Atomic Systems in 2-inner Product Spaces

Bahram Dastourian, Mohammad Janfada*
Department of Pure Mathematics
Ferdowsi University of Mashhad
Mashhad, 1159-91775, Iran
E-mail: bdastorian@gmail.com
E-mail: mjanfada@gmail.com

Abstract. In this paper, the concept of a family of local atoms in a 2-inner product space is introduced and then this concept is generalized to an atomic system for an operator. Next a characterization of atomic systems is proved. This characterization lead us to obtain a new frame which is a generalization of frames in 2-inner product spaces.

Keywords: 2-inner product space, 2-normed space, Family of local atoms, Atomic system, Frame.


1. Introduction and Preliminaries

Frames in Hilbert spaces were introduced by Duffin and Schaffer [9] in the context of nonharmonic Fourier series in 1952. In 1986, frames were brought to life by Daubechies et al. [7]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [13], sampling [10, 11], coding and communications [19], filter bank theory [2], system modeling [8], and so on.

Atomic systems for bounded linear operators on Hilbert spaces have been introduced by L. Găvruţa in [15] as a generalization of families of local atoms...

*Corresponding Author

Received 06 September 2016; Accepted 11 November 2017
©2018 Academic Center for Education, Culture and Research TMU
A sequence \( \{f_j\}_{j \in \mathbb{N}} \) in a Hilbert space \( \mathcal{H} \) is called an atomic system for a bounded linear operator \( K \) on \( \mathcal{H} \) if

1. the series \( \sum_{j \in \mathbb{N}} c_j f_j \) converges for all \( c = (c_j) \in \ell^2 := \{ (b_j)_{j \in \mathbb{N}} : \sum_{j \in \mathbb{N}} |b_j|^2 < \infty \} \);

2. there exists \( C > 0 \) such that for every \( f \in \mathcal{H} \) there exists \( a_f = (a_j) \in \ell^2 \) such that \( \|a_f\|_2 \leq C \|f\| \) and \( K f = \sum_{j \in \mathbb{N}} a_j f_j \).

It is proved that this concept is equivalent to \( K \)-frames, where \( K \) is a bounded linear operator on separable Hilbert space \( \mathcal{H} \) [15].

A sequence \( \{f_j\}_{j \in \mathbb{N}} \) is said to be a \( K \)-frame for \( \mathcal{H} \) if there exist constants \( A, B > 0 \) such that

\[
A \|K^* f\|^2 \leq \sum_{j \in \mathbb{N}} |\langle f, f_j \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.
\]

We refer to [20] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert modules [5] and Banach spaces [6]. Note that frames in Hilbert spaces are just a particular case of \( K \)-frames, when \( K \) is the identity operator on these Hilbert spaces.

The concepts of 2-inner product spaces and 2-normed spaces have been studied by many authors [3, 4, 14, 16, 17, 18]. In the sequel, we introduce 2-inner product and 2-normed spaces.

Definition 1.1. Suppose that \( X \) is a vector space of dimension grater than 1 over the field \( \mathbb{F} \) (either \( \mathbb{R} \) or \( \mathbb{C} \)). If there exists a mapping \( \langle \cdot, \cdot | \cdot, \cdot \rangle : X \times X \times X \rightarrow \mathbb{F} \) with the properties

1. \( \langle f, f | h \rangle \geq 0 \) and \( \langle f, f | h \rangle = 0 \) if and only if \( f \) and \( h \) are linearly dependent;
2. \( \langle f, f | h \rangle = \langle h, f | f \rangle \);
3. \( \langle g, f | h \rangle = \langle f, g | h \rangle \);
4. \( \langle \alpha f, g | h \rangle = \alpha \langle f, g | h \rangle \) for \( \alpha \in \mathbb{F} \);
5. \( \langle f_1 + f_2, g | h \rangle = \langle f_1, g | h \rangle + \langle f_2, g | h \rangle \),

then the pair \( (X, \langle \cdot, \cdot | \cdot, \cdot \rangle) \) is called a 2-inner product space. The map \( \langle \cdot, \cdot | \cdot, \cdot \rangle \) is said to be a 2-inner product on \( X \).

Some basic properties of 2-inner product \( \langle \cdot, \cdot | \cdot, \cdot \rangle \) can be immediately obtained as follows (see [3, 4]).

- \( \langle 0, g | h \rangle = \langle f, 0 | h \rangle = \langle f, g | 0 \rangle = 0 \);
- \( \langle f, \alpha g | h \rangle = \overline{\alpha} \langle f, g | h \rangle \);
- \( \langle f, g | \alpha h \rangle = |\alpha|^2 \langle f, g | h \rangle \);

for all \( f, g, h \in X \) and \( \alpha \in \mathbb{F} \).

One of the most important properties of 2-inner product is the Cauchy-Schwarz inequality

\[
|\langle f, g | h \rangle|^2 \leq \langle f, f | h \rangle \langle g, g | h \rangle, \quad f, g, h \in X.
\]
For a given 2-inner product space \((X, \langle ., . \rangle)\) we can define a function \(\| ., . \|\) on \(X \times X\) by
\[
\|f, h\| = \langle f, f \rangle^{\frac{1}{2}}
\] (1.1)
for all \(f, h \in X\).

The above mentioned function satisfies the following conditions:

a. \(\|f, h\| \geq 0\) and \(\|f, h\| = 0\) if and only if \(f\) and \(h\) are linearly dependent;

b. \(\|f, h\| = \|h, f\|\);

c. \(\|\alpha f, h\| = |\alpha|\|f, h\|, \alpha \in \mathbb{F}\);

d. \(\|f_1 + f_2, h\| \leq \|f_1, h\| + \|f_2, h\|\).

A 2-norm on a vector space \(X\) is a function \(\| ., . \|\) defined on \(X \times X\) satisfying the conditions (a) to (d) and \((X, \| ., . \|)\) is called a linear 2-normed space.

Whenever a 2-inner product space \((X, \langle ., . \rangle)\) is given, we consider it as a linear 2-normed space \((X, \| ., . \|)\) via the 2-norm defined by (1.1).

Let \(X\) be a 2-inner product space. A sequence \(\{f_j\}\) is called convergent if there exists \(f \in X\) such that \(\lim_{j \to \infty} \|f_j - f, h\| = 0\), for all \(h \in X\). Similarly, we can define a Cauchy sequence in \(X\). Also, \(X\) is said to be a 2-Hilbert space if it is complete (see [18]).

Now we are ready to state the concept of a 2-frame which was introduced in [1]. A sequence \(\{f_j\}\) in a 2-Hilbert space \((X, \langle ., . \rangle)\) is called a 2-frame associated to \(h \in X\) if there exist \(A, B > 0\) such that
\[
A\|f, h\|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f, h\|^2, \forall f \in X.
\] (1.2)

If the right side of (1.2) holds, then \(\{f_j\}\) is called a 2-Bessel sequence.

In this paper, we shall introduce 2-atomic systems as a generalization of families of local 2-atoms. A characterization of 2-atomic systems is given. This leads us to obtain a generalization of 2-frame.

2. Main Results

In this section we are going to define the concept of a family of local 2-atoms. Next we will generalize this concept to a 2-atomic system for a linear operator and then a generalization of 2-frames will be studied.

In the sequel we assumed that \((X, \langle ., . \rangle)\) is a 2-Hilbert space, \(h \in X\) and \(\langle h \rangle\) is the subspace generated by \(h\).

**Definition 2.1.** Let \(\{f_j\}\) be a 2-Bessel sequence in a 2-inner product space \(X, h \in X\) and \(Y\) be a closed subspace of \(X\). We say that \(\{f_j\}\) is a family of local 2-atoms for \(Y\) associated to \(h\) if there exists a sequence of bilinear functionals \(\{c_j\}\) on \(X \times \langle h \rangle\) such that

i) \(\sum_j |c_j(f, h)|^2 \leq C\|f, h\|^2\), for some \(C > 0\);

ii) \(f = \sum_j c_j(f, h) f_j\), for all \(f \in Y\).
Note that a map \( c_j : X \times \langle h \rangle \rightarrow \mathbb{F} \) is called a bilinear functional if the following conditions hold for every \( f, g \in X \) and \( \alpha \in \mathbb{F} \).

(i) \( c_j(\alpha f + g, h) = \alpha c_j(f, h) + c_j(g, h) \);
(ii) \( c_j(f, \alpha h) = \alpha c_j(f, h) \).

In the following proposition, it is proved that every family of local 2-atoms is indeed a 2-frame sequence.

**Proposition 2.2.** Suppose that \( \{ f_j \} \) is a family of local 2-atoms for \( Y \), a closed subspace of 2-inner product space \( X \), then \( \{ f_j \} \) is a 2-frame for \( Y \) associated to \( h \).

**Proof.** It is enough to show that \( \{ f_j \} \) has a lower bound. Since \( \{ f_j \} \) is a family of local 2-atoms, there exists a sequence of bilinear functionals \( \{ c_j \} \) such that

\[
\sum_j |c_j(f, h)|^2 \leq C \| f, h \|_2^2, \quad f \in Y, \text{ for some } C > 0.
\]

\[
\| f, h \|^4 = (\langle f, f | h \rangle)^2 = (\langle f, \sum_j c_j(f, h) f_j | h \rangle)^2 = (\sum_j c_j(f, h) \langle f, f_j | h \rangle)^2 \leq \sum_j |c_j(f, h)|^2 \sum_j |\langle f, f_j | h \rangle|^2 \leq C \| f, h \|_2^2 \sum_j |\langle f, f_j | h \rangle|^2,
\]

it means that

\[
\frac{1}{C} \| f, h \|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2.
\]

Assume that \( (X, \langle \cdot, \cdot; \cdot \rangle) \) is a 2-Hilbert space and \( h \in X \). The algebraic complement of \( \langle h \rangle \) in \( X \) is denoted by \( M_h \), i.e. \( \langle h \rangle \oplus M_h = X \).

One may see that

\[
\langle f, g | h \rangle = \langle f, g | h \rangle, \quad f, g \in X.
\]

defines a semi-inner product on \( X \) (see [1]). This semi-inner product induces the following inner product on the quotient space \( X/\langle h \rangle \) denoted by \( M_h \) as follows:

\[
\langle f + \langle h \rangle, g + \langle h \rangle | h \rangle = \langle f, g | h \rangle, \quad f, g \in X.
\]

So \( M_h \) with respect to \( \| f \|_h := \sqrt{\langle f, f | h \rangle}, f \in M_h \), is a normed space. The completion of the inner product space \( M_h \) is denoted by \( X_h \).

With these notations, one can rewrite (1.2) as follows:

\[
A \| f \|_h^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B \| f \|_h^2, \quad \forall f \in X_h.
\]

Now we are going to generalize the concept of a family of local 2-atoms.

**Definition 2.3.** Let \( X \) be a 2-inner product space and fix \( h \in X \). Let \( K_h \) be a bounded linear operator on the Hilbert space \( X_h \). A sequence \( \{ f_j \} \subseteq X \) is
called a 2-atomic system for \( K_h \) associated to \( h \) if

i) \( \{f_j\} \) is a 2-Bessel sequence;

ii) for any \( f \in X_h \) there exists \( a_f = \{a_j\} \in \ell^2 \) such that \( K_h f = \sum_j a_j f_j \), where \( \|a_f\|_{\ell^2} \leq C \|f, h\|_X \) and \( C \) is a positive constant.

Note that the convergence of the series \( \sum_j a_j f_j \) is in the topology of \( X \). Also if \( \{f_j\} \subseteq X_h \) then the convergence of the series \( \sum_j a_j f_j \) is in the topology of \( X \) implies its convergence in \( X_h \).

A characterization of a 2-atomic system corresponding to \( h \in X \) is given as follows which lead us to obtain a generalization of 2-frame.

**Theorem 2.4.** Let \( K_h \) be a bounded linear operator on \( X_h \). Then for a sequence \( \{f_j\} \subseteq X_h \) the following statements are equivalent:

(i) \( \{f_j\} \) is a 2-atomic system for \( K_h \);

(ii) there exist \( A, B > 0 \) such that

\[
A \|K_h^* f\|_h^2 \leq \sum_j |\langle f, f_j \rangle_h|^2 \leq B \|f\|_h^2, \forall f \in X_h;
\]

(iii) \( \{f_j\} \subseteq X_h \) is a 2-Bessel sequence and there exists a 2-Bessel sequence \( \{g_j\} \) such that

\[
K_h f = \sum_j \langle f, g_j \rangle_h f_j, f \in X_h;
\]

(iv) \( \{f_j\} \subseteq X_h \) is a 2-Bessel sequence and there exists a 2-Bessel sequence \( \{g_j\} \) such that

\[
K_h^* f = \sum_j \langle f, f_j \rangle_h g_j, f \in X_h;
\]

(v) \( \{Q_h f_j\} \) is a 2-atomic system for the bounded linear operator \( Q_h K_h \), where \( Q_h \) is an injective operator on \( X_h \).

**Proof.** \( i \rightarrow ii) \) For every \( f \in X_h \) we have

\[
\|K_h^* f\|^2 = \|K_h^* f, h\|^2
= \sup\{ |\langle K_h^* f, g \rangle_h|^2 : g \in X_h, \|g, h\| = 1 \}
= \sup\{ |\langle f, K_h g \rangle h|^2 : g \in X_h, \|g, h\| = 1 \}.
\]
By definition of a 2-atomic system for $K_h$, there exists $C > 0$ such that $K_h g = \sum_j b_j f_j$ with $\|b_j\|_{\ell^2} = ||\{b_j\}||_{\ell^2} \leq C \|g, h\|$ and so

$$\|K^*_h f\|^2 = \sup\{(|f, \sum_j b_j f_j|)_h^2 : g \in X_h, \|g, h\| = 1\}$$

$$= \sup\{(|\sum_j b_j (f, f_j)_h|)^2 : g \in X_h, \|g, h\| = 1\}$$

$$\leq \sup\{\sum_j |b_j|^2 \sum_j (f, f_j)_h^2 : g \in X_h, \|g, h\| = 1\}$$

$$\leq C^2 \|g, h\|^2 \sum_j (f, f_j)_h^2$$

$$= C^2 \sum_j (f, f_j)_h^2.$$

It means that $\frac{1}{C^2} \|K^*_h f\|^2 \leq \sum_j (f, f_j)_h^2$.

\(ii \rightarrow iii\) Similar to Theorem 3 of [15], there exists a 2-Bessel sequence \(\{g_j\} \in X_h\) such that

$$K_h f = \sum_j \langle f, g_j \rangle_h f_j = \sum_j \langle f, g_j \rangle_h f_j.$$

\(iii \rightarrow iv\) For $f, g \in X_h$ we have

$$\langle K_h f, g \rangle_h = \langle \sum_j \langle f, g_j \rangle_h f_j, g \rangle_h$$

$$= \sum_j \langle f, g_j \rangle_h \langle f_j, g \rangle_h$$

$$= \sum_j \langle f, g_j \rangle_h \langle f_j, g \rangle_h$$

$$= \langle f, \sum_j \langle g, f_j \rangle_h g_j \rangle_h.$$

that is $K^*_h f = \sum_j \langle f, f_j \rangle_h g_j$.

\(iv \rightarrow iii\) It is similar to $iii \rightarrow iv$ so we omit it.

\(i \rightarrow v\) Since \(\{f_j\}\) is a 2-atomic system for $K_h$, for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$ so $Q_h K_h f = \sum_j a_j Q_h f_j$, i.e. \(\{Q_h f_j\}\) is a 2-atomic system for $Q_h K_h$.

\(v \rightarrow i\) Since \(\{Q_h f_j\}\) is a 2-atomic system for $Q_h K_h$, for any $f \in X_h$ there exists $\{b_j\} \in \ell^2$ such that $Q_h K_h f = \sum_j b_j Q_h f_j$ so $Q_h (K_h f - \sum_j b_j f_j) = 0$.

Due to injectivity of $Q_h$, $K_h f = \sum_j b_j f_j$. □

As a result of Theorem 2.4 the following definition is given.
Definition 2.5. Let $K_h$ be a bounded linear operator on $X_h$. A sequence $\{f_j\}$ in $X$ is called a $2$-$K$-frame if there exist $A, B > 0$ such that

$$A\|K_h^*f\|^2_h \leq \sum_j |\langle f, f_j \rangle|^2 \leq B\|f\|^2_h, \forall f \in X_h.$$ 

Trivially a $2$-frame, which was defined in [1], is a special case of $2$-$K$-frames with $K_h = I$.

A consequence of Theorem 2.4 is given as follows.

Theorem 2.6. Let $P_{Y_h}$ be the orthogonal projection on $Y_h$ as a closed subspace of $X_h$. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:

(i) $\{f_j\}$ is a family of local $2$-atoms for $Y_h$;
(ii) $\{f_j\}$ is a $2$-atomic system for $P_{Y_h}$;
(iii) $\{f_j\}$ is a $2$-$P_{Y_h}$-frame;
(iv) $\{f_j\}$ is a $2$-Bessel sequence and there exists a $2$-Bessel sequence $\{g_j\}$ such that

$$P_{Y_h}f = \sum_j \langle f, g_j \rangle_h f_j = \sum_j \langle f, f_j \rangle_h g_j, f \in X_h;$$

(v) $\{Q_h f_j\}$ is a $2$-atomic system for bounded linear operator $Q_h P_{Y_h}$, where $Q_h$ is an injective operator on $X_h$.

Proof. $i \rightarrow ii$ is obvious.

$ii \leftrightarrow iii, iii \leftrightarrow iv, iv \leftrightarrow ii$ and $v \leftrightarrow i$ hold from Theorem 2.4.

$iv \rightarrow i$ Since $P_{Y_h}f = \sum_j \langle f, g_j \rangle_h f_j$, it is enough to put $c_j(f, h) = \langle f, g_j \rangle_h$ because it is linear and

$$\sum_j |c_j(f, h)|^2 = \sum_j |\langle f, g_j \rangle_h|^2 \leq D\|f, h\|^2,$$

where $D$ is the upper $2$-frame bound of $\{g_j\}$. \[\square\]

Example 2.7. Let $n \in \mathbb{N}$ be odd and consider $X = \mathbb{R}^n$ with the following standard two inner product

$$\langle x, y | z \rangle = \det \begin{pmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{pmatrix}$$

where $\langle ., . \rangle$ is the inner product of $\mathbb{R}^n$. Let $\{e_1, ... , e_n\}$ be the standard basis of $\mathbb{R}^n$ and $h = e_n$. Trivially in this case $X_h = \mathbb{R}^{n-1}$ and one can see that its induced inner product is the standard inner product of $\mathbb{R}^{n-1}$. Now define the operator $K_h$ on $X_h$ by

$$K_h(e_{2i}) = e_i, i = 1, 2, ..., \frac{n-1}{2}$$

and otherwise $K_h e_i = e_{i+1}, i \leq n - 1$.

Then one can see that $e_1, e_1, e_3, e_2, ..., e_{n-1}, e_{n-1}$ is a $2$-$K_h$-frame.
ACKNOWLEDGMENTS

The authors wish to thank the anonymous reviewers for their specific and useful comments.

REFERENCES

2. H. Bolcskei, F. Hlawatsch, H. G. Feichtinger, Frame-theoretic analysis of oversampled
5. B. Dastourian, M. Janfada, *-frames for operators on Hilbert modules, Wavelets and
6. B. Dastourian, M. Janfada, Frames for operators in Banach spaces via semi-inner prod-
8. N. E. Dudley Ward, J. R. Partington, A construction of rational wavelets and frames in
Hardy-Sobolev space with applications to system modelling, SIAM J. Control Optim.
10. Y. C. Eldar, Sampling with arbitrary sampling and reconstruction spaces and oblique
11. Y. C. Eldar, T. Werther, General framework for consistent sampling in Hilbert spaces,
12. H.G. Feichtinger, T. Werther, Atomic systems for subspaces, in: L. Zayed (Ed.), Pro-
13. P. J. S. G. Ferreira, Mathematics for multimedia signal processing II: Discrete finite
frames and signal reconstruction, In: Byrnes, J.S. (ed.) Signal processing for multimedia,
Soc. 16(2), (2010), 127–132.
17. H. Gunawan, Inner products on n-inner product spaces, Soochow J. Math. 28(4), (2002),
389–398.
18. Z. Lewandowska, Bounded 2-linear operators on 2-normed sets, Glas. Mat. Ser. III,
19. T. Strohmer, R. Jr. Heath, Grassmanian frames with applications to coding and com-