Atomic Systems in 2-inner Product Spaces

Bahram Dastourian, Mohammad Janfada*

Department of Pure Mathematics
Ferdowsi University of Mashhad
Mashhad, 1159-91775, Iran

E-mail: bdastorian@gmail.com
E-mail: mjanfada@gmail.com

Abstract. In this paper, the concept of a family of local atoms in a 2-inner product space is introduced and then this concept is generalized to an atomic system for an operator. Next a characterization of atomic systems is proved. This characterization lead us to obtain a new frame which is a generalization of frames in 2-inner product spaces.

Keywords: 2-inner product space, 2-normed space, Family of local atoms, Atomic system, Frame.


1. Introduction and Preliminaries

Frames in Hilbert spaces were introduced by Duffin and Schaffer [9] in the context of nonharmonic Fourier series in 1952. In 1986, frames were brought to life by Daubechies et al. [7]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [13], sampling [10, 11], coding and communications [19], filter bank theory [2], system modeling [8], and so on.

Atomic systems for bounded linear operators on Hilbert spaces have been introduced by L. Găvruta in [15] as a generalization of families of local atoms.
A sequence \( \{f_j\}_{j \in \mathbb{N}} \) in a Hilbert space \( \mathcal{H} \) is called an \textit{atomic system} for a bounded linear operator \( K \) on \( \mathcal{H} \) if

1. the series \( \sum_{j \in \mathbb{N}} c_j f_j \) converges for all \( c = (c_j) \in l^2 := \{ \{b_j\}_{j \in \mathbb{N}} : \sum_{j \in \mathbb{N}} |b_j|^2 < \infty \} \);
2. there exists \( C > 0 \) such that for every \( f \in \mathcal{H} \) there exists \( a_f = (a_j) \in l^2 \) such that \( \|a_f\|_2 \leq C\|f\| \) and \( Kf = \sum_{j \in \mathbb{N}} a_j f_j \).

It is proved that this concept is equivalent to \( K \)-frames, where \( K \) is a bounded linear operator on separable Hilbert space \( \mathcal{H} \) [15].

A sequence \( \{f_j\}_{j \in \mathbb{N}} \) is said to be a \( K \)-frame for \( \mathcal{H} \) if there exist constants \( A,B > 0 \) such that

\[
A\|K^*f\|_2^2 \leq \sum_{j \in \mathbb{N}} |\langle f,f_j \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in \mathcal{H}.
\]

We refer to [20] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert modules [5] and Banach spaces [6]. Note that frames in Hilbert spaces are just a particular case of \( K \)-frames, when \( K \) is the identity operator on these Hilbert spaces.

The concepts of 2-inner product spaces and 2-normed spaces have been studied by many authors [3, 4, 14, 16, 17, 18]. In the sequel, we introduce 2-inner product and 2-normed spaces.

\textbf{Definition 1.1.} Suppose that \( X \) is a vector space of dimension grater than 1 over the field \( \mathbb{F} \) (either \( \mathbb{R} \) or \( \mathbb{C} \)). If there exists a mapping \( \langle .,|.| \rangle : X \times X \times X \to \mathbb{F} \) with the properties

1. \( \langle f,f|h \rangle \geq 0 \) and \( \langle f,f|h \rangle = 0 \) if and only if \( f \) and \( h \) are linearly dependent;
2. \( \langle f,f|h \rangle = \langle h,f|f \rangle \);
3. \( \langle g,f|h \rangle = \langle f,g|h \rangle \);
4. \( \langle \alpha f,g|h \rangle = \alpha \langle f,g|h \rangle \) for \( \alpha \in \mathbb{F} \);
5. \( \langle f_1 + f_2,g|h \rangle = \langle f_1,g|h \rangle + \langle f_2,g|h \rangle \),

then the pair \( (X,\langle .,|.| \rangle) \) is called a 2-inner product space. The map \( \langle .,|.| \rangle \) is said to be a 2-inner product on \( X \).

Some basic properties of 2-inner product \( \langle .,|.| \rangle \) cane be immediately obtained as follows (see [3, 4]).

- \( \langle 0,g|h \rangle = \langle f,0|h \rangle = \langle f,g|0 \rangle = 0 \);
- \( \langle f,\alpha g|h \rangle = \overline{\alpha} \langle f,g|h \rangle \);
- \( \langle f,g|\alpha h \rangle = |\alpha|^2 \langle f,g|h \rangle \);

for all \( f,g,h \in X \) and \( \alpha \in \mathbb{F} \).

One of the most important properties of 2-inner product is the Cauchy-Schwarz inequality

\[
|\langle f,g|h \rangle|^2 \leq \langle f,f|h \rangle \langle g,g|h \rangle, \quad f,g,h \in X.
\]
For a given 2-inner product space \((X, \langle ., . \rangle)\) we can define a function \(\|., .\|\) on \(X \times X\) by
\[
\|f, h\| = \langle f, f | h \rangle^{\frac{1}{2}} \tag{1.1}
\]
for all \(f, h \in X\).

The above mentioned function satisfies the following conditions:

a. \(\|f, h\| \geq 0\) and \(\|f, h\| = 0\) if and only if \(f\) and \(h\) are linearly dependent;

b. \(\|f, h\| = \|h, f\|\);

c. \(\|\alpha f, h\| = |\alpha| \|f, h\|, \alpha \in \mathbb{F}\);

d. \(\|f_1 + f_2, h\| \leq \|f_1, h\| + \|f_2, h\|\).

A 2-norm on a vector space \(X\) is a function \(\|., .\|\) defined on \(X \times X\) satisfying the conditions (a) to (d) and \((X, \|., .\|)\) is called a linear 2-normed space.

Whenever a 2-inner product space \((X, \langle ., . \rangle)\) is given, we consider it as a linear 2-normed space \((X, \|., .\|)\) via the 2-norm defined by (1.1).

Let \(X\) be a 2-inner product space. A sequence \(\{f_j\}\) is called convergent if there exists \(f \in X\) such that \(\lim_{j \to \infty} \|f_j - f, h\| = 0\), for all \(h \in X\). Similarly, we can define a Cauchy sequence in \(X\). Also, \(X\) is said to be a 2-Hilbert space if it is complete (see [18]).

Now we are ready to state the concept of a 2-frame which was introduced in [1]. A sequence \(\{f_j\}\) in a 2-Hilbert space \((X, \langle ., . \rangle)\) is called a 2-frame associated to \(h \in X\) if there exist \(A, B > 0\) such that
\[
A \|f, h\|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B \|f, h\|^2, \forall f \in X. \tag{1.2}
\]

If the right side of (1.2) holds, then \(\{f_j\}\) is called a 2-Bessel sequence.

In this paper, we shall introduce 2-atomic systems as a generalization of families of local 2-atoms. A characterization of 2-atomic systems is given. This leads us to obtain a generalization of 2-frame.

2. Main Results

In this section we are going to define the concept of a family of local 2-atoms. Next we will generalize this concept to a 2-atomic system for a linear operator and then a generalization of 2-frames will be studied.

In the sequel we assumed that \((X, \langle ., . \rangle)\) is a 2-Hilbert space, \(h \in X\) and \(\langle h \rangle\) is the subspace generated by \(h\).

**Definition 2.1.** Let \(\{f_j\}\) be a 2-Bessel sequence in a 2-inner product space \(X, h \in X\) and \(Y\) be a closed subspace of \(X\). We say that \(\{f_j\}\) is a family of local 2-atoms for \(Y\) associated to \(h\) if there exists a sequence of bilinear functionals \(\{c_j\}\) on \(X \times \langle h \rangle\) such that

i) \(\sum_j |c_j(f, h)|^2 \leq C \|f, h\|^2\), for some \(C > 0\);

ii) \(f = \sum_j c_j(f, h)f_j\),

for all \(f \in Y\).
Note that a map $c_j : X \times \langle h \rangle \rightarrow \mathbb{F}$ is called a bilinear functional if the following conditions hold for every $f, g \in X$ and $\alpha \in \mathbb{F}$.

(i) $c_j(\alpha f + g, h) = \alpha c_j(f, h) + c_j(g, h)$;

(ii) $c_j(f, \alpha h) = \alpha c_j(f, h)$.

In the following proposition, it is proved that every family of local 2-atoms is indeed a 2-frame sequence.

**Proposition 2.2.** Suppose that $\{f_j\}$ is a family of local 2-atoms for $Y$, a closed subspace of 2-inner product space $X$, then $\{f_j\}$ is a 2-frame for $Y$ associated to $h$.

**Proof.** It is enough to show that $\{f_j\}$ has a lower bound. Since $\{f_j\}$ is a family of local 2-atoms, there exists a sequence of bilinear functionals $\{c_j\}$ such that

$$
\sum_j |c_j(f, h)|^2 \leq C \|f, h\|^2, f \in Y, \text{ for some } C > 0.
$$

One may see that

$$
\langle f, g \rangle_h = \langle f, g | h \rangle, f, g \in X.
$$

defines a semi-inner product on $X$ (see [1]). This semi-inner product induces the following inner product on the quotient space $X_h$ denoted by $M_h$ as follows:

$$
\langle f + \langle h \rangle, g + \langle h \rangle \rangle_h = \langle f, g \rangle_h, \text{ for } f, g \in X.
$$

So $M_h$ with respect to $\|f\|_h := \sqrt{(f, f)_h}, f \in M_h$, is a normed space. The completion of the inner product space $M_h$ is denoted by $X_h$.

With these notations, one can rewrite (1.2) as follows:

$$
A \|f\|^2_h \leq \sum_j |\langle f, f_j \rangle_h|^2 \leq B \|f\|^2_h, \forall f \in X_h.
$$

Now we are going to generalize the concept of a family of local 2-atoms.

**Definition 2.3.** Let $X$ be a 2-inner product space and fix $h \in X$. Let $K_h$ be a bounded linear operator on the Hilbert space $X_h$. A sequence $\{f_j\} \subseteq X$ is
called a 2-atomic system for $K_h$ associated to $h$ if
i) $\{f_j\}$ is a 2-Bessel sequence;
ii) for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$, where $\|a_f\|_{\ell^2} \leq C\|f,h\|_X$ and $C$ is a positive constant.

Note that the convergence of the series $\sum_j a_j f_j$ is in the topology of $X$. Also if $\{f_j\} \subseteq X_h$ then the convergence of the series $\sum_j a_j f_j$ is in the topology of $X$ implies its convergence in $X_h$.

A characterization of a 2-atomic system corresponding to $h \in X$ is given as follows which lead us to obtain a generalization of 2-frame.

**Theorem 2.4.** Let $K_h$ be a bounded linear operator on $X_h$. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:

(i) $\{f_j\}$ is a 2-atomic system for $K_h$;
(ii) there exist $A, B > 0$ such that

$$A\|K_h^*f\|_h^2 \leq \sum_j |(f, f_j|h)|^2 \leq B\|f\|_h^2, \forall f \in X_h;$$

(iii) $\{f_j\} \subseteq X_h$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that

$$K_h f = \sum_j (f, g_j|h)f_j, f \in X_h;$$

(iv) $\{f_j\} \subseteq X_h$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that

$$K_h^* f = \sum_j (f, f_j|h)g_j, f \in X_h;$$

(v) $\{Q_h f_j\}$ is a 2-atomic system for the bounded linear operator $Q_h K_h$, where $Q_h$ is an injective operator on $X_h$.

**Proof.** $i \to ii)$ For every $f \in X_h$ we have

$$\|K_h^* f\|_h^2 = \|K_h^* f, h\|_h^2 = \sup \{|(K_h^* f, g|h)|^2 : g \in X_h, \|g, h\| = 1\} = \sup \{|(f, K_h g|h)|^2 : g \in X_h, \|g, h\| = 1\}.$$
By definition of a 2-atomic system for $K_h$, there exists $C > 0$ such that $K_h g = \sum_j b_j f_j$ with $\| b_j \|_{\ell^2} = \| \{ b_j \} \|_{\ell^2} \leq C \| g, h \|$ and so

$$\| K_h^* f \|^2 = \sup \{ |\langle f, \sum_j b_j f_j | h \rangle|^2 : g \in X_h, \| g, h \| = 1 \}$$

$$= \sup \{ \left| \sum_j b_j \langle f, f_j | h \rangle \right|^2 : g \in X_h, \| g, h \| = 1 \}$$

$$\leq \sup \{ \sum_j |b_j|^2 \sum_j |\langle f, f_j | h \rangle|^2 : g \in X_h, \| g, h \| = 1 \}$$

$$\leq C^2 \| g, h \|^2 \sum_j |\langle f, f_j | h \rangle|^2$$

It means that $\frac{1}{C^2} \| K_h^* f \|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2$.

\(ii \rightarrow iii\) Similar to Theorem 3 of [15], there exists a 2-Bessel sequence $\{ g_j \} \in X_h$ such that

$$K_h f = \sum_j \langle f, g_j | h \rangle f_j = \sum_j \langle f, g_j | h \rangle f_j.$$  

\(iii \rightarrow iv\) For $f, g \in X_h$ we have

$$\langle K_h f, g \rangle_h = \langle \sum_j \langle f, g_j | h \rangle f_j, g \rangle_h$$

$$= \sum_j \langle f, g_j | h \rangle \langle f_j, g_h \rangle$$

$$= \sum_j \langle f, g_j \rangle_h \langle f_j, g_h \rangle$$

$$= \langle f, \sum_j \langle g, f_j | h \rangle g_j \rangle_h,$$

that is $K_h^* f = \sum_j \langle f, f_j | h \rangle g_j$.

\(iv \rightarrow iii\) It is similar to \(iii \rightarrow iv\) so we omit it.

\(i \rightarrow v\) Since $\{ f_j \}$ is a 2-atomic system for $K_h$, for any $f \in X_h$ there exists $a_f = \{ a_j \} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$ so $Q_h K_h f = \sum_j a_j Q_h f_j$, i.e. $\{ Q_h f_j \}$ is a 2-atomic system for $Q_h K_h$.

\(v \rightarrow i\) Since $\{ Q_h f_j \}$ is a 2-atomic system for $Q_h K_h$, for any $f \in X_h$ there exists $\{ b_j \} \in \ell^2$ such that $Q_h K_h f = \sum_j b_j Q_h f_j$ so $\sum_j b_j f_j = 0$. Due to injectivity of $Q_h$, $K_h f = \sum_j b_j f_j$. \(\square\)

As a result of Theorem 2.4 the following definition is given.
Definition 2.5. Let $K_h$ be a bounded linear operator on $X_h$. A sequence $\{f_j\}$ in $X$ is called a 2-$K$-frame if there exist $A, B > 0$ such that
\[
A\|K_h f\|_h^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f\|_h^2, \quad \forall f \in X_h.
\]

Trivially a 2-frame, which was defined in [1], is a special case of 2-$K$-frames with $K_h = I$.

A consequence of Theorem 2.4 is given as follows.

Theorem 2.6. Let $P_{Y_h}$ be the orthogonal projection on $Y_h$ as a closed subspace of $X_h$. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:

(i) $\{f_j\}$ is a family of local 2-atoms for $Y_h$;
(ii) $\{f_j\}$ is a 2-atomic system for $P_{Y_h}$;
(iii) $\{f_j\}$ is a 2-$P_{Y_h}$-frame;
(iv) $\{f_j\}$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that
\[
P_{Y_h} f = \sum_j \langle f, g_j | h \rangle f_j = \sum_j \langle f, f_j | h \rangle g_j, \quad f \in X_h;
\]
(v) $\{Q_h f_j\}$ is a 2-atomic system for bounded linear operator $Q_h P_{Y_h}$, where $Q_h$ is an injective operator on $X_h$.

Proof. $i \to ii$ is obvious.

ii $\leftrightarrow$ iii, iii $\leftrightarrow$ iv, iv $\leftrightarrow$ ii and v $\leftrightarrow$ i hold from Theorem 2.4.

iv $\to$ i) Since $P_{Y_h} f = \sum_j \langle f, g_j | h \rangle f_j$, it is enough to put $c_j(f, h) = \langle f, g_j | h \rangle$ because it is linear and
\[
\sum_j |c_j(f, h)|^2 = \sum_j |\langle f, g_j | h \rangle|^2 \leq D\|f, h\|_h^2,
\]
where $D$ is the upper 2-frame bound of $\{g_j\}$. \qed

Example 2.7. Let $n \in \mathbb{N}$ be odd and consider $X = \mathbb{R}^n$ with the following standard two inner product
\[
\langle x, y | z \rangle = \det \begin{pmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{pmatrix}
\]
where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathbb{R}^n$. Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$ and $h = e_n$. Trivially in this case $X_h = \mathbb{R}^{n-1}$ and one can see that its induced inner product is the standard inner product of $\mathbb{R}^{n-1}$. Now define the operator $K_h$ on $X_h$ by
\[
K_h(e_{2i}) = e_i, i = 1, 2, ..., \frac{n-1}{2} \quad \text{and otherwise} \quad K_h e_i = e_i, i \leq n - 1.
\]
Then one can see that $e_1, e_1, e_3, e_2, ..., e_{n-1}, e_{n-1}$ is a 2-$K_h$-frame.
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