Atomic Systems in 2-inner Product Spaces

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Abstract. In this paper, the concept of a family of local atoms in a
2-inner product space is introduced and then this concept is generalized
to an atomic system for an operator. Next a characterization of atomic
systems is proved. This characterization lead us to obtain a new frame
which is a generalization of frames in 2-inner product spaces.

Keywords: 2-inner product space, 2-normed space, Family of local atoms,
Atomic system, Frame.


1. Introduction and Preliminaries

Frames in Hilbert spaces were introduced by Duffin and Schaffer [9] in the
context of nonharmonic Fourier series in 1952. In 1986, frames were brought
to life by Daubechies et al. [7]. Now frames play an important role not only
in the theoretics but also in many kinds of applications, and have been widely
applied in signal processing [13], sampling [10, 11], coding and communications
[19], filter bank theory [2], system modeling [8], and so on.

Atomic systems for bounded linear operators on Hilbert spaces have been
introduced by L. Găvruta in [15] as a generalization of families of local atoms

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A sequence $\{f_j\}_{j \in \mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called an atomic system for a bounded linear operator $K$ on $\mathcal{H}$ if

1. the series $\sum_{j \in \mathbb{N}} c_j f_j$ converges for all $c = (c_j) \in l^2 := \{ (b_j)_{j \in \mathbb{N}} : \sum_{j \in \mathbb{N}} |b_j|^2 < \infty \}$;
2. there exists $C > 0$ such that for every $f \in \mathcal{H}$ there exists $a_f = (a_j) \in l^2$ such that $\|a_f\|_2 \leq C \|f\|_2$ and $Kf = \sum_{j \in \mathbb{N}} a_j f_j$.

It is proved that this concept is equivalent to $K$-frames, where $K$ is a bounded linear operator on separable Hilbert space $\mathcal{H}$ [15].

A sequence $\{f_j\}_{j \in \mathbb{N}}$ is said to be a $K$-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A \|K^* f\|^2 \leq \sum_{j \in \mathbb{N}} |\langle f, f_j \rangle|^2 \leq B \|f\|^2, \forall f \in \mathcal{H}.$$ 

We refer to [20] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert modules [5] and Banach spaces [6]. Note that frames in Hilbert spaces are just a particular case of $K$-frames, when $K$ is the identity operator on these Hilbert spaces.

The concepts of 2-inner product spaces and 2-normed spaces have been studied by many authors [3, 4, 14, 16, 17, 18]. In the sequel, we introduce 2-inner product and 2-normed spaces.

**Definition 1.1.** Suppose that $X$ is a vector space of dimension grater than 1 over the field $F$ (either $\mathbb{R}$ or $\mathbb{C}$). If there exists a mapping $\langle . , . | . , . \rangle : X \times X \times X \to F$ with the properties

1. $\langle f, f | h \rangle \geq 0$ and $\langle f, f | h \rangle = 0$ if and only if $f$ and $h$ are linearly dependent;
2. $\langle f, f | h \rangle = \langle h, h | f \rangle$;
3. $\langle g, f | h \rangle = \langle f, g | h \rangle$;
4. $\langle \alpha f, g | h \rangle = \alpha \langle f, g | h \rangle$ for $\alpha \in F$;
5. $\langle f_1 + f_2, g | h \rangle = \langle f_1, g | h \rangle + \langle f_2, g | h \rangle$,

then the pair $(X, \langle . , . | . , . \rangle)$ is called a 2-inner product space. The map $\langle . , . | . , . \rangle$ is said to be a 2-inner product on $X$.

Some basic properties of 2-inner product $\langle . , . | . , . \rangle$ cane be immediately obtained as follows (see [3, 4]).

- $\langle 0, g | h \rangle = \langle f, 0 | h \rangle = \langle f, g | 0 \rangle = 0$;
- $\langle f, \alpha g | h \rangle = \overline{\alpha} \langle f, g | h \rangle$;
- $\langle f, g | \alpha h \rangle = |\alpha|^2 \langle f, g | h \rangle$;

for all $f, g, h \in X$ and $\alpha \in F$.

One of the most important properties of 2-inner product is the Cauchy-Schwarz inequality

$$|\langle f, g | h \rangle|^2 \leq \langle f, f | h \rangle \langle g, g | h \rangle, \quad f, g, h \in X.$$
For a given 2-inner product space \((X, \langle ., . \rangle)\) we can define a function \(\| ., . \|\) on \(X \times X\) by
\[
\| f, h \| = \langle f, f | h \rangle^{\frac{1}{2}}
\] (1.1)
for all \(f, h \in X\).

The above mentioned function satisfies the following conditions:

a. \(\| f, h \| \geq 0\) and \(\| f, h \| = 0\) if and only if \(f\) and \(h\) are linearly dependent;

b. \(\| f, h \| = \| h, f \|\);

c. \(\| \alpha f, h \| = \| \alpha \| \| f, h \|, \alpha \in F\);

d. \(\| f_1 + f_2, h \| \leq \| f_1, h \| + \| f_2, h \|\).

A 2-norm on a vector space \(X\) is a function \(\| ., . \|\) defined on \(X \times X\) satisfying the conditions (a) to (d) and \((X, \| ., . \|)\) is called a linear 2-normed space. Whenever a 2-inner product space \((X, \langle ., . \rangle)\) is given, we consider it as a linear 2-normed space \((X, \| ., . \|)\) via the 2-norm defined by (1.1).

Let \(X\) be a 2-inner product space. A sequence \(\{f_j\}\) is called convergent if there exists \(f \in X\) such that \(\lim_{j \to \infty} \| f_j - f, h \| = 0\), for all \(h \in X\). Similarly, we can define a Cauchy sequence in \(X\). Also, \(X\) is said to be a 2-Hilbert space if it is complete (see [18]).

Now we are ready to state the concept of a 2-frame which was introduced in [1]. A sequence \(\{f_j\}\) in a 2-Hilbert space \((X, \langle ., . \rangle)\) is called a 2-frame associated to \(h \in X\) if there exist \(A, B > 0\) such that
\[
A \| f, h \|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B \| f, h \|^2, \forall f \in X.
\] (1.2)

If the right side of (1.2) holds, then \(\{f_j\}\) is called a 2-Bessel sequence.

In this paper, we shall introduce 2-atomic systems as a generalization of families of local 2-atoms. A characterization of 2-atomic systems is given. This leads us to obtain a generalization of 2-frame.

2. Main Results

In this section we are going to define the concept of a family of local 2-atoms. Next we will generalize this concept to a 2-atomic system for a linear operator and then a generalization of 2-frames will be studied.

In the sequel we assumed that \((X, \langle ., . \rangle)\) is a 2-Hilbert space, \(h \in X\) and \(\langle h \rangle\) is the subspace generated by \(h\).

**Definition 2.1.** Let \(\{f_j\}\) be a 2-Bessel sequence in a 2-inner product space \(X, h \in X\) and \(Y\) be a closed subspace of \(X\). We say that \(\{f_j\}\) is a family of local 2-atoms for \(Y\) associated to \(h\) if there exists a sequence of bilinear functionals \(\{c_j\}\) on \(X \times \langle h \rangle\) such that

i) \(\sum_j |c_j(f, h)|^2 \leq C \| f, h \|^2\), for some \(C > 0\);

ii) \(f = \sum_j c_j(f, h)f_j\),

for all \(f \in Y\).
Note that a map \( c_j : X \times \langle h \rangle \to \mathbb{F} \) is called a bilinear functional if the following conditions hold for every \( f, g \in X \) and \( \alpha \in \mathbb{F} \).

(i) \( c_j(\alpha f + g, h) = \alpha c_j(f, h) + c_j(g, h) \);

(ii) \( c_j(f, \alpha h) = \alpha c_j(f, h) \).

In the following proposition, it is proved that every family of local 2-atoms is indeed a 2-frame sequence.

**Proposition 2.2.** Suppose that \( \{ f_j \} \) is a family of local 2-atoms for \( Y \), a closed subspace of 2-inner product space \( X \), then \( \{ f_j \} \) is a 2-frame for \( Y \) associated to \( h \).

**Proof.** It is enough to show that \( \{ f_j \} \) has a lower bound. Since \( \{ f_j \} \) is a family of local 2-atoms, there exists a sequence of bilinear functionals \( \{ c_j \} \) such that

\[
\sum_j |c_j(f, h)|^2 \leq C \| f, h \|^2, \quad f \in Y, \text{ for some } C > 0.
\]

Assume that \((X, \langle \cdot, \cdot \rangle)\) is a 2-Hilbert space and \( h \in X \). The algebraic complement of \( \langle h \rangle \) in \( X \) is denoted by \( M_h \), i.e. \( \langle h \rangle \oplus M_h = X \).

One may see that

\[
\langle f, g \rangle_h = \langle f, g | h \rangle, \quad f, g \in X.
\]

defines a semi-inner product on \( X \) (see [1]). This semi-inner product induces the following inner product on the quotient space \( \frac{X}{\langle h \rangle} \) denoted by \( M_h \) as follows:

\[
\langle f + \langle h \rangle, g + \langle h \rangle \rangle_h = \langle f, g \rangle_h, \quad f, g \in X.
\]

So \( M_h \) with respect to \( \| f \|_h := \sqrt{\langle f, f \rangle_h}, f \in M_h \), is a normed space. The completion of the inner product space \( M_h \) is denoted by \( X_h \).

With these notations, one can rewrite (1.2) as follows:

\[
A \| f \|^2_h \leq \sum_j |\langle f, f_j \rangle_h|^2 \leq B \| f \|^2_h, \quad \forall f \in X_h.
\]

Now we are going to generalize the concept of a family of local 2-atoms.

**Definition 2.3.** Let \( X \) be a 2-inner product space and fix \( h \in X \). Let \( K_h \) be a bounded linear operator on the Hilbert space \( X_h \). A sequence \( \{ f_j \} \subseteq X \) is
called a 2-atomic system for $K_h$ associated to $h$ if
i) $\{f_j\}$ is a 2-Bessel sequence;

ii) for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$, where $\|a_f\|_{\ell^2} \leq C\|f, h\|_X$ and $C$ is a positive constant.

Note that the convergence of the series $\sum_j a_j f_j$ is in the topology of $X$. Also if $\{f_j\} \subseteq X_h$ then the convergence of the series $\sum_j a_j f_j$ is in the topology of $X$ implies its convergence in $X_h$.

A characterization of a 2-atomic system corresponding to $h \in X$ is given as follows which lead us to obtain a generalization of 2-frame.

**Theorem 2.4.** Let $K_h$ be a bounded linear operator on $X_h$. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:

(i) $\{f_j\}$ is a 2-atomic system for $K_h$;

(ii) there exist $A, B > 0$ such that

$$A\|K_h f\|_h^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h;$$

(iii) $\{f_j\} \subseteq X_h$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that

$$K_h f = \sum_j \langle f, g_j | h \rangle f_j, f \in X_h;$$

(iv) $\{f_j\} \subseteq X_h$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that

$$K_h^* f = \sum_j \langle f, f_j | h \rangle g_j, f \in X_h;$$

(v) $\{Q_h f_j\}$ is a 2-atomic system for the bounded linear operator $Q_h K_h$, where $Q_h$ is an injective operator on $X_h$.

**Proof.** i $\rightarrow$ ii) For every $f \in X_h$ we have

$$\|K_h^* f\|^2 = \|K_h^* f, h\|^2$$

$$= \sup\{|\langle K_h^* f, g | h \rangle|^2 : g \in X_h, \|g, h\| = 1\}$$

$$= \sup\{|\langle f, K_h g | h \rangle|^2 : g \in X_h, \|g, h\| = 1\}.$$
By definition of a 2-atomic system for $K_h$, there exists $C > 0$ such that $K_g = \sum b_j f_j$ with $\|b_j\|_{\ell^2} = ||\{b_j\}||_{\ell^2} \leq C \|g,h\|$ and so

$$
\||K^*_h f\|_2^2 = \sup\{\langle f, \sum b_j f_j \rangle_h^2 : g \in X_h, \|g,h\| = 1\}
= \sup\{\sum_j b_j^2 \|f_j\|_2^2 : g \in X_h, \|g,h\| = 1\}
\leq C^2 \|g,h\|_2^2 \sum_j \|f_j\|_2^2
= C^2 \sum_j |\langle f, f_j \rangle_h|^2.
$$

It means that $\frac{1}{C^2} \||K^*_h f\|_2^2 \leq \sum_j |\langle f, f_j \rangle_h|^2$.

$ii \rightarrow iii$) Similar to Theorem 3 of [15], there exists a 2-Bessel sequence $\{g_j\} \in X_h$ such that

$$
K_h f = \sum_j \langle f, g_j \rangle_h f_j = \sum_j \langle f, g_j |h\rangle f_j.
$$

$iii \rightarrow iv$) For $f, g \in X_h$ we have

$$
\langle K_h f, g \rangle_h = \langle \sum_j \langle f, g_j |h\rangle f_j, g \rangle_h
= \sum_j \langle f, g_j \rangle |h\rangle \langle f_j, g \rangle
= \sum_j \langle f, g_j |h\rangle \langle f_j, g \rangle
= \langle f, \sum_j \langle g, f_j |h\rangle g_j \rangle_h,
$$

that is $K^*_h f = \sum_j \langle f, f_j |h\rangle g_j$.

$iv \rightarrow iii$) It is similar to $iii \rightarrow iv$ so we omit it.

$i \rightarrow v$) Since $\{f_j\}$ is a 2-atomic system for $K_h$, for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$ so $Q_h K_h f = \sum_j a_j Q_h f_j$, i.e. $\{Q_h f_j\}$ is a 2-atomic system for $Q_h K_h$.

$v \rightarrow i$) Since $\{Q_h f_j\}$ is a 2-atomic system for $Q_h K_h$, for any $f \in X_h$ there exists $\{b_j\} \in \ell^2$ such that $Q_h K_h f = \sum_j b_j Q_h f_j$ so $Q_h (K_h f - \sum_j b_j f_j) = 0$. Due to injectivity of $Q_h$, $K_h f = \sum_j b_j f_j$.

As a result of Theorem 2.4 the following definition is given.
Definition 2.5. Let $K_h$ be a bounded linear operator on $X_h$. A sequence $\{f_j\}$ in $X$ is called 2-$K$-frame if there exist $A, B > 0$ such that

$$A\|K_h f\|_h^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h.$$ 

Trivially a 2-frame, which was defined in [1], is a special case of 2-$K$-frames with $K_h = I$.

A consequence of Theorem 2.4 is given as follows.

Theorem 2.6. Let $P_{Y_h}$ be the orthogonal projection on $Y_h$ as a closed subspace of $X_h$. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:

(i) $\{f_j\}$ is a family of local 2-atoms for $Y_h$;
(ii) $\{f_j\}$ is a 2-atomic system for $P_{Y_h}$;
(iii) $\{f_j\}$ is a 2-$P_{Y_h}$-frame;
(iv) $\{f_j\}$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that

$$P_{Y_h} f = \sum_j \langle f, g_j | h \rangle f_j = \sum_j \langle f, f_j | h \rangle g_j, f \in X_h;$$

(v) $\{Q_h f_j\}$ is a 2-atomic system for bounded linear operator $Q_h P_{Y_h}$, where $Q_h$ is an injective operator on $X_h$.

Proof. $i \rightarrow ii$ is obvious.

$ii \leftrightarrow iii, iii \leftrightarrow iv, iv \leftrightarrow ii$ and $v \leftrightarrow i$ hold from Theorem 2.4.

$iv \rightarrow i$ Since $P_{Y_h} f = \sum_j \langle f, g_j | h \rangle f_j$, it is enough to put $c_j(f, h) = \langle f, g_j | h \rangle$ because it is linear and

$$\sum_j |c_j(f, h)|^2 = \sum_j |\langle f, g_j | h \rangle|^2 \leq D\|f, h\|^2,$$

where $D$ is the upper 2-frame bound of $\{g_j\}$. $\square$

Example 2.7. Let $n \in \mathbb{N}$ be odd and consider $X = \mathbb{R}^n$ with the following standard two inner product

$$\langle x, y | z \rangle = \det \begin{pmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{pmatrix}$$

where $\langle ., . \rangle$ is the inner product of $\mathbb{R}^n$. Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$ and $h = e_n$. Trivially in this case $X_h = \mathbb{R}^{n-1}$ and one can see that its induced inner product is the standard inner product of $\mathbb{R}^{n-1}$. Now define the operator $K_h$ on $X_h$ by

$$K_h(e_{2i}) = e_i, i = 1, 2, ..., n-1 \quad \text{and otherwise } K_h e_i = e_i, i \leq n - 1.$$ 

Then one can see that $e_1, e_1, e_3, e_2, ..., e_{n-1}, e_{n-1}$ is a 2-$K_h$-frame.
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