

## Atomic Systems in 2-inner Product Spaces

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**ABSTRACT.** In this paper, the concept of a family of local atoms in a 2-inner product space is introduced and then this concept is generalized to an atomic system for an operator. Next a characterization of atomic systems is proved. This characterization lead us to obtain a new frame which is a generalization of frames in 2-inner product spaces.

**Keywords:** 2-inner product space, 2-normed space, Family of local atoms, Atomic system, Frame.

**2000 Mathematics subject classification:** 42C15, 46C50.

### 1. INTRODUCTION AND PRELIMINARIES

Frames in Hilbert spaces were introduced by Duffin and Schaffer [9] in the context of nonharmonic Fourier series in 1952. In 1986, frames were brought to life by Daubechies *et al.* [7]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [13], sampling [10, 11], coding and communications [19], filter bank theory [2], system modeling [8], and so on.

Atomic systems for bounded linear operators on Hilbert spaces have been introduced by L. Găvruta in [15] as a generalization of families of local atoms

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[12]. A sequence  $\{f_j\}_{j \in \mathbb{N}}$  in a Hilbert space  $\mathcal{H}$  is called an *atomic system* for a bounded linear operator  $K$  on  $\mathcal{H}$  if

i) the series  $\sum_{j \in \mathbb{N}} c_j f_j$  converges for all  $c = (c_j) \in l^2 := \{\{b_j\}_{j \in \mathbb{N}} : \sum_{j \in \mathbb{N}} |b_j|^2 < \infty\}$ ;

ii) there exists  $C > 0$  such that for every  $f \in \mathcal{H}$  there exists  $a_f = (a_j) \in l^2$  such that  $\|a_f\|_{l^2} \leq C\|f\|$  and  $Kf = \sum_{j \in \mathbb{N}} a_j f_j$ .

It is proved that this concept is equivalent to  $K$ -frames, where  $K$  is a bounded linear operator on separable Hilbert space  $\mathcal{H}$  [15].

A sequence  $\{f_j\}_{j \in \mathbb{N}}$  is said to be a  $K$ -frame for  $\mathcal{H}$  if there exist constants  $A, B > 0$  such that

$$A\|K^*f\|^2 \leq \sum_{j \in \mathbb{N}} |\langle f, f_j \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

We refer to [20] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert modules [5] and Banach spaces [6]. Note that frames in Hilbert spaces are just a particular case of  $K$ -frames, when  $K$  is the identity operator on these Hilbert spaces.

The concepts of 2-inner product spaces and 2-normed spaces have been studied by many authors [3, 4, 14, 16, 17, 18]. In the sequel, we introduce 2-inner product and 2-normed spaces.

**Definition 1.1.** Suppose that  $X$  is a vector space of dimension greater than 1 over the field  $\mathbb{F}$  (either  $\mathbb{R}$  or  $\mathbb{C}$ ). If there exists a mapping  $\langle \cdot, \cdot | \cdot \rangle : X \times X \times X \rightarrow \mathbb{F}$  with the properties

1.  $\langle f, f | h \rangle \geq 0$  and  $\langle f, f | h \rangle = 0$  if and only if  $f$  and  $h$  are linearly dependent;
2.  $\langle f, f | h \rangle = \langle h, h | f \rangle$ ;
3.  $\langle g, f | h \rangle = \overline{\langle f, g | h \rangle}$ ;
4.  $\langle \alpha f, g | h \rangle = \alpha \langle f, g | h \rangle$  for  $\alpha \in \mathbb{F}$ ;
5.  $\langle f_1 + f_2, g | h \rangle = \langle f_1, g | h \rangle + \langle f_2, g | h \rangle$ ,

then the pair  $(X, \langle \cdot, \cdot | \cdot \rangle)$  is called a 2-inner product space. The map  $\langle \cdot, \cdot | \cdot \rangle$  is said to be a 2-inner product on  $X$ .

Some basic properties of 2-inner product  $\langle \cdot, \cdot | \cdot \rangle$  can be immediately obtained as follows (see [3, 4]).

- $\langle 0, g | h \rangle = \langle f, 0 | h \rangle = \langle f, g | 0 \rangle = 0$ ;
- $\langle f, \alpha g | h \rangle = \overline{\alpha} \langle f, g | h \rangle$ ;
- $\langle f, g | \alpha h \rangle = |\alpha|^2 \langle f, g | h \rangle$ ;

for all  $f, g, h \in X$  and  $\alpha \in \mathbb{F}$ .

One of the most important properties of 2-inner product is the Cauchy-Schwarz inequality

$$|\langle f, g | h \rangle|^2 \leq \langle f, f | h \rangle \langle g, g | h \rangle, \quad f, g, h \in X.$$

For a given 2-inner product space  $(X, \langle \cdot, \cdot | \cdot \rangle)$  we can define a function  $\|\cdot, \cdot\|$  on  $X \times X$  by

$$\|f, h\| = \langle f, f | h \rangle^{\frac{1}{2}} \quad (1.1)$$

for all  $f, h \in X$ .

The above mentioned function satisfies the following conditions:

- a.  $\|f, h\| \geq 0$  and  $\|f, h\| = 0$  if and only if  $f$  and  $h$  are linearly dependent;
- b.  $\|f, h\| = \|h, f\|$ ;
- c.  $\|\alpha f, h\| = |\alpha| \|f, h\|, \alpha \in \mathbb{F}$ ;
- d.  $\|f_1 + f_2, h\| \leq \|f_1, h\| + \|f_2, h\|$ .

A 2-norm on a vector space  $X$  is a function  $\|\cdot, \cdot\|$  defined on  $X \times X$  satisfying the conditions (a) to (d) and  $(X, \|\cdot, \cdot\|)$  is called a linear 2-normed space. Whenever a 2-inner product space  $(X, \langle \cdot, \cdot | \cdot \rangle)$  is given, we consider it as a linear 2-normed space  $(X, \|\cdot, \cdot\|)$  via the 2-norm defined by (1.1).

Let  $X$  be a 2-inner product space. A sequence  $\{f_j\}$  is called convergent if there exists  $f \in X$  such that  $\lim_{j \rightarrow \infty} \|f_j - f, h\| = 0$ , for all  $h \in X$ . Similarly, we can define a Cauchy sequence in  $X$ . Also,  $X$  is said to be a 2-Hilbert space if it is complete (see [18]).

Now we are ready to state the concept of a 2-frame which was introduced in [1]. A sequence  $\{f_j\}$  in a 2-Hilbert space  $(X, \langle \cdot, \cdot | \cdot \rangle)$  is called a 2-frame associated to  $h \in X$  if there exist  $A, B > 0$  such that

$$A\|f, h\|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f, h\|^2, \forall f \in X. \quad (1.2)$$

If the right side of (1.2) holds, then  $\{f_j\}$  is called a 2-Bessel sequence.

In this paper, we shall introduce 2-atomic systems as a generalization of families of local 2-atoms. A characterization of 2-atomic systems is given. This leads us to obtain a generalization of 2-frame.

## 2. MAIN RESULTS

In this section we are going to define the concept of a family of local 2-atoms. Next we will generalize this concept to a 2-atomic system for a linear operator and then a generalization of 2-frames will be studied.

In the sequel we assumed that  $(X, \langle \cdot, \cdot | \cdot \rangle)$  is a 2-Hilbert space,  $h \in X$  and  $\langle h \rangle$  is the subspace generated by  $h$ .

**Definition 2.1.** Let  $\{f_j\}$  be a 2-Bessel sequence in a 2-inner product space  $X$ ,  $h \in X$  and  $Y$  be a closed subspace of  $X$ . We say that  $\{f_j\}$  is a family of local 2-atoms for  $Y$  associated to  $h$  if there exists a sequence of bilinear functionals  $\{c_j\}$  on  $X \times \langle h \rangle$  such that

$$i) \sum_j |c_j(f, h)|^2 \leq C\|f, h\|^2, \text{ for some } C > 0;$$

$$ii) f = \sum_j c_j(f, h)f_j,$$

for all  $f \in Y$ .

Note that a map  $c_j : X \times \langle h \rangle \rightarrow \mathbb{F}$  is called a bilinear functional if the following conditions hold for every  $f, g \in X$  and  $\alpha \in \mathbb{F}$ .

- (i)  $c_j(\alpha f + g, h) = \alpha c_j(f, h) + c_j(g, h)$ ;
- (ii)  $c_j(f, \alpha h) = \alpha c_j(f, h)$ .

In the following proposition, it is proved that every family of local 2-atoms is indeed a 2-frame sequence.

**Proposition 2.2.** *Suppose that  $\{f_j\}$  is a family of local 2-atoms for  $Y$ , a closed subspace of 2-inner product space  $X$ , then  $\{f_j\}$  is a 2-frame for  $Y$  associated to  $h$ .*

*Proof.* It is enough to show that  $\{f_j\}$  has a lower bound. Since  $\{f_j\}$  is a family of local 2-atoms, there exists a sequence of bilinear functionals  $\{c_j\}$  such that  $\sum_j |c_j(f, h)|^2 \leq C \|f, h\|^2, f \in Y$ , for some  $C > 0$ .

$$\begin{aligned} \|f, h\|^4 &= (\langle f, f|h \rangle)^2 \\ &= (\langle f, \sum_j c_j(f, h) f_j|h \rangle)^2 \\ &= (\sum_j \overline{c_j(f, h)} \langle f, f_j|h \rangle)^2 \\ &\leq \sum_j |c_j(f, h)|^2 \sum_j |\langle f, f_j|h \rangle|^2 \\ &\leq C \|f, h\|^2 \sum_j |\langle f, f_j|h \rangle|^2, \end{aligned}$$

it means that  $\frac{1}{C} \|f, h\|^2 \leq \sum_j |\langle f, f_j|h \rangle|^2$ . □

Assume that  $(X, \langle \cdot, \cdot \rangle, \cdot)$  is a 2-Hilbert space and  $h \in X$ . The algebraic complement of  $\langle h \rangle$  in  $X$  is denoted by  $M_h$ , i.e.  $\langle h \rangle \oplus M_h = X$ .

One may see that

$$\langle f, g \rangle_h = \langle f, g|h \rangle, \quad f, g \in X.$$

defines a semi-inner product on  $X$  (see [1]). This semi-inner product induces the following inner product on the quotient space  $\frac{X}{\langle h \rangle}$  denoted by  $M_h$  as follows:

$$\langle f + \langle h \rangle, g + \langle h \rangle \rangle_h = \langle f, g \rangle_h, \quad f, g \in X.$$

So  $M_h$  with respect to  $\|f\|_h := \sqrt{\langle f, f \rangle_h}, f \in M_h$ , is a normed space. The completion of the inner product space  $M_h$  is denoted by  $X_h$ .

With these notations, one can rewrite (1.2) as follows:

$$A \|f\|_h^2 \leq \sum_j |\langle f, f_j \rangle_h|^2 \leq B \|f\|_h^2, \quad \forall f \in X_h.$$

Now we are going to generalize the concept of a family of local 2-atoms.

**Definition 2.3.** Let  $X$  be a 2-inner product space and fix  $h \in X$ . Let  $K_h$  be a bounded linear operator on the Hilbert space  $X_h$ . A sequence  $\{f_j\} \subseteq X$  is

called a 2-atomic system for  $K_h$  associated to  $h$  if

- i)  $\{f_j\}$  is a 2-Bessel sequence;
- ii) for any  $f \in X_h$  there exists  $a_f = \{a_j\} \in \ell^2$  such that  $K_h f = \sum_j a_j f_j$ , where  $\|a_f\|_{\ell^2} \leq C\|f, h\|_X$  and  $C$  is a positive constant.

Note that the convergence of the series  $\sum_j a_j f_j$  is in the topology of  $X$ . Also if  $\{f_j\} \subseteq X_h$  then the convergence of the series  $\sum_j a_j f_j$  is in the topology of  $X$  implies its convergence in  $X_h$ .

A characterization of a 2-atomic system corresponding to  $h \in X$  is given as follows which lead us to obtain a generalization of 2-frame.

**Theorem 2.4.** *Let  $K_h$  be a bounded linear operator on  $X_h$ . Then for a sequence  $\{f_j\} \subseteq X_h$  the following statements are equivalent:*

- (i)  $\{f_j\}$  is a 2-atomic system for  $K_h$ ;
- (ii) there exist  $A, B > 0$  such that

$$A\|K_h^* f\|_h^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h;$$

- (iii)  $\{f_j\} \subseteq X_h$  is a 2-Bessel sequence and there exists a 2-Bessel sequence  $\{g_j\}$  such that

$$K_h f = \sum_j \langle f, g_j | h \rangle f_j, f \in X_h;$$

- (iv)  $\{f_j\} \subseteq X_h$  is a 2-Bessel sequence and there exists a 2-Bessel sequence  $\{g_j\}$  such that

$$K_h^* f = \sum_j \langle f, f_j | h \rangle g_j, f \in X_h;$$

- (v)  $\{Q_h f_j\}$  is a 2-atomic system for the bounded linear operator  $Q_h K_h$ , where  $Q_h$  is an injective operator on  $X_h$ .

*Proof.*  $i \rightarrow ii)$  For every  $f \in X_h$  we have

$$\begin{aligned} \|K_h^* f\|^2 &= \|K_h^* f, h\|^2 \\ &= \sup\{|\langle K_h^* f, g | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &= \sup\{|\langle f, K_h g | h \rangle|^2 : g \in X_h, \|g, h\| = 1\}. \end{aligned}$$

By definition of a 2-atomic system for  $K_h$ , there exists  $C > 0$  such that  $K_h g = \sum_j b_j f_j$  with  $\|b_g\|_{\ell^2} = \|\{b_j\}\|_{\ell^2} \leq C\|g, h\|$  and so

$$\begin{aligned} \|K_h^* f\|^2 &= \sup\{|\langle f, \sum_j b_j f_j | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &= \sup\{|\sum_j \bar{b}_j \langle f, f_j | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &\leq \sup\{\sum_j |b_j|^2 \sum_j |\langle f, f_j | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &\leq C^2 \|g, h\|^2 \sum_j |\langle f, f_j | h \rangle|^2 \\ &= C^2 \sum_j |\langle f, f_j | h \rangle|^2. \end{aligned}$$

It means that  $\frac{1}{C^2} \|K_h^* f\|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2$ .

*ii*  $\rightarrow$  *iii*) Similar to Theorem 3 of [15], there exists a 2-Bessel sequence  $\{g_j\} \in X_h$  such that

$$K_h f = \sum_j \langle f, g_j \rangle_h f_j = \sum_j \langle f, g_j | h \rangle f_j.$$

*iii*  $\rightarrow$  *iv*) For  $f, g \in X_h$  we have

$$\begin{aligned} \langle K_h f, g \rangle_h &= \langle \sum_j \langle f, g_j | h \rangle f_j, g \rangle_h \\ &= \sum_j \langle f, g_j | h \rangle \langle f_j, g | h \rangle \\ &= \sum_j \langle f, g_j \rangle_h \langle f_j, g \rangle_h \\ &= \langle f, \sum_j \langle g, f_j | h \rangle g_j \rangle_h, \end{aligned}$$

that is  $K_h^* f = \sum_j \langle f, f_j | h \rangle g_j$ .

*iv*  $\rightarrow$  *iii*) It is similar to *iii*  $\rightarrow$  *iv* so we omit it.

*i*  $\rightarrow$  *v*) Since  $\{f_j\}$  is a 2-atomic system for  $K_h$ , for any  $f \in X_h$  there exists  $a_f = \{a_j\} \in \ell^2$  such that  $K_h f = \sum_j a_j f_j$  so  $Q_h K_h f = \sum_j a_j Q_h f_j$ , i.e.  $\{Q_h f_j\}$  is a 2-atomic system for  $Q_h K_h$ .

*v*  $\rightarrow$  *i*) Since  $\{Q_h f_j\}$  is a 2-atomic system for  $Q_h K_h$ , for any  $f \in X_h$  there exists  $\{b_j\} \in \ell^2$  such that  $Q_h K_h f = \sum_j b_j Q_h f_j$  so  $Q_h(K_h f - \sum_j b_j f_j) = 0$ . Due to injectivity of  $Q_h$ ,  $K_h f = \sum_j b_j f_j$ .  $\square$

As a result of Theorem 2.4 the following definition is given.

**Definition 2.5.** Let  $K_h$  be a bounded linear operator on  $X_h$ . A sequence  $\{f_j\}$  in  $X$  is called 2- $K$ -frame if there exist  $A, B >$  such that

$$A\|K_h^*f\|_h^2 \leq \sum_j |\langle f, f_j|_h \rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h.$$

Trivially a 2-frame, which was defined in [1], is a special case of 2- $K$ -frames with  $K_h = I$ .

A consequence of Theorem 2.4 is given as follows.

**Theorem 2.6.** Let  $P_{Y_h}$  be the orthogonal projection on  $Y_h$  as a closed subspace of  $X_h$ . Then for a sequence  $\{f_j\} \subseteq X_h$  the following statements are equivalent:

- (i)  $\{f_j\}$  is a family of local 2-atoms for  $Y_h$ ;
- (ii)  $\{f_j\}$  is a 2-atomic system for  $P_{Y_h}$ ;
- (iii)  $\{f_j\}$  is a 2- $P_{Y_h}$ -frame;
- (iv)  $\{f_j\}$  is a 2-Bessel sequence and there exists a 2-Bessel sequence  $\{g_j\}$  such that

$$P_{Y_h}f = \sum_j \langle f, g_j|_h \rangle f_j = \sum_j \langle f, f_j|_h \rangle g_j, f \in X_h;$$

- (v)  $\{Q_h f_j\}$  is a 2-atomic system for bounded linear operator  $Q_h P_{Y_h}$ , where  $Q_h$  is an injective operator on  $X_h$ .

*Proof.*  $i \rightarrow ii$  is obvious.

$ii \leftrightarrow iii$ ,  $iii \leftrightarrow iv$ ,  $iv \leftrightarrow ii$  and  $v \leftrightarrow i$  hold from Theorem 2.4.

$iv \rightarrow i$ ) Since  $P_{Y_h}f = \sum_j \langle f, g_j|_h \rangle f_j$ , it is enough to put  $c_j(f, h) = \langle f, g_j|_h \rangle$  because it is linear and

$$\sum_j |c_j(f, h)|^2 = \sum_j |\langle f, g_j|_h \rangle|^2 \leq D\|f, h\|^2,$$

where  $D$  is the upper 2-frame bound of  $\{g_j\}$ . □

**EXAMPLE 2.7.** Let  $n \in \mathbb{N}$  be odd and consider  $X = \mathbb{R}^n$  with the following standard two inner product

$$\langle x, y|z \rangle = \det \begin{pmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{pmatrix}$$

where  $\langle \cdot, \cdot \rangle$  is the inner product of  $\mathbb{R}^n$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbb{R}^n$  and  $h = e_n$ . Trivially in this case  $X_h = \mathbb{R}^{n-1}$  and one can see that its induced inner product is the standard inner product of  $\mathbb{R}^{n-1}$ . Now define the operator  $K_h$  on  $X_h$  by

$$K_h(e_{2i}) = e_i, i = 1, 2, \dots, \frac{n-1}{2} \text{ and otherwise } K_h e_i = e_i, i \leq n-1.$$

Then one can see that  $e_1, e_1, e_3, e_2, \dots, e_{\frac{n-1}{2}}, e_{n-1}$  is a 2- $K_h$ -frame.

## ACKNOWLEDGMENTS

The authors wish to thank the anonymous reviewers for their specific and useful comments.

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