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Serre Subcategories and Local Cohomology Modules with Respect to a Pair of Ideals

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Abstract. This paper is concerned with the relation between local cohomology modules defined by a pair of ideals and the Serre subcategories of the category of modules. We characterize the membership of local cohomology modules in a certain Serre subcategory from lower range or upper range.

Keywords: Local cohomology modules, pair of ideals, Serre subcategory.

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1. Introduction

Throughout this paper, R is denoted a commutative Noetherian ring, I and J are denoted two ideals of R, and M is an R-module. We refer the reader to [2] and [4] for any unexplained terminology.

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa [16] introduced the local cohomology modules with respect to a pair of ideals (I, J). To be more precise, let $W(I, J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J$ for some positive integer $n\}$. Then for an R-module M, the (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M, which consists of all elements x of M with $\mathrm{Supp}Rx \subseteq W(I, J)$, is considered. It is known, $\Gamma_{I,J}$ is a left exact additive functor from the category of all R-modules and R-homomorphism to itself. For all integer i, the i-th local cohomology functor $H^i_{I,J}$ with respect

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to (I, J) is defined to be the i-th right derived functor of $\Gamma_{I,J}$. The i-th local cohomology module of M with respect to (I, J) is denoted by $H^i_{I,J}(M)$. When $J = 0$, then $H^i_{I,J}$ coincides with the usual local cohomology functor H^i_I with the support in the closed subset $V(I)$.
The study of this generalized local cohomology modules was continued by many authors (see for example [5], [6], [9] and [10], [14]).

Recall that a class of R-modules is a Serre subcategory of the category of R-modules when it is closed under taking submodules, quotients and extensions. Always, S stands for a Serre subcategory of the category of R-modules.

Using the generalized local cohomology modules, we can define $T^s_{I,J}(M)$ (resp. $T^{I,J}_s(M)$) of the R-module M relative to a pair (I, J) of ideals of R by

$$T^s_{I,J}(M) = \inf\{i \in \mathbb{N} \mid H^i_{I,J}(M) \text{ is not in } S\},$$

$$(\text{resp.} \quad T^{I,J}_s(M) = \text{Sup}\{i \in \mathbb{N} \mid H^i_{I,J}(M) \text{ is not in } S\})$$

with the usual convention that the infimum (resp. Supremum) of the empty set of integers interpreted as $+\infty$ (resp. $-\infty$).

Our objective in this paper is to investigate the notions $T^s_{I,J}(M)$ and $T^{I,J}_s(M)$. we prove the following,

Theorem 1.1. *Let S be a Melkersson subcategory with respect to (I, J). Suppose M is Weakly Laskerian module. Then $T^s_{I,J}(M) = \inf\{T^s_{\mathfrak{a}}(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$, where $T^s_{\mathfrak{a}}(M)$ is the least non-negative integer i such that $H^i_{\mathfrak{a}}(M)$ is not in S.*

Theorem 1.2. *Let S be a Serre subcategory and let M be a Weakly Laskerian module. Then $T^{I,J}_s(M) = Sup\{T^{I,J}_s(R/\mathfrak{p}) \mid \mathfrak{p} \in Supp(M)\}$.*

One can see that the subcategories of finitely generated R-modules, minimax R-modules, minimax and (I, J)-cofinite R-modules, weakly Laskerian R-modules, and Matlis reflexive R-modules are examples of Serre subcategory. So, this paper recovers some results regarding the local cohomology R-modules that have appeared in different papers (see for instance [3], [5] and [13]).

2. The Results

This section is started with the following definition.

Definition 2.1. (see [1, Definition 3.1]) A Serre subcategory of the category of R-modules is said to be a Melkersson subcategory with respect to the ideal $\mathfrak{a}$, if for any $\mathfrak{a}$-torsion R-module X, $(0 :_X \mathfrak{a})$ is in S implies that X is in S. Examples are given by the class of Artinian modules, minimax and $\mathfrak{a}$-cofinite modules.

Also, we say that S is a Melkersson subcategory with respect to the pair of ideals (I, J), if for an (I, J)-torsion R-module X, $(0 :_X I)$ is in S implies that X is in S. Obviously, if S is a Melkersson subcategory with respect to (I, J), then S is a Melkersson subcategory with respect to all ideals of $\widetilde{W}(I,J)$, where $\widetilde{W}(I,J)$ denote the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n.

Proposition 2.2. *Let S be a Melkersson subcategory with respect to the ideal I and t an integer. Let T be an R-module such that $\mathrm{Ext}^i_R(R/I,T)$ is in S for all $i < t$. Then $H^i_I(T)$ is in S for all $i < t$. Particularly, for an R-module M, the module $H^i_I\big(H^j_{I,J}(M)\big)$ is in S for all i and $j < T^s_{I,J}(M)$.*

Proof. We prove the theorem by induction on i. It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It easily follows from the exact sequence

$$0 \longrightarrow \Gamma_I(T) \longrightarrow T \longrightarrow T/\Gamma_I(T) \longrightarrow 0,$$

that $\mathrm{Ext}^i_R(R/I,T)$ is in S if and only if $\mathrm{Ext}^i_R(R/I,T/\Gamma_I(T))$ is in S. Also, by [2, Corollary 2.1.7], $H^i_I(M) \cong H^i_I\big(M/\Gamma_I(M)\big)$ for all $i > 0$. Therefore we assume that $\Gamma_I(T) = 0$. Now, we apply Melkerson's technic [12], so let E be an injective envelope of T. Then $\Gamma_I(E) = \mathrm{Hom}(R/I,E) = 0$. Put $L = E/T$ and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow L \longrightarrow 0.$$

We obtain isomorphisms; $H^i_I(T) \cong H^{i-1}_I(L)$ and $\mathrm{Ext}^i_R(R/I,T) \cong \mathrm{Ext}^{i-1}_I(R/I,L)$ for all $i > 0$. Use the induction hypothesis applied to L, and conclude that the $H^i_I(T)$ is in S for all $i < t$. It therefore follows, in view of the definition of $T^s_{I,J}(M)$, that $H^i_I(H^j_{I,J}(M))$ is in S for all $j < T^s_{I,J}(M)$. □

Example 2.3. In Theorem 2.2, the assumption S is Melkersson subcategory is necessary. To see this, let (R,m) be a local ring, and let M be a non-zero, finitely generated R-module of dimension $n > 0$. Then $H^n_{\mathfrak{m}}(M)$ is not finitely generated (see [2, Corollary 7.3.3]).

The next theorem recovers the Theorem 2.5 of [3] and Theorem 2.2 of [17].

Theorem 2.4. *Let M be in S and $j \leq T^s_{I,J}(M) = t$. Then*

(i) *$\mathrm{Ext}^i_R\big(R/I,H^j_{I,J}(M)\big)$ is in S for all $i = 0,1$.*
(ii) *$H^i_I\big(H^j_{I,J}(M)\big)$ is in S for all $i = 0,1$, if S is a Melkersson subcategory with respect to I.*

Proof. (i) Consider the functors $F(-) = \mathrm{Hom}_R(R/I,-\,)$ and $G(-) = \Gamma_{I,J}(-\,)$. Then one has $FG(-) = \mathrm{Hom}(R/I,-\,)$. So, by [14, Theorem 11.38], there is a Grothendieck's spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R\big(R/I,H^q_{I,J}(M)\big) \overset{p}{\Longrightarrow} \mathrm{Ext}^{p+q}_R(R/I,M).$$

By using an argument similar to the proof of [8, Theorem 2.2], we obtain that $\mathrm{Ext}^i_R\big(R/I,H^t_{I,J}(M)\big)$ is in S for all $i = 0,1$. This completes the proof.

(ii) Using [15, Theorem 11.38] there exists a Grothendieck's spectral sequence

$$E_2^{p,q} = H_I^p\big(H_{I,J}^q(M)\big) \overset{p}{\Longrightarrow} H_I^{p+q}(M).$$

Also, there is a bound filtration $0 = \varphi^{t+1}H^t \subseteq \varphi^t H^t \subseteq \cdots \subseteq \varphi^0 H^t = H_I^t(M)$ such that $E_\infty^{i,t-i} \cong \frac{\varphi^i H^t}{\varphi^{i+1}H^t}$ for all $0 \leq i \leq t$. By the hypotheses with proposition 2.2 $H_I^i(M)$ is in S for all i and hence $E_\infty^{p,q}$ is in S for all p, q. Note that $E_\infty^{p,q} = E_r^{p,q}$ for large r and each p, q. It follows that there is an integer $\ell \geq 2$ such that $E_r^{p,q}$ is in S for all $r \geq \ell$. We now argue by descending induction on ℓ. Now, assume that $2 < \ell < r$ and that the claim holds for ℓ. Since $E_r^{p,q}$ is in a subquotient of $E_2^{p,q}$ for all $p, q \in \mathbb{N}_0$, the hypotheses give $E_r^{p+r,t-r+1}$ is in S for all $r \geq 2$. In addition, $E_\ell^{p,t} = \frac{\ker d_{\ell-1}^{p,t}}{\mathrm{im} d^{p-\ell+1,t+\ell-2}}$ and $\mathrm{im} d_{\ell-1}^{p-\ell+1,t+\ell-2} = 0$ for $p = 0, 1$, it follows that $\ker d_{\ell-1}^{p,t}$ is in S for all $\ell > 2$ and $p = 0, 1$. Let $r \geq 2$ and $p = 0, 1$, we consider the sequence

$$0 \longrightarrow \ker d_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow E_r^{p+r,t-r+1}.$$

Since both $\ker d_{\ell-1}^{p,t}$ and $E_{\ell-1}^{p+r,t-r+1}$ are in S, it follows that $E_{\ell-1}^{p,t}$ is in S for $p = 0, 1$. This completes the inductive step. □

Proposition 2.5. *Let M be in S such that* $\dim(M/JM) \leq 1$. *Then* $\mathrm{Ext}_R^i\big(R/I, H_{I,J}^j(M)\big)$ *is in S*

Proof. Consider the following spectral sequence

$$E_2^{p,q} := \mathrm{Ext}_R^p\big(R/I, H_{I,J}^q(M)\big) \overset{p}{\Longrightarrow} \mathrm{Ext}_R^{p+q}(R/I, M) = H^{p+q}.$$

In view of [16, Theorem 4.3] $E_2^{p,q} = 0$ unless $q = 0, 1$. It follows that the exact sequence

$$H^{p+1} \longrightarrow E_2^{p+1,0} \longrightarrow E_2^{p-1,1} \longrightarrow H^p \longrightarrow E_2^{p,0} \longrightarrow E_2^{p-2,1} \longrightarrow H^{p-1}$$

which in turn yields the exact sequence

$$\mathrm{Ext}_R^{p+1}(R/I, M) \longrightarrow \mathrm{Ext}_R^{p+1}(R/I, \Gamma_{I,J}(M)) \longrightarrow \mathrm{Ext}_R^{p-1}(R/I, H_{I,J}^1(M)) \longrightarrow$$
$$\mathrm{Ext}_R^p(R/I, M) \longrightarrow \mathrm{Ext}_R^p(R/I, \Gamma_{I,J}(M)) \longrightarrow \mathrm{Ext}_2^{p-2}(R/I, H_{I,J}^1(M)).$$

Since, by our assumption, the R-modules $\mathrm{Ext}_R^i(R/I, \Gamma_{I,J}(M))$ and $\mathrm{Ext}_R^i(R/I, M)$ are in S for all i and hence the result follows. □

Corollary 2.6. *Let S be a Melkersson subcategory of the category of R-modules. Let M be in S and* $\dim(M/JM) \leq 1$. *Then* $H_I^i\big(H_{I,J}^j(M)\big)$ *is in S for all i and j.*

Proof. The result follows from the part (ii) of Theorem (2.4) and argument similar to the proof of Proposition 2.5.

Recall that an R-module M is weakly Laskerian if any quotient of M has a finitely many associated prime ideals. This holds, by employing a method of proof which is similar to that used in [2, Lemma 2.1.1], M is a $\mathfrak{a}$-torsion-free if

and only if $\mathfrak{a}$ contains a non-zerodivisor on M. Clearly, any finitely generated module and any minimax module are weakly Laskerian modules.
In addition, by using an argument similar to the proof of [11, Theorem 6.4], there exists a chain $0 = M_0 \subset M_1 \cdots \subset M_n = M$ of submodules of M such that for each i we have $M_i/M_{i-1} = \mathfrak{p}_i$ with $\mathfrak{p}_i \in Supp(M)$. □

Theorem 2.7. *Let $\mathfrak{a} \in \widetilde{W}(I, J)$ and M be a Weakly Laskerian module. Then $Ext^i_R(R/\mathfrak{a}, M)$ is in S for all $0 \leq i < T^s_{I,J}(M)$.*

Proof. It follows by using induction on $T^s_{I,J}(M)$. □

Theorem 2.8. *Let $\mathfrak{a} \in \widetilde{W}(I, J)$ and S be a Melkersson subcategory with respect to the ideal $\mathfrak{a}$. Suppose M is Weakly Laskerian module. Then $H^i_{\mathfrak{a}}(M) \in S$ for all $i < T^s_{I,J}(M)$.*

Proof. By using the induction on $t = T^s_{I,J}(M)$, the theorem is proved. It is straightforward to see that the result is true when $t = 1$. Suppose that $t > 1$, and the result holds for the case $t - 1$. Since $H^i_{I,J}(M) \cong H^i_{I,J}\big(M/\Gamma_{I,J}(M)\big)$ for all $i > 0$, we may replace M by $M/\Gamma_{I,J}(M)$ and hence assume that there is an element $x \in \mathfrak{a}$, such that x is a non-zero divisor on M. The exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ induces two exact sequences

$$\longrightarrow H^{i-1}_{I,J}(M/xM) \longrightarrow H^i_{I,J}(M) \xrightarrow{x} H^i_{I,J}(M) \longrightarrow H^i_{I,J}(M/xM) \qquad \text{and}$$

$$\longrightarrow H^{i-1}_{\mathfrak{a}}(M/xM) \longrightarrow H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/xM) \qquad (*)$$

of local cohomology modules. The induction hypothesis and the above sequences yield that the R-modules $H^i_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{a}}(M/xM)$ are in S for all $i < t - 1$. It suffices to show that $H^{t-1}_{\mathfrak{a}}(M)$ is in S. Now, the exactness of $(*)$, in conjunction with the fact $\big(0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}\big) \subseteq (0 :_{H^{t-1}_{\mathfrak{a}}(M)} x)$ and our hypotheses, show that $H^{t-1}_{\mathfrak{a}}(M)$ is in S, this proves our claim. □

Theorem 2.9. *Let S be a Melkersson subcategory with respect to (I, J). Suppose M is Weakly Laskerian module. Then $T^s_{I,J}(M) = \inf\{T^s_{\mathfrak{a}}(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$, where $T^s_{\mathfrak{a}}(M)$ is the least non-negative integer i such that $H^i_{\mathfrak{a}}(M)$ is not in S.*

Proof. It is enough, in view Definition 2.1 and Theorem 2.8, to show that, $T^s_{I,J}(M) > i$ if $i < T^s_{\mathfrak{a}}(M)$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$. To do this, let $\mathfrak{a}$ be an arbitrary ideal in $\widetilde{W}(I, J)$. We prove this by induction on i. It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It follows from [16, Proposition 1.4] that $H^i_{I,J}(M) \cong H^i_{I,J}\big(M/\Gamma_{I,J}(M)\big)$ for all $i \geq 1$. Hence, by replacing M with $M/\Gamma_{I,J}(M)$, we may assume that there exists an element $x \in I$ which is a non-zero divisor on M. Now, we may consider the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ to obtain the exact sequences

$$H^{i-1}_{\mathfrak{a}}(M/xM) \longrightarrow H^i_{\mathfrak{a}}(M) \xrightarrow{x} H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/xM) \quad \text{and}$$

$$H^{i-1}_{I,J}(M/xM) \longrightarrow H^i_{I,J}(M) \xrightarrow{x} H^i_{I,J}(M) \longrightarrow H^i_{I,J}(M/xM).$$

Now, one can use the above exact sequences in conjunction with the inductive hypothesis to see that $\big(0 :_{H^i_{I,J}(M)} x\big)$ is in S. Since $\big(0 :_{H^i_{I,J}(M)} I\big) \subseteq \big(0 :_{H^i_{I,J}(M)} x\big)$, and hence the result follows.

This shows that the study of generalized local cohomology in a Melkersson subcategory in the upper range depends on the ideals of $\widetilde{W}(I,J)$. □

Theorem 2.10. *Let M be Weakly Laskerian module and let r be a non-negative integer such that $H^r_{I,J}(R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in Supp(M)$. Then $H^r_{I,J}(M) \in S$.*

Proof. Clearly, there exists a filtration of the submodules of M

$$0 \subseteq M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$$

such that for each $1 \leq j \leq \ell$, then $M_j/M_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \in \mathrm{Supp}_R M$. We use induction on ℓ. When $\ell = 1$, $H^r_{I,J}(R/\mathfrak{p}) = H^r_{I,J}(M)$ is in S, where we put $\mathfrak{p} = \mathfrak{p}_j$. Now Suppose that $\ell > 1$ and the result has been proved for $\ell - 1$. The exact sequence

$$0 \longrightarrow M_{\ell-1} \longrightarrow M_\ell \longrightarrow M_\ell/M_{\ell-1} \longrightarrow 0$$

induces the long exact sequence

$$H^r_{I,J}(M_{\ell-1}) \longrightarrow H^r_{I,J}(M_\ell) \longrightarrow H^r_{I,J}(M_\ell/M_{\ell-1}).$$

It follows that $H^r_{I,J}(M_\ell)$ is in S. This completes the proof. □

Definition 2.11. (see [17, Definition 2.1]) An R-module T is called (I,J)-cofinite if $\mathrm{Supp}T \subseteq W(I,J)$ and $\mathrm{Ext}^i_R(R/I,T)$ is a finite R-modules, for every $i \geq 0$. Whence according to [9, Lemma 2], the class of (I,J)-cofinite minimax (Artinian) modules is closed under taking submodules, quotients and extensions, it is a Serre subcategory of the category of R-modules.

The following result is an application of the Theorem 2.10.

Corollary 2.12. *Let M be a finitely generated R-module with $dimM = d$. Then $H^d_{I,J}(M)$ is Artinian and (I,J)-cofinite.*

Proof. Let S be the class of (I,J)-cofinite Artinian modules. It is enough, in view Theorem 2.10, to show that R-module $H^d_{I,J}(R/\mathfrak{p})$ is Artinian and (I,J)-cofinite for all $\mathfrak{p} \in \mathrm{Supp}M$. If $J \subseteq \mathfrak{p}$, then $R/\mathfrak{p}$ is J-torsion and then $H^d_{I,J}(R/\mathfrak{p}) \cong H^d_I(R/\mathfrak{p})$. Since $\dim R/\mathfrak{p} \leq d$, then, in view of [12, Proposition 5.1], $H^d_I(R/\mathfrak{p})$ is Artinian and I-cofinite. If $J \nsubseteq \mathfrak{p}$, then $\dim(R/\mathfrak{p})/J(R/\mathfrak{p}) < \dim(R/\mathfrak{p}) \leq d$ and so $H^d_{I,J}(R/\mathfrak{p}) = 0$ by [16, Theorem 4.3]. The proof is completed. □

Theorem 2.13. *Let M be a Weakly Laskerian module. Then $H^i_{I,J}(R/\mathfrak{p})$ is in S for all $i > T^{I,J}_s(M)$ and $\mathfrak{p} \in Supp(M)$.*

Proof. We use descending induction on i. Now, assume that $i > T_s^{I,J}(M)$ and that the claim holds for $i+1$. We want to show that $H^i_{I,J}(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \mathrm{Supp}(M)$. Suppose the contrary. We set: $A = \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}M, H^i_{I,J}(R/\mathfrak{p}) \text{ is not in } S\}$. Clearly $A \neq \emptyset$; it follows that the set A has a maximal element, let $\mathfrak{p}$ be one such. Since $\mathfrak{p} \in \mathrm{Supp}(M)$, there is a non-zero map $f : M \to R/\mathfrak{p}$. The exact sequence $0 \longrightarrow \mathrm{Ker}f \longrightarrow M \longrightarrow \mathrm{Im}f \longrightarrow 0$, yields the exact sequence

$$H^i_{I,J}(M) \longrightarrow H^i_{I,J}(\mathrm{Im}f) \longrightarrow H^{i+1}_{I,J}(\mathrm{Ker}f).$$

Since $\mathrm{Supp}(\mathrm{Ker}f) \subset \mathrm{Supp}(M)$, it follows from the inductive hypothesis that the R-module $H^{i+1}_{I,J}(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \mathrm{Supp}(\ker f)$, so, that, in view of the Theorem 2.10, and the above exact sequence, the R-module $H^i_{I,J}(\mathrm{Im}f)$ is in S. There is a filtration

$$0 = N_t \subset N_{t-1} \subset N_{t-2} \subset \cdots \subset N_0 = \mathrm{Coker}f$$

of submodules of $\mathrm{Coker}f$, such that for each $0 \leq i \leq t$, $N_{i-1}/N_i \cong R/\mathfrak{q}_i$ where $\mathfrak{q}_i \in \mathrm{Supp}(\mathrm{Coker}f)$. Then by maximality of $\mathfrak{p}$, $H^i_{I,J}(R/\mathfrak{q}_i)$ is in S. Next the exact sequence $0 \longrightarrow \mathrm{Im}f \longrightarrow R/\mathfrak{p} \longrightarrow \mathrm{Coker}f \longrightarrow 0$, yields the exact sequence

$$H^i_{I,J}(\mathrm{Im}f) \longrightarrow H^i_{I,J}(R/\mathfrak{p}) \longrightarrow H^i_{I,J}(\mathrm{Coker}f).$$

It follows that $H^i_{I,J}(R/\mathfrak{p})$ is in S, which is a contradiction. □

Lemma 2.14. *If N and M are Weakly Laskerian modules such that Supp$(N) \subseteq$ Supp(M), then $T_s^{I,J}(N) \leq T_s^{I,J}(M)$. In particular, if Supp$(N)$ = Supp(M) then $T_s^{I,J}(N) = T_s^{I,J}(M)$.*

Proof. It is enough to show that $H^i_{I,J}(N)$ is in S for all finite R-module N with $\mathrm{Supp}N \subseteq \mathrm{Supp}M$ and for all $i > T_s^{I,J}(M)$. In view of the previous theorem, $H^i_{I,J}(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \mathrm{Supp}(M)$. Now. since $\mathrm{Supp}(N) \subseteq \mathrm{Supp}(M)$, the result follows by Theorem 2.10. □

As an immediate result of Theorems (2.13) and (2.10), we have the following Corollary. This shows that the study of generalized local cohomology of Weakly Laskerian module M in a Serre subcategory in the lower range depends just on the support of module M.

Corollary 2.15. *Let M be a Weakly Laskerian module. Then*

$$T_s^{I,J}(M) = Sup\{T_s^{I,J}(R/\mathfrak{p}) \mid \mathfrak{p} \in Supp(M)\}.$$

Theorem 2.16. *Let $T^s_{I,J}(M) > 0$ and $\mathfrak{a} \in \widetilde{W}(I,J)$. If M has finite krull dimension, then $H^i_{I,J}(M)/\mathfrak{a}H^i_{I,J}(M)$ is in S for all $i \geq T_s^{I,J}(M) = t$.*

Proof. When $i > T_s^{I,J}(M)$, the result is clearly, it is enough to show that $H^t_{I,J}(M)/\mathfrak{a}H^t_{I,J}(M)$ is in S. We proceed by induction on $\dim M = n$. If $n = 0$, then M is $\mathfrak{m}$-torsion and there is nothing to prove. So let $n > 0$ and suppose that the result has been proved for any finitely generated module N with $\dim(N) = n-1$. Since $H^i_{I,J}(M) \cong H^i_{I,J}\Big(M/\Gamma_{I,J}(M)\Big)$ for $i > 0$, we can assume that M is (I,J)-torsion. Thus, there is an element $x \in \mathfrak{a}$, such that x is a non-zero divisor on M. Now, one can complete the proof by using an argument similar to the proof of [7, Theorem 3.3]. $\square$

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