Some Algebraic and Combinatorial Properties of the Complete T-Partite Graphs

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ABSTRACT. In this paper, we characterize the shellable complete t-partite graphs. It is also shown that for these types of graphs the concepts vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent. Furthermore, we give a combinatorial condition for the Cohen-Macaulay complete t-partite graphs.

Keywords: Cohen-Macaulay, shellable, Vertex decomposable, Edge ideal.

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1. INTRODUCTION

Let G be a finite simple graph on n vertices. Let V_G and E_G denote, respectively, the vertex set and the edge set of G. An independent set in G is a subset of V_G which none of elements are adjacent and the independence complex \Delta_G of a graph G is defined by
\[ \Delta_G = \{ A \subseteq V_G : A \text{ is an independent set in } A \}. \]
We denote by MIS(G) the set of all maximal independent sets in G (the set of facets of \Delta_G); see [16]. In this paper, we obtain a lower bound for |MIS(G)|. Recently, researchers studied the algebraic properties of a commutative ring by

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131
its associated combinatorial structure, like for instance zero divisor graph; see [1, 8, 13]. Now let \( R = k[x_1, \ldots, x_n] \) be the polynomial ring over a field \( k \) in \( n \) variable \( x_1, \ldots, x_n \). Identifying \( i \in V_G \) with the variable \( x_i \) in \( R \), the edge ideal \( I(G) \) of \( G \) will be defined as the monomial ideal generated by all of monomials \( x_i x_j \), such that \( \{x_i, x_j\} \in E_G \).

In recent years, researchers tried to identify Cohen-Macaulay graphs in terms of their combinatorial properties. Estrada and Villarreal in [3] showed that Cohen-Macaulayness and shellability of a bipartite graph \( G \) are the same. Herzog and Hibi in [6] proved that a bipartite graph \( G \) is Cohen-Macaulay if and only if \( |V_1| = |V_2| \) and there is an order on vertices of \( V_1 \) and \( V_2 \) as \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \), respectively, such that:

i) \( x_i \sim y_i \) for \( i = 1, \ldots, n \),
ii) if \( x_i \sim y_j \), then \( i \leq j \),
iii) if \( x_i \sim y_j \) and \( x_j \sim y_k \), then \( x_i \sim y_k \).

Here, we present a necessary and sufficient condition under which shellability of a complete \( t \)-partite graph is equivalent to Cohen-Macaulayness.

It is known that any vertex decomposable graph is shellable (hence sequentially Cohen-Macaulay), but the converse is not valid in general. So, it is interesting to know a family of graphs in which the property of shellability, vertex decomposability and sequentially Cohen-Macaulayness are the same. Francisco and Van Tuyl in [4] showed that \( n \)-cycles for \( n = 3, 5 \) belong to this family of graphs. F. Mohammadi and D. Kiani in [10] proved that in \( \theta_{n_1, \ldots, n_k} \) for \( \{n_1, \ldots, n_k\} \neq \{2, 5\} \), vertex decomposability, shellability and sequentially Cohen-Macaulayness are coincide. Van Tuyl in [14] showed that in bipartite graphs, three concepts are equivalent. In this paper, attempts have been particularly made to introduce another member of this family, that is, complete \( t \)-partite graphs.

Herzog and Hibi, in [6], showed that a bipartite graph without isolated vertices \( G \) is unmixed if and only if there exists a bipartition \( V_1 = \{x_1, \ldots, x_g\} \) and \( V_2 = \{y_1, \ldots, y_g\} \) of \( V_G \) such that:

i) \( \{x_i, y_i\} \in E_G \) for all \( i \), and
ii) if \( \{x_i, y_j\} \) and \( \{x_j, y_k\} \) are in \( E_G \) and \( i, j, k \) are distinct, then \( \{x_i, y_k\} \in E_G \).

In the current paper, among other results, we provide a condition for identifying all the unmixed complete \( t \)-partite graphs.

2. Main Results

A graph \( G \) is \( t \)-partite if its vertex set can be partitioned into disjoint independent subsets \( V_1, \ldots, V_t \). Moreover, in this paper, we consider \( t \) as the smallest number that has this property. The graph \( G \) is called complete \( t \)-partite graph if its vertex set can be partitioned into disjoint independent subsets \( V_1, \ldots, V_t \) such that for all \( u \) and \( v \) in different partition sets, \( uv \in E_G \). A
Some algebraic and combinatorial properties of the complete T-partite graphs 133

A \textit{k-coloring} of a graph \(G\) is a labeling \(f : V(G) \to S\) where \(|S| = k\). The labels are considered as colors and the set of vertices of one given color form a color class. A \(k\)-coloring is said to be proper if adjacent vertices have different labels. A graph is \(k\)-\textit{colorable} if it has a proper \(k\)-coloring. The chromatic number of graph \(G\), \(\chi(G)\), is the least \(k\) such that \(G\) is \(k\)-colorable [11].

The following proposition gives a lower bound for cardinality of the set of all maximal independent sets in \(G\), \(|MIS(G)|\), in terms of chromatic number.

\textbf{Proposition 2.1.} Let \(G\) be a graph. If \(\chi(G) = t\), then \(|MIS(G)| \geq t\).

\textit{Proof.} Since \(\chi(G) = t\), there exist \(t\) color classes for \(G\). Hence, the graph \(G\) can be considered as a \(t\)-partite graph. Suppose that \(V_i\) is the set of elements of \(i\)-th color class. Then \(V_i\) is an independent set of \(G\) and there exists a maximal independent set \(F_i\) in \(G\) such that \(V_i \subseteq F_i\) for all \(1 \leq i \leq t\). Thus, there exists at least \(t\) maximal independent set for \(G\). \(\square\)

\textit{Remark 2.2.} Using the definition of a complete \(t\)-partite graph, it follows that \(|MIS(G)| = t\) for any complete \(t\)-partite graph \(G\).

\textbf{Definition 2.3.} A simplicial complex \(\Delta\) is called \textit{shellable} if the facets (maximal faces) of \(\Delta\) can be ordered \(F_1, \ldots, F_s\) such that for all \(1 \leq i < j \leq s\), there exist some \(v \in F_j \setminus F_i\) and some \(l \in \{1, \ldots, j - 1\}\) with \(F_j \setminus F_i = \{v\}\). We call \(F_1, \ldots, F_s\) to be a \textit{shelling} of a shellable complex, \(\Delta\), when the facets are ordered as in the definition.

Now by the next theorem, all shellable complete \(t\)-partite graphs can be classified.

\textbf{Theorem 2.4.} Let \(G\) be a complete \(t\)-partite graph. \(G\) is shellable if and only if \(G\) is \(t\)-colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.

\textit{Proof.} \(\Leftarrow\) Assume that \(V_G = \{x_1, \ldots, x_n\}\) is the set of vertices. To prove that \(G\) is shellable, we have to find a shelling \(F_1, \ldots, F_t\) for \(\Delta_G\). Since proper \(t\)-vertex coloring gives a partition of \(V_G\) into \(t\) color classes, we suppose that the set of elements of \(i\)-th color class is \(V_i\). By assumption, \(V_i = \{x_1, \ldots, x_m\}\) where \(m = n - t + 1\) and \(V_i = \{x_{m+i-1}\}\) for all \(2 \leq i \leq t\). We know that each \(V_i\) is an independent set. Now, if \(x_{m+i-1} \in V_i\) for \(2 \leq i \leq t\), then we can replace \(V_i\) by \(V_1 \cup V_i\) and obtain \((t-1)\)-partition for \(G\) that is a contradiction. Therefore, \(V_i\) is a maximal independent set of \(G\) and hence a facet of \(\Delta_G\). By the same argument, each \(V_i\) is a maximal independent set. We put \(F_i = V_i\). Thus, we find an ordering on the facets of \(\Delta_G\) as follows:

\[
F_1 = \{x_1, \ldots, x_m\}, F_2 = \{x_{m+1}\}, \ldots, F_t = \{x_{m+t-1}\}.
\]

Since \(F_i \setminus F_1 = \{x_{m+i-1}\}\) for all \(2 \leq i \leq t\), \(F_1, \ldots, F_t\) is a shelling of \(\Delta_G\).

\(\Rightarrow\) Suppose that \(V_1, \ldots, V_t\) is a partition of \(V_G\). According to Remark 2.2,
we have $|\text{MIS}(G)| = t$, so $\Delta G$ has exactly $t$ facets. On the other hand, $G$ is shellable and we can consider $F_1, \ldots, F_t$ as a shelling of $\Delta G$. Since $V_i$'s are maximal independent sets, we obtain $\{V_1, \ldots, V_t\} = \{F_1, \ldots, F_t\}$. Without loss of generality, put $F_i = V_i$ for all $1 \leq i \leq t$. There exists $x_2 \in F_2 \setminus F_1$ such that $F_2 \setminus F_1 = \{x_2\}$ because $F_1, \ldots, F_t$ is a shelling. Thus $F_2 = (F_2 \setminus F_1) \cup (F_2 \cap F_1) = \{x_2\}$. 

Now, suppose by induction that $F_1 = \{x_1, \ldots, x_m\}$, $F_2 = \{x_{m+1}\}$, $F_i = \{x_{m+i-1}\}$. Since $F_1, \ldots, F_i$ is a shelling of $\Delta G$, there exists $x_{i+1} \in F_{i+1} \setminus F_1$ and $l \in \{1, \ldots, i\}$ such that $F_{i+1} \setminus F_l = \{x_{i+1}\}$, then $F_{i+1} = (F_{i+1} \setminus F_l) \cup (F_{i+1} \cap F_l) = \{x_{i+1}\}$. Hence, one of the color classes has arbitrary elements and the other classes have exactly one element. □

**Definition 2.5.** A simplicial complex $\Delta$ is recursively defined to be vertex decomposable if it is either a simplex, or else has some vertex $v$ so that

i) both $\Delta \setminus v$ and $\text{link}^v_{\Delta}$ are vertex decomposable and

ii) no face of $\text{link}^v_{\Delta}$ is a facet of $\Delta \setminus v$, where

$$\text{link}^v_{\Delta} = \{G : G \cap F = \emptyset, G \cup F \in \Delta\}.$$  

A graph $G$ is called vertex decomposable if the simplicial complex $\Delta G$ is vertex decomposable.

The following theorem is one of the main results of this paper which characterizes all vertex decomposable complete $t$-partite graphs.

**Theorem 2.6.** Let $G$ be a complete $t$-partite graph. Then, $G$ is vertex decomposable if and only if $G$ is $t$-colorable such that exactly one of the color classes has arbitrary elements and the other classes have only one element.

**Proof.** $\Rightarrow$ Assume that for any proper $t$-vertex coloring of $G$, there exists at least two classes with at least two elements. By Theorem 2.4, $G$ is not shellable and hence is not vertex decomposable, by ([17], Corollary 7).

$\Leftarrow$ By assumption, it follows that $G$ is a chordal graph, so $G$ is vertex decomposable by ([17], Corollary 7). □

**Definition 2.7.** A subset $C \subset V_G$ is a minimal vertex cover of $G$ if:

i) every edge of $G$ is incident with one vertex in $C$, and

ii) there is no proper subset of $C$ with the first property.

If $C$ satisfies only condition (i), it is called a vertex cover of $G$. A graph $G$ is said to be unmixed if all the minimal vertex covers of $G$ have the same number of elements.

By the next theorem, we present a combinatorial characterization of all the unmixed complete $t$-partite graphs.

**Theorem 2.8.** Let $G$ be a complete $t$-partite graph. $G$ is unmixed if and only if $G$ is $t$-colorable such that all color classes have the same number of elements.
Proof. Since any minimal vertex cover of a complete t-partite graph $G$ contains all the elements of $(t - 1)$ classes, then each selected $(t - 1)$ of color classes has the same number of elements if and only if all the classes have the same cardinality. □

Definition 2.9. ([15], Definition 3.3.8) A pure $d$-dimensional complex $\Delta$ is called strongly connected if each pair of facets $F, G$ can be connected by a sequence of facets $F = F_0, F_1, \ldots, F_s = G$ such that $\dim(F_i \cap F_{i-1}) = d - 1$ for $1 \leq i \leq s$.

Lemma 2.10. ([2], Proposition 11.7) Every Cohen-Macaulay complex is strongly connected.

Lemma 2.11. ([15], Corollary 3.3.7) A Cohen-Macaulay simplicial complex $\Delta$ is pure, that is, all its maximal faces have the same dimension.

The following theorem is an effective combinatorial criterion for Cohen-Macaulayness of the complete $t$-partite graphs.

Theorem 2.12. Let $G$ be a complete $t$-partite graph. $G$ is Cohen-Macaulay if and only if $G$ is $t$-colorable such that all color classes have exactly one element.

Proof. $\Leftarrow$) By assumption, we have $G = K_t$, so $G$ is a chordal graph. By [7], $G$ is Cohen-Macaulay if and only if $G$ is unmixed. Thus, the assertion follows from Theorem 2.8.

$\Rightarrow$) Suppose that for any proper $t$-vertex coloring of $G$, there exists at least two classes with at least two elements. Since $G$ is Cohen-Macaulay, $\Delta_G$ is pure, by Lemma 2.11. Therefore, all the color classes have the same number of elements. Assume that their size is $d + 1$, hence all facets of $\Delta_G$ are of dimension $d$ and then $\dim(\Delta_G) = d$. We will show that it is not possible that $d + 1 \geq 2$. For any two facets $F$ and $E$, ($F \neq E$), we have $F \cap E = \phi$, then $\dim(F \cap E) = -1 \neq d - 1$, because of $d + 1 \geq 2$. Therefore, $\Delta_G$ is not strongly connected, a contradiction to Lemma 2.10. It follows that all the color classes have exactly one element. □

Now, we give a special condition for complete $t$-partite graphs under which shellability is equal to Cohen-Macaulayness.

Corollary 2.13. Let $G$ be a complete $t$-partite graph. The property of being Cohen-Macaulay for $G$ is equivalent to being shellable if and only if $G$ is $t$-colorable such that all color classes have exactly one element.

Definition 2.14. Let $k$ be a field and $R = k[x_1, \ldots, x_n]$ be the polynomial ring over $k$. A graded $R$-module $M$ is called sequentially Cohen-Macaulay (over $k$) if there exists finite filtration of graded $R$-modules

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$
such that each $M_i/M_{i-1}$ is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_r/M_{r-1}).$$

We call a graph $G$ sequentially Cohen-Macaulay over the field $k$ if $R/I(G)$ is sequentially Cohen-Macaulay.

Suppose that $I$ is a homogeneous ideal of $R$. The ideal generated by all homogeneous degree $d$ elements of $I$ is denoted by $(I_d)$. The concept of componentwise linear was introduced by Herzog and Hibi [5]. A homogeneous ideal $I$ is componentwise linear if $(I_d)$ has a linear resolution for all $d$. Let $I$ be a square-free monomial ideal of $R$ and $I_{[d]}$ be the ideal generated by the square-free monomials of degree $d$ of $I$. Herzog and Hibi ([5], Proposition 1.5) have shown that the square-free monomial ideal $I$ is componentwise linear if and only if $I_{[d]}$ has a linear resolution for all $d$. In [5], it is also shown that:

**Theorem 2.15.** Let $I$ be a square-free monomial ideal in a polynomial ring. Then $I$ is componentwise linear if and only if $R/I$ is sequentially Cohen-Macaulay.

In [12], Stanley showed that shellability implies the sequentially Cohen-Macaulayness.

**Theorem 2.16.** Let $\Delta$ be a simplicial complex, and suppose that $R/I_\Delta$ is the associated Stanley Reisner ring. If $\Delta$ is shellable, then $R/ I_\Delta$ is sequentially Cohen-Macaulay.

In the following theorem, we show that being sequentially Cohen-Macaulay of the complete $t$-partite graph is really a combinatorial property.

**Theorem 2.17.** Let $G$ be a complete $t$-partite graph. $G$ is sequentially Cohen-Macaulay if and only if $G$ is $t$-colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.

**Proof.** $\Leftarrow$ It follows from Theorems 2.4 and 2.16.

$\Rightarrow$ Assume that $V_G = \{x_1, \ldots, x_n\}$ and $V_1, \ldots, V_t$ is a partition of $V_G$ where $V_i$ is the set of elements in $i$-th color class. We proceed by contradiction. One may consider the following cases:

**Case (1):** Suppose that there exist at least two parts $V_i$ and $V_j$ with $|V_i| \geq 2$ and $|V_j| \geq 3$ and $r$ is the maximum cardinality of parts of $G$ which is at least 3. Let $J = I(G)_{[d]}$ where $d = n - r + 1$. Using Theorem 2.15, to show that $G$ is not sequentially Cohen-Macaulay, it suffices to prove that $J$ does not have a linear resolution.

We use simplicial homology to compute the Betti numbers of $J$. A square-free vector is a vector that its entries are in $\{0, 1\}$. For a monomial ideal $I$ and
Some algebraic and combinatorial properties of the complete $T$-partite graphs

a degree $b \in \mathbb{N}^n$, define

$$K^b(I) = \{\text{squarefree vectors } c \in \{0, 1\}^n \text{ such that } \frac{x^b}{x^c} \in I\}$$

to be the upper Koszul simplicial complex of $I$ in degree $b$ ([9], Definition 1.33). For a vector $b \in \mathbb{N}^n$, the Betti numbers of $I$ in degree $b$ can be expressed as $\beta_{1,b}(I) = \dim_k H^{-1}_\beta(K^b(I), k)$ ([9], Theorem 1.34). The sum of $\beta_{1,b}(I)$ over all square-free vectors $b$ of degree $j$ is equal to $\beta_{1,j}(I)$.

To prove that $J$ does not have a linear resolution, we will show that $\beta_{1,n}(J) \neq 0$. We associate to the monomial $m = x_1 \ldots x_n$ a unique square-free vector $b = (1, \ldots, 1)$. We have a chain complex

$$\ldots \rightarrow C_2(K^b(J)) \xrightarrow{\partial_3} C_1(K^b(J)) \xrightarrow{\partial_4} C_0(K^b(J)) \xrightarrow{\partial_5} C_{-1}(K^b(J)) \rightarrow 0.$$

The $s$-dimensional faces $[x_{i_0}, \ldots, x_{i_s}]$ of $K^b(J)$ are the basis of $C_s(K^b(J))$ and

$$\partial_s([x_{i_0}, \ldots, x_{i_s}]) = \sum_{t=0}^s (-1)^t [x_{i_0}, \ldots, \hat{x}_{i_t}, \ldots, x_{i_s}].$$

The above chain complex is introduced in ([4], Proposition 4.1). To obtain $\beta_{1,n}(J)$, we need to compute $\dim_k H^{-d}_\beta(K^b(M), k) = \dim_k (\ker \partial_d / \im \partial_d)$. If we can find an element in $\ker \partial_d$ that is not in $\im \partial_1$, we have shown that $\beta_{1,n}(J) > 0$. We suppose that $x_1, x_2, x_3$ belong to the part with maximum cardinality. Put $f = [x_1]$. Then, $\partial_0(f) = 0$ and hence $f$ is in the $\ker$ of $\partial_0$. We claim that $f$ is not in the image of $\partial_1$. To prove, assume that $\partial_1([x_1, x_i]) = [x_1]$. Then, $[x_s] - [x_1] = [x_1]$, hence $[x_s] - [x_1] - [x_1] = 0$ that is a contradiction because $[x_1], [x_s], [x_1]$ are linear independent.

**Case (2) :** Assume that for any $1 \leq i \leq t$, we have $|V_i| = 2$. The proof of this case is similar to that of case 1. It suffices to consider $J = I(G)|_{[d]}$ where $d = n - 2$.

**Case (3) :** Suppose that there exist at least two parts $V_i$ and $V_j$ with $|V_i| = |V_j| = 2$ and at least one part $V_r$ with $|V_r| = 1$. One can apply the same argument to case 2. \hfill $\square$

**Corollary 2.18.** Let $G$ be a complete $t$-partite graph. The followings are equivalent:

1. $G$ is shellable.
2. $G$ is vertex decomposable.

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