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Atomic Systems in 2-inner Product Spaces

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Abstract. In this paper, the concept of a family of local atoms in a 2-inner product space is introduced and then this concept is generalized to an atomic system for an operator. Next a characterization of atomic systems is proved. This characterization lead us to obtain a new frame which is a generalization of frames in 2-inner product spaces.

Keywords: 2-inner product space, 2-normed space, Family of local atoms, Atomic system, Frame.

2000 Mathematics subject classification: 42C15, 46C50.

1. Introduction and Preliminaries

Frames in Hilbert spaces were introduced by Duffin and Schaffer [9] in the context of nonharmonic Fourier series in 1952. In 1986, frames were brought to life by Daubechies *et al.* [7]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [13], sampling [10, 11], coding and communications [19], filter bank theory [2], system modeling [8], and so on.

Atomic systems for bounded linear operators on Hilbert spaces have been introduced by L. Găvruţa in [15] as a generalization of families of local atoms

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[12]. A sequence $\{f_j\}_{j\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called an *atomic system* for a bounded linear operator K on $\mathcal{H}$ if
$i)$ the series $\sum_{j\in\mathbb{N}} c_j f_j$ converges for all $c = (c_j) \in l^2 := \{\{b_j\}_{j\in\mathbb{N}} : \sum_{j\in\mathbb{N}} |b_j|^2 < \infty\}$;
$ii)$ there exists $C > 0$ such that for every $f \in \mathcal{H}$ there exists $a_f = (a_j) \in l^2$ such that $\|a_f\|_{l^2} \leq C\|f\|$ and $Kf = \sum_{j\in\mathbb{N}} a_j f_j$.
It is proved that this concept is equivalent to K-frames, where K is a bounded linear operator on separable Hilbert space $\mathcal{H}$ [15].
A sequence $\{f_j\}_{j\in\mathbb{N}}$ is said to be a K-*frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{j\in\mathbb{N}} |\langle f, f_j\rangle|^2 \leq B\|f\|^2, \ \forall f \in \mathcal{H}.$$

We refer to [20] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert modules [5] and Banach spaces [6]. Note that frames in Hilbert spaces are just a particular case of K-frames, when K is the identity operator on these Hilbert spaces.

The concepts of 2-inner product spaces and 2-normed spaces have been studied by many authors [3, 4, 14, 16, 17, 18]. In the sequel, we introduce 2-inner product and 2-normed spaces.

Definition 1.1. Suppose that X is a vector space of dimension grater than 1 over the field $\mathbb{F}$ (*either* $\mathbb{R}$ *or* $\mathbb{C}$). If there exists a mapping $\langle ., .|.\rangle : X\times X\times X \to \mathbb{F}$ with the properties

1. $\langle f, f|h\rangle \geq 0$ and $\langle f, f|h\rangle = 0$ if and only if f and h are linearly dependent;
2. $\langle f, f|h\rangle = \langle h, h|f\rangle$;
3. $\langle g, f|h\rangle = \overline{\langle f, g|h\rangle}$;
4. $\langle \alpha f, g|h\rangle = \alpha\langle f, g|h\rangle$ for $\alpha \in \mathbb{F}$;
5. $\langle f_1 + f_2, g|h\rangle = \langle f_1, g|h\rangle + \langle f_2, g|h\rangle$,

then the pair $(X, \langle ., .|.\rangle)$ is called a 2-inner product space. The map $\langle ., .|.\rangle$ is said to be a 2-inner product on X.

Some basic properties of 2-inner product $\langle ., .|.\rangle$ cane be immediately obtained as follows (see [3, 4]).

- $\langle 0, g|h\rangle = \langle f, 0|h\rangle = \langle f, g|0\rangle = 0$;
- $\langle f, \alpha g|h\rangle = \overline{\alpha}\langle f, g|h\rangle$;
- $\langle f, g|\alpha h\rangle = |\alpha|^2\langle f, g|h\rangle$;

for all $f, g, h \in X$ and $\alpha \in \mathbb{F}$.
One of the most important properties of 2-inner product is the Cauchy-Schwarz inequality

$$|\langle f, g|h\rangle|^2 \leq \langle f, f|h\rangle\langle g, g|h\rangle, \quad f, g, h \in X.$$

For a given 2-inner product space $(X, \langle ., .|.\rangle)$ we can define a function $\|., .\|$ on $X \times X$ by

$$\|f, h\| = \langle f, f|h\rangle^{\frac{1}{2}} \tag{1.1}$$

for all $f, h \in X$.

The above mentioned function satisfies the following conditions:

a. $\|f, h\| \geq 0$ and $\|f, h\| = 0$ if and only if f and h are linearly dependent;
b. $\|f, h\| = \|h, f\|$;
c. $\|\alpha f, h\| = |\alpha|\|f, h\|, \alpha \in \mathbb{F}$;
d. $\|f_1 + f_2, h\| \leq \|f_1, h\| + \|f_2, h\|$.

A 2-norm on a vector space X is a function $\|., .\|$ defined on $X \times X$ satisfying the conditions (a) to (d) and $(X, \|., .\|)$ is called a linear 2-normed space. Whenever a 2-inner product space $(X, \langle ., .|.\rangle)$ is given, we consider it as a linear 2-normed space $(X, \|., .\|)$ via the 2-norm defined by (1.1).

Let X be a 2-inner product space. A sequence $\{f_j\}$ is called convergent if there exists $f \in X$ such that $\lim_{j\to\infty} \|f_j - f, h\| = 0$, for all $h \in X$. Similarly, we can define a Cauchy sequence in X. Also, X is said to be a 2-Hilbert space if it is complete (see [18]).

Now we are ready to state the concept of a 2-frame which was introduced in [1]. A sequence $\{f_j\}$ in a 2-Hilbert space $(X, \langle ., .|.\rangle)$ is called a 2-frame associated to $h \in X$ if there exist $A, B > 0$ such that

$$A\|f, h\|^2 \leq \sum_j |\langle f, f_j|h\rangle|^2 \leq B\|f, h\|^2, \forall f \in X. \tag{1.2}$$

If the right side of (1.2) holds, then $\{f_j\}$ is called a 2-Bessel sequence.

In this paper, we shall introduce 2-atomic systems as a generalization of families of local 2-atoms. A characterization of 2-atomic systems is given. This leads us to obtain a generalization of 2-frame.

2. Main Results

In this section we are going to define the concept of a family of local 2-atoms. Next we will generalize this concept to a 2-atomic system for a linear operator and then a generalization of 2-frames will be studied.

In the sequel we assumed that $(X, \langle ., .|.\rangle)$ is a 2-Hilbert space, $h \in X$ and $\langle h\rangle$ is the subspace generated by h.

Definition 2.1. Let $\{f_j\}$ be a 2-Bessel sequence in a 2-inner product space X, $h \in X$ and Y be a closed subspace of X. We say that $\{f_j\}$ is a family of local 2-atoms for Y associated to h if there exists a sequence of bilinear functionals $\{c_j\}$ on $X \times \langle h\rangle$ such that
i) $\sum_j |c_j(f, h)|^2 \leq C\|f, h\|^2$, for some $C > 0$;
ii) $f = \sum_j c_j(f, h) f_j$,
for all $f \in Y$.

Note that a map $c_j : X \times \langle h \rangle \to \mathbb{F}$ is called a bilinear functional if the following conditions hold for every $f, g \in X$ and $\alpha \in \mathbb{F}$.
(i) $c_j(\alpha f + g, h) = \alpha c_j(f, h) + c_j(g, h)$;
(ii) $c_j(f, \alpha h) = \alpha c_j(f, h)$.

In the following proposition, it is proved that every family of local 2-atoms is indeed a 2-frame sequence.

Proposition 2.2. *Suppose that $\{f_j\}$ is a family of local 2-atoms for Y, a closed subspace of 2-inner product space X, then $\{f_j\}$ is a 2-frame for Y associated to h.*

Proof. It is enough to show that $\{f_j\}$ has a lower bound. Since $\{f_j\}$ is a family of local 2-atoms, there exists a sequence of bilinear functionals $\{c_j\}$ such that $\sum_j |c_j(f, h)|^2 \leq C\|f, h\|^2, f \in Y$, for some $C > 0$.

$$\begin{aligned}
\|f, h\|^4 &= (\langle f, f|h\rangle)^2 \\
&= (\langle f, \sum_j c_j(f, h) f_j|h\rangle)^2 \\
&= (\sum_j \overline{c_j(f, h)} \langle f, f_j|h\rangle)^2 \\
&\leq \sum_j |c_j(f, h)|^2 \sum_j |\langle f, f_j|h\rangle|^2 \\
&\leq C\|f, h\|^2 \sum_j |\langle f, f_j|h\rangle|^2,
\end{aligned}$$

it means that $\frac{1}{C}\|f, h\|^2 \leq \sum_j |\langle f, f_j|h\rangle|^2$. $\square$

Assume that $(X, \langle ., .; .\rangle)$ is a 2-Hilbert space and $h \in X$. The algebraic complement of $\langle h \rangle$ in X is denoted by M_h, i.e. $\langle h \rangle \oplus M_h = X$.

One may see that

$$\langle f, g\rangle_h = \langle f, g|h\rangle, \ f, g \in X.$$

defines a semi-inner product on X (see [1]). This semi-inner product induces the following inner product on the quotient space $\frac{X}{\langle h \rangle}$ denoted by M_h as follows:

$$\langle f + \langle h \rangle, g + \langle h \rangle\rangle_h = \langle f, g\rangle_h, \ f, g \in X.$$

So M_h with respect to $\|f\|_h := \sqrt{\langle f, f\rangle_h}, f \in M_h$, is a normed space. The completion of the inner product space M_h is denoted by X_h.
With these notations, one can rewrite (1.2) as follows:

$$A\|f\|_h^2 \leq \sum_j |\langle f, f_j\rangle_h|^2 \leq B\|f\|_h^2, \ \forall f \in X_h.$$

Now we are going to generalize the concept of a family of local 2-atoms.

Definition 2.3. Let X be a 2-inner product space and fix $h \in X$. Let K_h be a bounded linear operator on the Hilbert space X_h. A sequence $\{f_j\} \subseteq X$ is

called a 2-atomic system for K_h associated to h if
$i)$ $\{f_j\}$ is a 2-Bessel sequence;
$ii)$ for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$, where $\|a_f\|_{\ell^2} \leq C\|f, h\|_X$ and C is a positive constant.

Note that the convergence of the series $\sum_j a_j f_j$ is in the topology of X. Also if $\{f_j\} \subseteq X_h$ then the convergence of the series $\sum_j a_j f_j$ is in the topology of X implies its convergence in X_h.

A characterization of a 2-atomic system corresponding to $h \in X$ is given as follows which lead us to obtain a generalization of 2-frame.

Theorem 2.4. *Let K_h be a bounded linear operator on X_h. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:*

- (i) *$\{f_j\}$ is a 2-atomic system for K_h;*
- (ii) *there exist $A, B > 0$ such that*

$$A\|K_h^* f\|_h^2 \leq \sum_j |\langle f, f_j|h\rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h;$$

- (iii) *$\{f_j\} \subseteq X_h$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that*

$$K_h f = \sum_j \langle f, g_j|h\rangle f_j, f \in X_h;$$

- (iv) *$\{f_j\} \subseteq X_h$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that*

$$K_h^* f = \sum_j \langle f, f_j|h\rangle g_j, f \in X_h;$$

- (v) *$\{Q_h f_j\}$ is a 2-atomic system for the bounded linear operator $Q_h K_h$, where Q_h is an injective operator on X_h.*

Proof. $i \to ii)$ For every $f \in X_h$ we have

$$\begin{aligned} \|K_h^* f\|^2 &= \|K_h^* f, h\|^2 \\ &= \sup\{|\langle K_h^* f, g|h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\ &= \sup\{|\langle f, K_h g|h\rangle|^2 : g \in X_h, \|g, h\| = 1\}. \end{aligned}$$

By definition of a 2-atomic system for K_h, there exists $C > 0$ such that $K_h g = \sum_j b_j f_j$ with $\|b_g\|_{\ell^2} = \|\{b_j\}\|_{\ell^2} \leq C\|g, h\|$ and so

$$\begin{aligned}
\|K_h^* f\|^2 &= \sup\{|\langle f, \sum_j b_j f_j | h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\
&= \sup\{|\sum_j \overline{b_j}\langle f, f_j | h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\
&\leq \sup\{\sum_j |b_j|^2 \sum_j |\langle f, f_j | h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\
&\leq C^2\|g, h\|^2 \sum_j |\langle f, f_j | h\rangle|^2 \\
&= C^2 \sum_j |\langle f, f_j | h\rangle|^2.
\end{aligned}$$

It means that $\frac{1}{C^2}\|K_h^* f\|^2 \leq \sum_j |\langle f, f_j | h\rangle|^2$.

$ii \to iii$) Similar to Theorem 3 of [15], there exists a 2-Bessel sequence $\{g_j\} \in X_h$ such that

$$K_h f = \sum_j \langle f, g_j\rangle_h f_j = \sum_j \langle f, g_j | h\rangle f_j.$$

$iii \to iv$) For $f, g \in X_h$ we have

$$\begin{aligned}
\langle K_h f, g\rangle_h &= \langle \sum_j \langle f, g_j | h\rangle f_j, g\rangle_h \\
&= \sum_j \langle f, g_j | h\rangle \langle f_j, g | h\rangle \\
&= \sum_j \langle f, g_j\rangle_h \langle f_j, g\rangle_h \\
&= \langle f, \sum_j \langle g, f_j | h\rangle g_j\rangle_h,
\end{aligned}$$

that is $K_h^* f = \sum_j \langle f, f_j | h\rangle g_j$.

$iv \to iii$) It is similar to $iii \to iv$ so we omit it.

$i \to v$) Since $\{f_j\}$ is a 2-atomic system for K_h, for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$ so $Q_h K_h f = \sum_j a_j Q_h f_j$, i.e. $\{Q_h f_j\}$ is a 2-atomic system for $Q_h K_h$.

$v \to i$) Since $\{Q_h f_j\}$ is a 2-atomic system for $Q_h K_h$, for any $f \in X_h$ there exists $\{b_j\} \in \ell^2$ such that $Q_h K_h f = \sum_j b_j Q_h f_j$ so $Q_h(K_h f - \sum_j b_j f_j) = 0$. Due to injectivity of Q_h, $K_h f = \sum_j b_j f_j$. $\square$

As a result of Theorem 2.4 the following definition is given.

Definition 2.5. Let K_h be a bounded linear operator on X_h. A sequence $\{f_j\}$ in X is called 2-K-frame if there exist $A, B >$ such that

$$A\|K_h^* f\|_h^2 \leq \sum_j |\langle f, f_j|h\rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h.$$

Trivially a 2-frame, which was defined in [1], is a special case of 2-K-frames with $K_h = I$.
A consequence of Theorem 2.4 is given as follows.

Theorem 2.6. *Let P_{Y_h} be the orthogonal projection on Y_h as a closed subspace of X_h. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:*

- (i) *$\{f_j\}$ is a family of local 2-atoms for Y_h;*
- (ii) *$\{f_j\}$ is a 2-atomic system for P_{Y_h};*
- (iii) *$\{f_j\}$ is a 2-P_{Y_h}-frame;*
- (iv) *$\{f_j\}$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that*

$$P_{Y_h} f = \sum_j \langle f, g_j|h\rangle f_j = \sum_j \langle f, f_j|h\rangle g_j, f \in X_h;$$

- (v) *$\{Q_h f_j\}$ is a 2-atomic system for bounded linear operator $Q_h P_{Y_h}$, where Q_h is an injective operator on X_h.*

Proof. $i \to ii$ is obvious.
$ii \longleftrightarrow iii$, $iii \longleftrightarrow iv$, $iv \longleftrightarrow ii$ and $v \longleftrightarrow i$ hold from Theorem 2.4.
$iv \to i$) Since $P_{Y_h} f = \sum_j \langle f, g_j|h\rangle f_j$, it is enough to put $c_j(f,h) = \langle f, g_j|h\rangle$ because it is linear and

$$\sum_j |c_j(f,h)|^2 = \sum_j |\langle f, g_j|h\rangle|^2 \leq D\|f,h\|^2,$$

where D is the upper 2-frame bound of $\{g_j\}$. □

Example 2.7. Let $n \in \mathbb{N}$ be odd and consider $X = \mathbb{R}^n$ with the following standard two inner product

$$\langle x, y|z\rangle = \det \begin{pmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{pmatrix}$$

where $\langle ., .\rangle$ is the inner product of $\mathbb{R}^n$. Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$ and $h = e_n$. Trivially in this case $X_h = \mathbb{R}^{n-1}$ and one can see that its induced inner product is the standard inner product of $\mathbb{R}^{n-1}$. Now define the operator K_h on X_h by

$$K_h(e_{2i}) = e_i, i = 1, 2, ..., \frac{n-1}{2} \text{ and otherwise } K_h e_i = e_i, i \leq n-1.$$

Then one can see that $e_1, e_1, e_3, e_2, ..., e_{\frac{n-1}{2}}, e_{n-1}$ is a 2-K_h-frame.

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