$ color classes, we suppose that the set of elements of $i$-th color class is $V_i$. By assumption, $V_i = \{x_1, \ldots, x_m\}$ where $m = n - t + 1$ and $V_i = \{x_{m+i-1}\}$ for all $2 \leq i \leq t$. We know that each $V_i$ is an independent set. Now, if $x_{m+i-1} \in V_i$ for $2 \leq i \leq t$, then we can replace $V_1$ by $V_1 \cup V_i$ and obtain $(t-1)$-partition for $G$ that is a contradiction. Therefore, $V_i$ is a maximal independent set of $G$ and hence a facet of $\Delta_G$. By the same argument, each $V_i$ is a maximal independent set. We put $F_i = V_i$.

Thus, we find an ordering on the facets of $\Delta_G$ as follows:

$$F_1 = \{x_1, \ldots, x_m\}, F_2 = \{x_{m+1}\}, \ldots, F_t = \{x_{m+t-1}\}.$$ 

Since $F_i \setminus F_1 = \{x_{m+i-1}\}$ for all $2 \leq i \leq t$, $F_1, \ldots, F_t$ is a shelling of $\Delta_G$.

$\Rightarrow$) Suppose that $V_1, \ldots, V_t$ is a partition of $V_G$. According to Remark 2.2,
we have $|\text{MIS}(G)| = t$, so $\Delta_G$ has exactly $t$ facets. On the other hand, $G$ is shellable and we can consider $F_1, \ldots, F_t$ as a shelling of $\Delta_G$. Since $V_i$'s are maximal independent sets, we obtain $\{V_1, \ldots, V_t\} = \{F_1, \ldots, F_t\}$. Without loss of generality, put $F_i = V_i$ for all $1 \leq i \leq t$. There exists $x_2 \in F_2 \setminus F_1$ such that $F_2 \setminus F_1 = \{x_2\}$ because $F_1, \ldots, F_t$ is a shelling. Thus $F_2 = (F_2 \setminus F_1) \cup (F_2 \cap F_1) = \{x_2\}$.

Now, suppose by induction that $F_1 = \{x_1, \ldots, x_m\}$, $F_2 = \{x_{m+1}\}$, $F_i = \{x_{m+i-1}\}$. Since $F_1, \ldots, F_i$ is a shelling of $\Delta_G$, there exists $x_{i+1} \in F_{i+1} \setminus F_i$ and $l \in \{1, \ldots, i\}$ such that $F_{i+1} \setminus F_i = \{x_{i+1}\}$, then $F_{i+1} = (F_{i+1} \setminus F_i) \cup (F_{i+1} \cap F_i) = \{x_{i+1}\}$. Hence, one of the color classes has arbitrary elements and the other classes have exactly one element.

**Definition 2.5.** A simplicial complex $\Delta$ is recursively defined to be vertex decomposable if it is either a simplex, or else has some vertex $v$ so that

i) both $\Delta \setminus v$ and $\text{link}_v^\Delta$ are vertex decomposable and

ii) no face of $\text{link}_v^\Delta$ is a facet of $\Delta \setminus v$, where

$$\text{link}_v^\Delta = \{G : G \cap F = \emptyset, G \cup F \in \Delta\}.$$  

A graph $G$ is called vertex decomposable if the simplicial complex $\Delta_G$ is vertex decomposable.

The following theorem is one of the main results of this paper which characterizes all vertex decomposable complete $t$-partite graphs.

**Theorem 2.6.** Let $G$ be a complete $t$-partite graph. Then, $G$ is vertex decomposable if and only if $G$ is $t$-colorable such that exactly one of the color classes has arbitrary elements and the other classes have only one element.

**Proof.** $\Rightarrow$) Assume that for any proper $t$-vertex coloring of $G$, there exists at least two classes with at least two elements. By Theorem 2.4, $G$ is not shellable and hence is not vertex decomposable, by ([17], Corollary 7).

$\Leftarrow$) By assumption, it follows that $G$ is a chordal graph, so $G$ is vertex decomposable by ([17], Corollary 7).

**Definition 2.7.** A subset $C \subset V_G$ is a minimal vertex cover of $G$ if:

i) every edge of $G$ is incident with one vertex in $C$, and

ii) there is no proper subset of $C$ with the first property.

If $C$ satisfies only condition (i), it is called a vertex cover of $G$. A graph $G$ is said to be unmixed if all the minimal vertex covers of $G$ have the same number of elements.

By the next theorem, we present a combinatorial characterization of all the unmixed complete $t$-partite graphs.

**Theorem 2.8.** Let $G$ be a complete $t$-partite graph. $G$ is unmixed if and only if $G$ is $t$-colorable such that all color classes have the same number of elements.
Proof. Since any minimal vertex cover of a complete t-partite graph $G$ contains all the elements of $(t - 1)$ classes, then each selected $(t - 1)$ of color classes has the same number of elements if and only if all the classes have the same cardinality. □

Definition 2.9. ([15], Definition 3.3.8) A pure $d$-dimensional complex $\Delta$ is called strongly connected if each pair of facets $F, G$ can be connected by a sequence of facets $F = F_0, F_1, \ldots, F_s = G$ such that $\dim(F_i \cap F_{i-1}) = d - 1$ for $1 \leq i \leq s$.

Lemma 2.10. ([2], Proposition 11.7) Every Cohen-Macaulay complex is strongly connected.

Lemma 2.11. ([15], Corollary 3.3.7) A Cohen-Macaulay simplicial complex $\Delta$ is pure, that is, all its maximal faces have the same dimension.

The following theorem is an effective combinatorial criterion for Cohen-Macaulayness of the complete $t$-partite graphs.

Theorem 2.12. Let $G$ be a complete $t$-partite graph. $G$ is Cohen-Macaulay graph if and only if $G$ is $t$-colorable such that all color classes have exactly one element.

Proof. $\Leftarrow$ By assumption, we have $G = K_t$, so $G$ is a chordal graph. By [7], $G$ is Cohen-Macaulay if and only if $G$ is unmixed. Thus, the assertion follows from Theorem 2.8.

$\Rightarrow$ Suppose that for any proper $t$-vertex coloring of $G$, there exists at least two classes with at least two elements. Since $G$ is Cohen-Macaulay, $\Delta_G$ is pure, by Lemma 2.11. Therefore, all the color classes have the same number of elements. Assume that their size is $d + 1$, hence all facets of $\Delta_G$ are of dimension $d$ and then $\dim(\Delta_G) = d$. We will show that it is not possible that $d + 1 \geq 2$. For any two facets $F$ and $E$, $(F \neq E)$, we have $F \cap E = \emptyset$, then $\dim(F \cap E) = -1 \neq d - 1$, because of $d + 1 \geq 2$. Therefore, $\Delta_G$ is not strongly connected, a contradiction to Lemma 2.10. It follows that all the color classes have exactly one element. □

Now, we give a special condition for complete $t$-partite graphs under which shellability is equal to Cohen-Macaulayness.

Corollary 2.13. Let $G$ be a complete $t$-partite graph. The property of being Cohen-Macaulay for $G$ is equivalent to being shellable if and only if $G$ is $t$-colorable such that all color classes have exactly one element.

Definition 2.14. Let $k$ be a field and $R = k[x_1, \ldots, x_n]$ be the polynomial ring over $k$. A graded $R$-module $M$ is called sequentially Cohen-Macaulay (over $k$) if there exists finite filtration of graded $R$-modules

\[ 0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M \]
such that each $M_i/M_{i-1}$ is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_r/M_{r-1}).$$

We call a graph $G$ sequentially Cohen-Macaulay over the field $k$ if $R/I(G)$ is sequentially Cohen-Macaulay.

Suppose that $I$ is a homogeneous ideal of $R$. The ideal generated by all homogeneous degree $d$ elements of $I$ is denoted by $(I_d)$. The concept of componentwise linear was introduced by Herzog and Hibi [5]. A homogeneous ideal $I$ is componentwise linear if $(I_d)$ has a linear resolution for all $d$. Let $I$ be a square-free monomial ideal of $R$ and $I_{[d]}$ be the ideal generated by the square-free monomials of degree $d$ of $I$. Herzog and Hibi ([5], Proposition 1.5) have shown that the square-free monomial ideal $I$ is componentwise linear if and only if $I_{[d]}$ has a linear resolution for all $d$. In [5], it is also shown that:

**Theorem 2.15.** Let $I$ be a square-free monomial ideal in a polynomial ring. Then $I^\vee$ is componentwise linear if and only if $R/I$ is sequentially Cohen-Macaulay.

In [12], Stanley showed that shellability implies the sequentially Cohen-Macaulayness.

**Theorem 2.16.** Let $\Delta$ be a simplicial complex, and suppose that $R/I_\Delta$ is the associated Stanley Reisner ring. If $\Delta$ is shellable, then $R/I_\Delta$ is sequentially Cohen-Macaulay.

In the following theorem, we show that being sequentially Cohen-Macaulay of the complete $t$-partite graph is really a combinatorial property.

**Theorem 2.17.** Let $G$ be a complete $t$-partite graph. $G$ is sequentially Cohen-Macaulay if and only if $G$ is $t$-colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.

**Proof.** $\Leftarrow)$ It follows from Theorems 2.4 and 2.16.

$\Rightarrow)$ Assume that $V_G = \{x_1, \ldots, x_n\}$ and $V_1, \ldots, V_t$ is a partition of $V_G$ where $V_i$ is the set of elements in $i$-th color class. We proceed by contradiction. One may consider the following cases:

**Case (1):** Suppose that there exist at least two parts $V_i$ and $V_j$ with $|V_i| \geq 2$ and $|V_j| \geq 3$ and $r$ is the maximum cardinality of parts of $G$ which is at least 3. Let $J = I(G)^\vee_{[d]}$ where $d = n - r + 1$. Using Theorem 2.15, to show that $G$ is not sequentially Cohen-Macaulay, it suffices to prove that $J$ does not have a linear resolution.

We use simplicial homology to compute the Betti numbers of $J$. A square-free vector is a vector that its entries are in $\{0, 1\}$. For a monomial ideal $I$ and
a degree \( b \in \mathbb{N}^n \), define

\[
K^b(I) = \{ \text{squarefree vectors } c \in \{0,1\}^n \text{ such that } \frac{x^b}{x^c} \in I \}
\]

to be the upper Koszul simplicial complex of \( I \) in degree \( b \) ([9], Definition 1.33).

For a vector \( b \in \mathbb{N}^n \), the Betti numbers of \( I \) in degree \( b \) can be expressed as

\[
\beta_{i,b}(I) = \dim_k H_{i-1}^C(K^b(I), k) \quad ([9], \text{Theorem 1.34}).
\]

The sum of \( \beta_{i,b}(I) \) over all square-free vectors \( b \) of degree \( j \) is equal to \( \beta_{i,j}(I) \).

To prove that \( J \) does not have a linear resolution, we will show that \( \beta_{1,n}(J) \neq 0 \). We associate to the monomial \( m = x_1 \ldots x_n \) a unique square-free vector \( b = (1, \ldots, 1) \). We have a chain complex

\[
\ldots \rightarrow C_2(K^b(J)) \xrightarrow{\partial_3} C_1(K^b(J)) \xrightarrow{\partial_3} C_0(K^b(J)) \xrightarrow{\partial_3} C_{-1}(K^b(J)) \rightarrow 0.
\]

The \( s \)-dimensional faces \([x_{i_0}, \ldots, x_{i_s}]\) of \( K^b(J) \) are the basis of \( C_s(K^b(J)) \) and

\[
\partial_s([x_{i_0}, \ldots, x_{i_s}]) = \sum_{t=0}^{s} (-1)^t[x_{i_0}, \ldots, \hat{x}_{i_t}, \ldots, x_{i_s}].
\]

The above chain complex is introduced in ([4], Proposition 4.1). To obtain \( \beta_{1,n}(J) \), we need to compute \( \dim_k H^C_0(K^b(M), k) = \dim_k(\ker \partial_0/\im \partial_1) \). If we can find an element in \( \ker \partial_0 \) that is not in \( \im \partial_1 \), we have shown that \( \beta_{1,n}(J) > 0 \). We suppose that \( x_1, x_2, x_3 \) belong to the part with maximum cardinality. Put \( f = [x_1] \). Then, \( \partial_0(f) = 0 \) and hence \( f \) is in the \( \ker \) of \( \partial_0 \). We claim that \( f \) is not in the image of \( \partial_1 \). To prove, assume that \( \partial_1([x_1, x_s]) = [x_1] \).

Then, \([x_s]-[x_1] = [x_1] \), hence \([x_s]-[x_1]-[x_1] = 0 \) that is a contradiction because \([x_1], [x_s], [x_1] \) are linear independent.

**Case(2)**: Assume that for any \( 1 \leq i \leq t \), we have \( |V_i| = 2 \). The proof of this case is similar to that of case 1. It suffices to consider \( J = I(G)^{\vee}_{[d]} \) where \( d = n-2 \).

**Case(3)**: Suppose that there exist at least two parts \( V_i \) and \( V_j \) with \( |V_i| = |V_j| = 2 \) and at least one part \( V_r \) with \( |V_r| = 1 \). One can apply the same argument to case 2.

\[ \square \]

**Corollary 2.18.** Let \( G \) be a complete \( t \)-partite graph. The followings are equivalent:

1. \( G \) is shellable.
2. \( G \) is vertex decomposable.
3. \( G \) is sequentially Cohen-Macaulay.

**Acknowledgments**

The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.
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