Serre Subcategories and Local Cohomology Modules with Respect to a Pair of Ideals

Fatemeh Dehghani-Zadeh
Department of Mathematics, Islamic Azad University, Yazd Branch, Yazd, Iran.
E-mail: fdzadeh@gmail.com, dehghanizadeh@iauyazd.ac.ir

Abstract. This paper is concerned with the relation between local cohomology modules defined by a pair of ideals and the Serre subcategories of the category of modules. We characterize the membership of local cohomology modules in a certain Serre subcategory from lower range or upper range.

Keywords: Local cohomology modules, pair of ideals, Serre subcategory.


1. Introduction

Throughout this paper, $R$ is denoted a commutative Noetherian ring, $I$ and $J$ are denoted two ideals of $R$, and $M$ is an $R$-module. We refer the reader to [2] and [4] for any unexplained terminology.

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa [16] introduced the local cohomology modules with respect to a pair of ideals $(I, J)$. To be more precise, let $W(I, J) = \{ p \in \text{Spec}(R) \mid I^n \subseteq p + J \text{ for some positive integer } n \}$. Then for an $R$-module $M$, the $(I, J)$-torsion submodule $\Gamma_{I,J}(M)$ of $M$, which consists of all elements $x$ of $M$ with $\text{Supp}Rx \subseteq W(I, J)$, is considered. It is known, $\Gamma_{I,J}$ is a left exact additive functor from the category of all $R$-modules and $R$-homomorphism to itself. For all integer $i$, the $i$-th local cohomology functor $H^i_{I,J}$ with respect
to \((I, J)\) is defined to be the \(i\)-th right derived functor of \(\Gamma_{I,J}\). The \(i\)-th local cohomology module of \(M\) with respect to \((I, J)\) is denoted by \(H^i_{I,J}(M)\). When \(J = 0\), then \(H^i_{I,J}\) coincides with the usual local cohomology functor \(H^i_I\) with the support in the closed subset \(V(I)\).

The study of this generalized local cohomology modules was continued by many authors (see for example [5], [6], [9] and [10], [14]).

Recall that a class of \(R\)-modules is a Serre subcategory of the category of \(R\)-modules when it is closed under taking submodules, quotients and extensions. Always, \(S\) stands for a Serre subcategory of the category of \(R\)-modules.

Using the generalized local cohomology modules, we can define \(T^s_{I,J}(M)\) (resp. \(T^{I,J}_s(M)\)) of the \(R\)-module \(M\) relative to a pair \((I, J)\) of ideals of \(R\) by

\[
T^s_{I,J}(M) = \inf \{i \in \mathbb{N} \mid H^i_{I,J}(M) \text{ is not in } S\},
\]

(resp. \(T^{I,J}_s(M) = \sup \{i \in \mathbb{N} \mid H^i_{I,J}(M) \text{ is not in } S\}\))

with the usual convention that the infimum (resp. supremum) of the empty set of integers interpreted as \(+\infty\) (resp. \(-\infty\)).

Our objective in this paper is to investigate the notions \(T^s_{I,J}(M)\) and \(T^{I,J}_s(M)\). we prove the following,

**Theorem 1.1.** Let \(S\) be a Melkersson subcategory with respect to \((I, J)\). Suppose \(M\) is Weakly Laskerian module. Then \(T^s_{I,J}(M) = \inf \{T^s_a(M) \mid a \in \tilde{W}(I, J)\}\), where \(T^s_a(M)\) is the least non-negative integer \(i\) such that \(H^i_a(M)\) is not in \(S\).

**Theorem 1.2.** Let \(S\) be a Serre subcategory and let \(M\) be a Weakly Laskerian module. Then \(T^{I,J}_s(M) = \sup \{T^{I,J}_s(R/p) \mid p \in \text{Supp}(M)\}\).

One can see that the subcategories of finitely generated \(R\)-modules, minimax \(R\)-modules, minimax and \((I, J)\)-cofinite \(R\)-modules, weakly Laskerian \(R\)-modules, and Matlis reflexive \(R\)-modules are examples of Serre subcategory. So, this paper recovers some results regarding the local cohomology \(R\)-modules that have appeared in different papers (see for instance [3], [5] and [13]).

2. The Results

This section is started with the following definition.

**Definition 2.1.** (see [1, Definition 3.1]) A Serre subcategory of the category of \(R\)-modules is said to be a Melkersson subcategory with respect to the ideal \(a\), if for any \(a\)-torsion \(R\)-module \(X\), \((0 :_X a)\) is in \(S\) implies that \(X\) is in \(S\). Examples are given by the class of Artinian modules, minimax and \(a\)-cofinite modules.

Also, we say that \(S\) is a Melkersson subcategory with respect to the pair of ideals \((I, J)\), if for an \((I, J)\)-torsion \(R\)-module \(X\), \((0 :_X I)\) is in \(S\) implies that \(X\) is in \(S\). Obviously, if \(S\) is a Melkersson subcategory with respect to \((I, J)\),
then $\mathcal{S}$ is a Melkersson subcategory with respect to all ideals of $\mathcal{W}(I, J)$, where $\mathcal{W}(I, J)$ denote the set of ideals $\mathfrak{a}$ of $R$ such that $I^n \subseteq \mathfrak{a} + J$ for some integer $n$.

**Proposition 2.2.** Let $\mathcal{S}$ be a Melkersson subcategory with respect to the ideal $I$ and $t$ an integer. Let $T$ be an $R$-module such that $\text{Ext}^i_R(R/I, T)$ is in $\mathcal{S}$ for all $i < t$. Then $H^i_T(T)$ is in $\mathcal{S}$ for all $i < t$. Particularly, for an $R$-module $M$, the module $H^i_T(H^j_{I, J}(M))$ is in $\mathcal{S}$ for all $i$ and $j < T^s_{I, J}(M)$.

**Proof.** We prove the theorem by induction on $i$. It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It easily follows from the exact sequence

$$0 \rightarrow \Gamma_i(T) \rightarrow T \rightarrow T/\Gamma_i(T) \rightarrow 0,$$

that $\text{Ext}^i_R(R/I, T)$ is in $\mathcal{S}$ if and only if $\text{Ext}^i_R(R/I, T/\Gamma_i(T))$ is in $\mathcal{S}$. Also, by [2, Corollary 2.1.7], $H^i_T(M) \cong H^i_T(M/\Gamma_i(M))$ for all $i > 0$. Therefore we assume that $\Gamma_i(T) = 0$. Now, we apply Melkerson’s technic [12], so let $E$ be an injective envelope of $T$. Then $\Gamma_i(E) = \text{Hom}(R/I, E) = 0$. Put $L = E/T$ and consider the exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow L \rightarrow 0.$$

We obtain isomorphisms; $H^i_T(T) \cong H^{i-1}_T(L)$ and $\text{Ext}^i_R(R/I, T) \cong \text{Ext}^{i-1}_R(R/I, L)$ for all $i > 0$. Use the induction hypothesis applied to $L$, and conclude that the $H^i_T(T)$ is in $\mathcal{S}$ for all $i < t$. It therefore follows, in view of the definition of $T^s_{I, J}(M)$, that $H^i_T(H^j_{I, J}(M))$ is in $\mathcal{S}$ for all $j < T^s_{I, J}(M)$. 

**Example 2.3.** In Theorem 2.2, the assumption $\mathcal{S}$ is Melkersson subcategory is necessary. To see this, let $(R, m)$ be a local ring, and let $M$ be a non-zero, finitely generated $R$-module of dimension $n > 0$. Then $H^n_{\mathfrak{m}}(M)$ is not finitely generated (see [2, Corollary 7.3.3]).

The next theorem recovers the Theorem 2.5 of [3] and Theorem 2.2 of [17].

**Theorem 2.4.** Let $M$ be in $\mathcal{S}$ and $j \leq T^s_{I, J}(M) = t$. Then

(i) $\text{Ext}^i_R(R/I, H^j_{I, J}(M))$ is in $\mathcal{S}$ for all $i = 0, 1$.

(ii) $H^i_T(H^j_{I, J}(M))$ is in $\mathcal{S}$ for all $i = 0, 1$, if $\mathcal{S}$ is a Melkersson subcategory with respect to $I$.

**Proof.** (i) Consider the functors $F(-) = \text{Hom}_R(R/I, - )$ and $G(-) = \Gamma_{I, J}(- )$. Then one has $FG(-) = \text{Hom}(R/I, - )$. So, by [14, Theorem 11.38], there is a Grothendieck's spectral sequence

$$E^2_{i, j} = \text{Ext}^i_R(R/I, H^j_{I, J}(M)) \Rightarrow \text{Ext}^{i+j}_R(R/I, M).$$

By using an argument similar to the proof of [8, Theorem 2.2], we obtain that $\text{Ext}^i_R(R/I, H^j_{I, J}(M))$ is in $\mathcal{S}$ for all $i = 0, 1$. This completes the proof.
(ii) Using [15, Theorem 11.38] there exists a Grothendieck’s spectral sequence
\[ E_2^{p,q} = H_1^p(H_{1,j}^q(M)) \Rightarrow H_r^{p+q}(M). \]
Also, there is a bound filtration 0 = \( \varphi^{t+1}H^t \subseteq \varphi^tH^t \subseteq \cdots \subseteq \varphi^0H^t = H_1^j(M) \) such that \( E_i^{t-i} \cong \frac{\varphi^iH^t}{\varphi^{i+1}H^t} \) for all \( 0 \leq i \leq t \). By the hypotheses with proposition 2.2 \( H_1^j(M) \) is in \( S \) for all \( i \) and hence \( E_\infty^{p,q} \) is in \( S \) for all \( p,q \). Note that \( E_\infty^{p,q} = E_r^{p,q} \) for large \( r \) and each \( p,q \). It follows that there is an integer \( \ell \geq 2 \) such that \( E_\ell^{p,q} \) is in \( S \) for all \( r \geq \ell \). We now argue by descending induction on \( \ell \). Now, assume that \( 2 < \ell < r \) and that the claim holds for \( \ell \). Since \( E_\ell^{p,q} \) is in a subquotient of \( E_2^{p,q} \) for all \( p,q \in \mathbb{N}_0 \), the hypotheses give \( E_\ell^{p+r,t-r+1} \) is in \( S \) for all \( r \geq 2 \). In addition, \( E_\ell^{p,t} = \frac{\ker d^{p,t}_\ell}{\text{im} d^{p,t-1}_\ell} \) and \( \text{im} d^{p-\ell+1,t+\ell-2}_\ell = 0 \) for \( p = 0,1 \), it follows that \( \ker d^{p,t}_{\ell-1} \) is in \( S \) for all \( \ell > 2 \) and \( p = 0,1 \). Let \( r \geq 2 \) and \( p = 0,1 \), we consider the sequence
\[ 0 \longrightarrow \ker d^{p,t}_{\ell-1} \longrightarrow E_\ell^{p,t} \longrightarrow E_\ell^{p+r,t-r+1}. \]
Since both \( \ker d^{p,t}_{\ell-1} \) and \( E_\ell^{p+r,t-r+1} \) are in \( S \), it follows that \( E_\ell^{p,t} \) is in \( S \) for \( p = 0,1 \). This completes the inductive step. \( \square \)

**Proposition 2.5.** Let \( M \) be in \( S \) such that \( \text{dim}(M/JM) \leq 1 \). Then \( \text{Ext}_R^j(R/I, H_{1,j}^1(M)) \) is in \( S \).

**Proof.** Consider the following spectral sequence
\[ E_2^{p,q} := \text{Ext}_R^p(R/I, H_{1,j}^q(M)) \Rightarrow \text{Ext}_R^{p+q}(R/I, M) = H_r^{p+q}. \]
In view of [16, Theorem 4.3] \( E_2^{p,q} = 0 \) unless \( q = 0,1 \). It follows that the exact sequence
\[ H^{p+1} \longrightarrow E_2^{p+1,0} \longrightarrow E_2^{p-1,1} \longrightarrow H^p \longrightarrow E_2^{p,0} \longrightarrow E_2^{p-2,1} \longrightarrow H^{p-1} \]
which in turn yields the exact sequence
\[ \text{Ext}_R^{p+1}(R/I, M) \longrightarrow \text{Ext}_R^{p+1}(R/I, \Gamma_{1,j}(M)) \longrightarrow \text{Ext}_R^{p-1}(R/I, H_{1,j}^1(M)) \longrightarrow \text{Ext}_R^p(R/I, \Gamma_{1,j}(M)) \longrightarrow \text{Ext}_R^p(R/I, H_{1,j}^1(M)). \]
Since, by our assumption, the \( R \)-modules \( \text{Ext}_R^i(R/I, \Gamma_{1,j}(M)) \) and \( \text{Ext}_R^i(R/I, M) \) are in \( S \) for all \( i \) and hence the result follows. \( \square \)

**Corollary 2.6.** Let \( S \) be a Melkersson subcategory of the category of \( R \)-modules. Let \( M \) be in \( S \) and \( \text{dim}(M/JM) \leq 1 \). Then \( H_{1,j}^1(H_{1,j}^1(M)) \) is in \( S \) for all \( i \) and \( j \).

**Proof.** The result follows from the part (ii) of Theorem (2.4) and argument similar to the proof of Proposition 2.5.

Recall that an \( R \)-module \( M \) is weakly Laskerian if any quotient of \( M \) has a finitely many associated prime ideals. This holds, by employing a method of proof which is similar to that used in [2, Lemma 2.1.1], \( M \) is a \( \alpha \)-torsion-free if
and only if $a$ contains a non-zerodivisor on $M$. Clearly, any finitely generated module and any minimax module are weakly Laskerian modules.

In addition, by using an argument similar to the proof of [11, Theorem 6.4], there exists a chain $0 = M_0 \subset M_1 \cdots \subset M_n = M$ of submodules of $M$ such that for each $i$ we have $M_i/M_{i-1} = p_i$ with $p_i \in \text{Supp}(M)$.

**Theorem 2.7.** Let $a \in \widetilde{W}(I,J)$ and $M$ be a Weakly Laskerian module. Then $\text{Ext}_R^i(R/a,M)$ is in $S$ for all $0 \leq i < T^a_{I,J}(M)$.

**Proof.** It follows by using induction on $T^a_{I,J}(M)$. □

**Theorem 2.8.** Let $a \in \widetilde{W}(I,J)$ and $S$ be a Melkersson subcategory with respect to the ideal $a$. Suppose $M$ is Weakly Laskerian module. Then $H^a_i(M) \in S$ for all $i < T^a_{I,J}(M)$.

**Proof.** By using the induction on $t = T^a_{I,J}(M)$, the theorem is proved. It is straightforward to see that the result is true when $t = 1$. Suppose that $t > 1$, and the result holds for the case $t - 1$. Since $H^a_{I,J}(M) \cong H^a_{I,J}(M/\Gamma_{I,J}(M))$ for all $i > 0$, we may replace $M$ by $M/\Gamma_{I,J}(M)$ and hence assume that there is an element $x \in a$, such that $x$ is a non-zero divisor on $M$. The exact sequence $0 \to M \to M/xM \to 0$ induces two exact sequences

\[ \begin{align*}
H^a_{I,J}(M/xM) & \to H^a_{I,J}(M) \to H^a_{I,J}(M/xM) \\
H^a_{I,J}(M) & \to H^a_{I,J}(M/xM)
\end{align*} \]

of local cohomology modules. The induction hypothesis and the above sequences yield that the $R$-modules $H^a_i(M)$ and $H^a_i(M/xM)$ are in $S$ for all $i < t - 1$. It suffices to show that $H^{a-1}_i(M)$ is in $S$. Now, the exactness of $(\ast)$, in conjunction with the fact $(0 : H^{a-1}_i(M), a) \subseteq (0 : H^{a-1}_i(M), x)$ and our hypotheses, show that $H^{a-1}_i(M)$ is in $S$, this proves our claim. □

**Theorem 2.9.** Let $S$ be a Melkersson subcategory with respect to $(I,J)$. Suppose $M$ is Weakly Laskerian module. Then $T^a_{I,J}(M) = \inf \{ T^a_n(M) \mid a \in \widetilde{W}(I,J) \}$, where $T^a_n(M)$ is the least non-negative integer $i$ such that $H^a_i(M)$ is not in $S$.

**Proof.** It is enough, in view Definition 2.1 and Theorem 2.8, to show that, $T^a_{I,J}(M) > i$ if $i < T^a_n(M)$ for all $a \in \widetilde{W}(I,J)$. To do this, let $a$ be an arbitrary ideal in $\widetilde{W}(I,J)$. We prove this by induction on $i$. It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It follows from [16, Proposition 1.4] that $H^a_{I,J}(M) \cong H^a_{I,J}(M/\Gamma_{I,J}(M))$ for all $i \geq 1$. Hence, by replacing $M$ with $M/\Gamma_{I,J}(M)$, we may assume that there exists an element $x \in I$ which is a non-zero divisor on $M$. Now, we may consider the exact sequence $0 \to M \to M/xM \to 0$ to obtain the exact sequences

\[ \begin{align*}
H^{a-1}_i(M/xM) & \to H^a_i(M) \to H^a_i(M/xM) \\
H^a_i(M) & \to H^a_i(M/xM)
\end{align*} \]
$H_{I,J}^{i-1}(M/xM) \rightarrow H_{I,J}^{i}(M) \rightarrow H_{I,J}^{i}(M) \rightarrow H_{I,J}^{i}(M/xM)$.

Now, one can use the above exact sequences in conjunction with the inductive hypothesis to see that $(0:_{H_{I,J}^{i}(M)} x)$ is in $S$. Since $(0:_{H_{I,J}^{i}(M)} I) \subseteq (0:_{H_{I,J}^{i}(M)} x)$, and hence the result follows.

This shows that the study of generalized local cohomology in a Melkersson subcategory in the upper range depends on the ideals of $\tilde{W}(I,J)$.

**Theorem 2.10.** Let $M$ be Weakly Laskerian module and let $r$ be a non-negative integer such that $H_{I,J}^{i}(R/p) \in S$ for all $p \in \text{Supp}(M)$. Then $H_{I,J}^{i}(M) \in S$.

**Proof.** Clearly, there exists a filtration of the submodules of $M$

$$0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$$

such that for each $1 \leq j \leq \ell$, then $M_j/M_{j-1} \cong R/p_j$, where $p_j \in \text{Supp}_RM$. We use induction on $\ell$. When $\ell = 1$, $H_{I,J}^{i}(R/p) = H_{I,J}^{i}(M)$ is in $S$, where we put $p = p_j$. Now Suppose that $\ell > 1$ and the result has been proved for $\ell - 1$. The exact sequence

$$0 \rightarrow M_{\ell-1} \rightarrow M_{\ell} \rightarrow M_{\ell}/M_{\ell-1} \rightarrow 0$$

induces the long exact sequence

$$H_{I,J}^{i}(M_{\ell-1}) \rightarrow H_{I,J}^{i}(M_{\ell}) \rightarrow H_{I,J}^{i}(M_{\ell}/M_{\ell-1})$$

It follows that $H_{I,J}^{i}(M_{\ell})$ is in $S$. This completes the proof. \qed

**Definition 2.11.** (see [17, Definition 2.1]) An $R$-module $T$ is called $(I,J)$-cofinite if $\text{Supp}T \subseteq \tilde{W}(I,J)$ and $\text{Ext}_R^i(R/I,T)$ is a finite $R$-modules, for every $i \geq 0$. Whence according to [9, Lemma 2], the class of $(I,J)$-cofinite minimax (Artinian) modules is closed under taking submodules, quotients and extensions, it is a Serre subcategory of the category of $R$-modules.

The following result is an application of the Theorem 2.10.

**Corollary 2.12.** Let $M$ be a finitely generated $R$-module with dim$M = d$. Then $H_{I,J}^{i}(M)$ is Artinian and $(I,J)$-cofinite.

**Proof.** Let $S$ be the class of $(I,J)$-cofinite Artinian modules. It is enough, in view Theorem 2.10, to show that $R$-module $H_{I,J}^{i}(R/p)$ is Artinian and $(I,J)$-cofinite for all $p \in \text{Supp}M$. If $J \subseteq p$, then $R/p$ is $J$-torsion and then $H_{I,J}^{i}(R/p) \cong H_{I}^{i}(R/p)$. Since dim$R/p \leq d$, then, in view of [12, Proposition 5.1], $H_{I}^{i}(R/p)$ is Artinian and $I$-cofinite. If $J \nsubseteq p$, then dim$(R/p)/J(R/p) < \dim(R/p) \leq d$ and so $H_{I,J}^{i}(R/p) = 0$ by [16, Theorem 4.3]. The proof is completed. \qed

**Theorem 2.13.** Let $M$ be a Weakly Laskerian module. Then $H_{I,J}^{i}(R/p)$ is in $S$ for all $i > T_{s}^{I,J}(M)$ and $p \in \text{Supp}(M)$. 

Proof: We use descending induction on $i$. Now, assume that $i > T^I_{i,J}(M)$ and that the claim holds for $i + 1$. We want to show that $H^i_{I,J}(R/p)$ is in $S$ for all $p \in \text{Supp}(M)$. Suppose the contrary. We set: $A = \{ p \mid p \in \text{Supp}M, H^i_{I,J}(R/p) \text{ is not in } S \}$. Clearly $A \neq \emptyset$; it follows that the set $A$ has a maximal element, let $p$ be one such. Since $p \in \text{Supp}(M)$, there is a non-zero map $f : M \to R/p$. The exact sequence $0 \to \text{Ker}f \to M \to \text{Im}f \to 0$, yields the exact sequence

$$H^i_{I,J}(M) \to H^i_{I,J}(\text{Im}f) \to H^{i+1}_{I,J}(\text{Ker}f).$$

Since $\text{Supp}(\text{Ker}f) \subseteq \text{Supp}(M)$, it follows from the inductive hypothesis that the $R$-module $H^{i+1}_{I,J}(R/p)$ is in $S$ for all $p \in \text{Supp}(\text{Ker}f)$, so, that, in view of the Theorem 2.10, and the above exact sequence, the $R$-module $H^i_{I,J}(\text{Im}f)$ is in $S$. There is a filtration

$$0 = N_i \subseteq N_{i-1} \subseteq N_{i-2} \subseteq \cdots \subseteq N_0 = \text{Coker}f$$

of submodules of $\text{Coker}f$, such that for each $0 \leq i \leq t$, $N_{i-1}/N_i \cong R/q_i$, where $q_i \in \text{Supp}(\text{Ker}f)$. Then by maximality of $p$, $H^i_{I,J}(R/q_i)$ is in $S$. Next the exact sequence $0 \to \text{Im}f \to R/p \to \text{Coker}f \to 0$, yields the exact sequence

$$H^i_{I,J}(\text{Im}f) \to H^i_{I,J}(R/p) \to H^i_{I,J}(\text{Coker}f).$$

It follows that $H^i_{I,J}(R/p)$ is in $S$, which is a contradiction. \hfill $\square$

Lemma 2.14. If $N$ and $M$ are Weakly Laskerian modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$, then $T^I_{s,J}(N) \leq T^I_{s,J}(M)$. In particular, if $\text{Supp}(N) = \text{Supp}(M)$ then $T^I_{s,J}(N) = T^I_{s,J}(M)$.

Proof. It is enough to show that $H^i_{I,J}(N)$ is in $S$ for all finite $R$-module $N$ with $\text{Supp}N \subseteq \text{Supp}M$ and for all $i > T^I_{s,J}(M)$. In view of the previous theorem, $H^i_{I,J}(R/p)$ is in $S$ for all $p \in \text{Supp}(M)$. Now, since $\text{Supp}(N) \subseteq \text{Supp}(M)$, the result follows by Theorem 2.10. \hfill $\square$

As an immediate result of Theorems (2.13) and (2.10), we have the following Corollary. This shows that the study of generalized local cohomology of Weakly Laskerian module $M$ in a Serre subcategory in the lower range depends just on the support of module $M$.

Corollary 2.15. Let $M$ be a Weakly Laskerian module. Then

$$T^I_{s,J}(M) = \text{Sup}(T^I_{s,J}(R/p) \mid p \in \text{Supp}(M)).$$

Theorem 2.16. Let $T^I_{s,J}(M) > 0$ and $a \in \widehat{W}(I,J)$. If $M$ has finite krull dimension, then $H^i_{I,J}(M)/aH^i_{I,J}(M)$ is in $S$ for all $i \geq T^I_{s,J}(M) = t$.

Proof. When $i > T^I_{s,J}(M)$, the result is clearly, it is enough to show that $H^i_{I,J}(M)/aH^i_{I,J}(M)$ is in $S$. We proceed by induction on $\text{dim}M = n$. If $n = 0$, then $M$ is $m$-torsion and there is nothing to prove. So let $n > 0$ and
suppose that the result has been proved for any finitely generated module $N$ with $\dim(N) = n - 1$. Since $H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$ for $i > 0$, we can assume that $M$ is $(I,J)$-torsion. Thus, there is an element $x \in \mathfrak{a}$, such that $x$ is a non-zero divisor on $M$. Now, one can complete the proof by using an argument similar to the proof of [7, Theorem 3.3]. □

Acknowledgments

I would like to thank the referee for his/her careful reading of the paper.

References