

Serre Subcategories and Local Cohomology Modules with Respect to a Pair of Ideals

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ABSTRACT. This paper is concerned with the relation between local cohomology modules defined by a pair of ideals and the Serre subcategories of the category of modules. We characterize the membership of local cohomology modules in a certain Serre subcategory from lower range or upper range.

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1. INTRODUCTION

Throughout this paper, R is denoted a commutative Noetherian ring, I and J are denoted two ideals of R , and M is an R -module. We refer the reader to [2] and [4] for any unexplained terminology.

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa [16] introduced the local cohomology modules with respect to a pair of ideals (I, J) . To be more precise, let $W(I, J) = \{\mathfrak{p} \in \text{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}$. Then for an R -module M , the (I, J) -torsion submodule $\Gamma_{I,J}(M)$ of M , which consists of all elements x of M with $\text{Supp}Rx \subseteq W(I, J)$, is considered. It is known, $\Gamma_{I,J}$ is a left exact additive functor from the category of all R -modules and R -homomorphism to itself. For all integer i , the i -th local cohomology functor $H_{I,J}^i$ with respect to (I, J) is defined to be the i -th right derived functor of $\Gamma_{I,J}$. The i -th local

cohomology module of M with respect to (I, J) is denoted by $H_{I,J}^i(M)$. When $J = 0$, then $H_{I,J}^i$ coincides with the usual local cohomology functor H_I^i with the support in the closed subset $V(I)$.

The study of this generalized local cohomology modules was continued by many authors (see for example [5], [6], [9] and [10], [14]).

Recall that a class of R -modules is a Serre subcategory of the category of R -modules when it is closed under taking submodules, quotients and extensions. Always, S stands for a Serre subcategory of the category of R -modules.

Using the generalized local cohomology modules, we can define $T_{I,J}^s(M)$ (resp. $T_s^{I,J}(M)$) of the R -module M relative to a pair (I, J) of ideals of R by

$$T_{I,J}^s(M) = \inf\{i \in \mathbb{N} \mid H_{I,J}^i(M) \text{ is not in } S\},$$

$$\text{(resp. } T_s^{I,J}(M) = \text{Sup}\{i \in \mathbb{N} \mid H_{I,J}^i(M) \text{ is not in } S\})$$

with the usual convention that the infimum (resp. Supremum) of the empty set of integers interpreted as $+\infty$ (resp. $-\infty$).

Our objective in this paper is to investigate the notions $T_{I,J}^s(M)$ and $T_s^{I,J}(M)$. we prove the following,

Theorem 1.1. *Let S be a Melkersson subcategory with respect to (I, J) . Suppose M is Weakly Laskerian module. Then $T_{I,J}^s(M) = \inf\{T_{\mathfrak{a}}^s(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$, where $T_{\mathfrak{a}}^s(M)$ is the least non-negative integer i such that $H_{\mathfrak{a}}^i(M)$ is not in S .*

Theorem 1.2. *Let S be a Serre subcategory and let M be a Weakly Laskerian module. Then $T_s^{I,J}(M) = \text{Sup}\{T_s^{I,J}(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Supp}(M)\}$.*

One can see that the subcategories of finitely generated R -modules, minimax R -modules, minimax and (I, J) -cofinite R -modules, weakly Laskerian R -modules, and Matlis reflexive R -modules are examples of Serre subcategory. So, this paper recovers some results regarding the local cohomology R -modules that have appeared in different papers (see for instance [3], [5] and [13]).

2. THE RESULTS

This section is started with the following definition.

Definition 2.1. (see [1, Definition 3.1]) A Serre subcategory of the category of R -modules is said to be a Melkersson subcategory with respect to the ideal \mathfrak{a} , if for any \mathfrak{a} -torsion R -module X , $(0 :_X \mathfrak{a})$ is in S implies that X is in S . Examples are given by the class of Artinian modules, minimax and \mathfrak{a} -cofinite modules.

Also, we say that S is a Melkersson subcategory with respect to the pair of ideals (I, J) , if for an (I, J) -torsion R -module X , $(0 :_X I)$ is in S implies that X is in S . Obviously, if S is a Melkersson subcategory with respect to (I, J) , then S is a Melkersson subcategory with respect to all ideals of $\widetilde{W}(I, J)$, where

$\widetilde{W}(I, J)$ denote the set of ideals \mathfrak{a} of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n .

Proposition 2.2. *Let S be a Melkersson subcategory with respect to the ideal I and t an integer. Let T be an R -module such that $\text{Ext}_R^i(R/I, T)$ is in S for all $i < t$. Then $H_I^i(T)$ is in S for all $i < t$. Particularly, for an R -module M , the module $H_I^i(H_{I,J}^j(M))$ is in S for all i and $j < T_{I,J}^s(M)$.*

Proof. We prove the theorem by induction on i . It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It easily follows from the exact sequence

$$0 \longrightarrow \Gamma_I(T) \longrightarrow T \longrightarrow T/\Gamma_I(T) \longrightarrow 0,$$

that $\text{Ext}_R^i(R/I, T)$ is in S if and only if $\text{Ext}_R^i(R/I, T/\Gamma_I(T))$ is in S . Also, by [2, Corollary 2.1.7], $H_I^i(M) \cong H_I^i(M/\Gamma_I(M))$ for all $i > 0$. Therefore we assume that $\Gamma_I(T) = 0$. Now, we apply Melkersson's technic [12], so let E be an injective envelope of T . Then $\Gamma_I(E) = \text{Hom}(R/I, E) = 0$. Put $L = E/T$ and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow L \longrightarrow 0.$$

We obtain isomorphisms; $H_I^i(T) \cong H_I^{i-1}(L)$ and $\text{Ext}_R^i(R/I, T) \cong \text{Ext}_R^{i-1}(R/I, L)$ for all $i > 0$. Use the induction hypothesis applied to L , and conclude that the $H_I^i(T)$ is in S for all $i < t$. It therefore follows, in view of the definition of $T_{I,J}^s(M)$, that $H_I^i(H_{I,J}^j(M))$ is in S for all $j < T_{I,J}^s(M)$. \square

EXAMPLE 2.3. In Theorem 2.2, the assumption S is Melkersson subcategory is necessary. To see this, let (R, \mathfrak{m}) be a local ring, and let M be a non-zero, finitely generated R -module of dimension $n > 0$. Then $H_{\mathfrak{m}}^n(M)$ is not finitely generated (see [2, Corollary 7.3.3]).

The next theorem recovers the Theorem 2.5 of [3] and Theorem 2.2 of [17].

Theorem 2.4. *Let M be in S and $j \leq T_{I,J}^s(M) = t$. Then*

- (i) $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$ is in S for all $i = 0, 1$.
- (ii) $H_I^i(H_{I,J}^j(M))$ is in S for all $i = 0, 1$, if S is a Melkersson subcategory with respect to I .

Proof. (i) Consider the functors $F(-) = \text{Hom}_R(R/I, -)$ and $G(-) = \Gamma_{I,J}(-)$. Then one has $FG(-) = \text{Hom}(R/I, -)$. So, by [14, Theorem 11.38], there is a Grothendieck's spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(R/I, H_{I,J}^q(M)) \xrightarrow{p} \text{Ext}_R^{p+q}(R/I, M).$$

By using an argument similar to the proof of [8, Theorem 2.2], we obtain that $\text{Ext}_R^i(R/I, H_{I,J}^t(M))$ is in S for all $i = 0, 1$. This completes the proof.

(ii) Using [15, Theorem 11.38] there exists a Grothendieck's spectral sequence

$$E_2^{p,q} = H_I^p(H_{I,J}^q(M)) \xrightarrow{p} H_I^{p+q}(M).$$

Also, there is a bound filtration $0 = \varphi^{t+1}H^t \subseteq \varphi^tH^t \subseteq \dots \subseteq \varphi^0H^t = H_I^t(M)$ such that $E_\infty^{i,t-i} \cong \frac{\varphi^iH^t}{\varphi^{i+1}H^t}$ for all $0 \leq i \leq t$. By the hypotheses with proposition 2.2 $H_I^i(M)$ is in S for all i and hence $E_\infty^{p,q}$ is in S for all p, q . Note that $E_\infty^{p,q} = E_r^{p,q}$ for large r and each p, q . It follows that there is an integer $\ell \geq 2$ such that $E_r^{p,q}$ is in S for all $r \geq \ell$. We now argue by descending induction on ℓ . Now, assume that $2 < \ell < r$ and that the claim holds for ℓ . Since $E_r^{p,q}$ is in a subquotient of $E_2^{p,q}$ for all $p, q \in \mathbb{N}_0$, the hypotheses give $E_r^{p+r,t-r+1}$ is in S for all $r \geq 2$. In addition, $E_\ell^{p,t} = \frac{\ker d_{\ell-1}^{p,t}}{\text{im} d_{\ell-1}^{p-\ell+1,t+\ell-2}}$ and $\text{im} d_{\ell-1}^{p-\ell+1,t+\ell-2} = 0$ for $p = 0, 1$, it follows that $\ker d_{\ell-1}^{p,t}$ is in S for all $\ell > 2$ and $p = 0, 1$. Let $r \geq 2$ and $p = 0, 1$, we consider the sequence

$$0 \longrightarrow \ker d_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow E_r^{p+r,t-r+1}.$$

Since both $\ker d_{\ell-1}^{p,t}$ and $E_{\ell-1}^{p+r,t-r+1}$ are in S , it follows that $E_{\ell-1}^{p,t}$ is in S for $p = 0, 1$. This completes the inductive step. \square

Proposition 2.5. *Let M be in S such that $\dim(M/JM) \leq 1$. Then $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$ is in S*

Proof. Consider the following spectral sequence

$$E_2^{p,q} := \text{Ext}_R^p(R/I, H_{I,J}^q(M)) \xrightarrow{p} \text{Ext}_R^{p+q}(R/I, M) = H^{p+q}.$$

In view of [16, Theorem 4.3] $E_2^{p,q} = 0$ unless $q = 0, 1$. It follows that the exact sequence

$$H^{p+1} \longrightarrow E_2^{p+1,0} \longrightarrow E_2^{p-1,1} \longrightarrow H^p \longrightarrow E_2^{p,0} \longrightarrow E_2^{p-2,1} \longrightarrow H^{p-1}$$

which in turn yields the exact sequence

$$\begin{aligned} \text{Ext}_R^{p+1}(R/I, M) &\longrightarrow \text{Ext}_R^{p+1}(R/I, \Gamma_{I,J}(M)) \longrightarrow \text{Ext}_R^{p-1}(R/I, H_{I,J}^1(M)) \longrightarrow \\ &\text{Ext}_R^p(R/I, M) \longrightarrow \text{Ext}_R^p(R/I, \Gamma_{I,J}(M)) \longrightarrow \text{Ext}_R^{p-2}(R/I, H_{I,J}^1(M)). \end{aligned}$$

Since, by our assumption, the R -modules $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M))$ and $\text{Ext}_R^i(R/I, M)$ are in S for all i and hence the result follows. \square

Corollary 2.6. *Let S be a Melkersson subcategory of the category of R -modules. Let M be in S and $\dim(M/JM) \leq 1$. Then $H_I^i(H_{I,J}^j(M))$ is in S for all i and j .*

Proof. The result follows from the part (ii) of Theorem (2.4) and argument similar to the proof of Proposition 2.5.

Recall that an R -module M is weakly Laskerian if any quotient of M has a finitely many associated prime ideals. This holds, by employing a method of proof which is similar to that used in [2, Lemma 2.1.1], M is a \mathfrak{a} -torsion-free if and only if it contains a non-zero-divisor on M . Clearly, any finitely generated module and any minimax module are weakly Laskerian modules.

In addition, by using an argument similar to the proof of [11, Theorem 6.4],

there exists a chain $0 = M_0 \subset M_1 \cdots \subset M_n = M$ of submodules of M such that for each i we have $M_i/M_{i-1} = \mathfrak{p}_i$ with $\mathfrak{p}_i \in \text{Supp}(M)$. \square

Theorem 2.7. *Let $\mathfrak{a} \in \widetilde{W}(I, J)$ and M be a Weakly Laskerian module. Then $\text{Ext}_R^i(R/\mathfrak{a}, M)$ is in S for all $0 \leq i < T_{I,J}^s(M)$.*

Proof. It follows by using induction on $T_{I,J}^s(M)$. \square

Theorem 2.8. *Let $\mathfrak{a} \in \widetilde{W}(I, J)$ and S be a Melkersson subcategory with respect to the ideal \mathfrak{a} . Suppose M is Weakly Laskerian module. Then $H_{\mathfrak{a}}^i(M) \in S$ for all $i < T_{I,J}^s(M)$.*

Proof. By using the induction on $t = T_{I,J}^s(M)$, the theorem is proved. It is straightforward to see that the result is true when $t = 1$. Suppose that $t > 1$, and the result holds for the case $t - 1$. Since $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$ for all $i > 0$, we may replace M by $M/\Gamma_{I,J}(M)$ and hence assume that there is an element $x \in \mathfrak{a}$, such that x is a non-zero divisor on M . The exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ induces two exact sequences

$$\begin{aligned} \rightarrow H_{I,J}^{i-1}(M/xM) \rightarrow H_{I,J}^i(M) \xrightarrow{x} H_{I,J}^i(M) \rightarrow H_{I,J}^i(M/xM) \quad \text{and} \\ \rightarrow H_{\mathfrak{a}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x} H_{\mathfrak{a}}^i(M) \rightarrow H_{\mathfrak{a}}^i(M/xM) \quad (*) \end{aligned}$$

of local cohomology modules. The induction hypothesis and the above sequences yield that the R -modules $H_{\mathfrak{a}}^i(M)$ and $H_{\mathfrak{a}}^i(M/xM)$ are in S for all $i < t - 1$. It suffices to show that $H_{\mathfrak{a}}^{t-1}(M)$ is in S . Now, the exactness of $(*)$, in conjunction with the fact $(0 :_{H_{\mathfrak{a}}^{t-1}(M)} \mathfrak{a}) \subseteq (0 :_{H_{\mathfrak{a}}^{t-1}(M)} x)$ and our hypotheses, show that $H_{\mathfrak{a}}^{t-1}(M)$ is in S , this proves our claim. \square

Theorem 2.9. *Let S be a Melkersson subcategory with respect to (I, J) . Suppose M is Weakly Laskerian module. Then $T_{I,J}^s(M) = \inf\{T_{\mathfrak{a}}^s(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$, where $T_{\mathfrak{a}}^s(M)$ is the least non-negative integer i such that $H_{\mathfrak{a}}^i(M)$ is not in S .*

Proof. It is enough, in view Definition 2.1 and Theorem 2.8, to show that, $T_{I,J}^s(M) > i$ if $i < T_{\mathfrak{a}}^s(M)$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$. To do this, let \mathfrak{a} be an arbitrary ideal in $\widetilde{W}(I, J)$. We prove this by induction on i . It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It follows from [16, Proposition 1.4] that $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$ for all $i \geq 1$. Hence, by replacing M with $M/\Gamma_{I,J}(M)$, we may assume that there exists an element $x \in I$ which is a non-zero divisor on M . Now, we may consider the exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ to obtain the exact sequences

$$\begin{aligned} H_{\mathfrak{a}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x} H_{\mathfrak{a}}^i(M) \rightarrow H_{\mathfrak{a}}^i(M/xM) \quad \text{and} \\ H_{I,J}^{i-1}(M/xM) \rightarrow H_{I,J}^i(M) \xrightarrow{x} H_{I,J}^i(M) \rightarrow H_{I,J}^i(M/xM). \end{aligned}$$

Now, one can use the above exact sequences in conjunction with the inductive hypothesis to see that $(0 :_{H_{I,J}^i(M)} x)$ is in S . Since $(0 :_{H_{I,J}^i(M)} I) \subseteq (0 :_{H_{I,J}^i(M)} x)$, and hence the result follows.

This shows that the study of generalized local cohomology in a Melkersson subcategory in the upper range depends on the ideals of $\widetilde{W}(I, J)$. \square

Theorem 2.10. *Let M be Weakly Laskerian module and let r be a non-negative integer such that $H_{I,J}^r(R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \text{Supp}(M)$. Then $H_{I,J}^r(M) \in S$.*

Proof. Clearly, there exists a filtration of the submodules of M

$$0 \subseteq M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$$

such that for each $1 \leq j \leq \ell$, then $M_j/M_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \in \text{Supp}_R M$. We use induction on ℓ . When $\ell = 1$, $H_{I,J}^r(R/\mathfrak{p}) = H_{I,J}^r(M)$ is in S , where we put $\mathfrak{p} = \mathfrak{p}_j$. Now Suppose that $\ell > 1$ and the result has been proved for $\ell - 1$. The exact sequence

$$0 \longrightarrow M_{\ell-1} \longrightarrow M_\ell \longrightarrow M_\ell/M_{\ell-1} \longrightarrow 0$$

induces the long exact sequence

$$H_{I,J}^r(M_{\ell-1}) \longrightarrow H_{I,J}^r(M_\ell) \longrightarrow H_{I,J}^r(M_\ell/M_{\ell-1}).$$

It follows that $H_{I,J}^r(M_\ell)$ is in S . This completes the proof. \square

Definition 2.11. (see [17, Definition 2.1]) An R -module T is called (I, J) -cofinite if $\text{Supp} T \subseteq W(I, J)$ and $\text{Ext}_R^i(R/I, T)$ is a finite R -modules, for every $i \geq 0$. Whence according to [9, Lemma 2], the class of (I, J) -cofinite minimax (Artinian) modules is closed under taking submodules, quotients and extensions, it is a Serre subcategory of the category of R -modules.

The following result is an application of the Theorem 2.10.

Corollary 2.12. *Let M be a finitely generated R -module with $\dim M = d$. Then $H_{I,J}^d(M)$ is Artinian and (I, J) -cofinite.*

Proof. Let S be the class of (I, J) -cofinite Artinian modules. It is enough, in view Theorem 2.10, to show that R -module $H_{I,J}^d(R/\mathfrak{p})$ is Artinian and (I, J) -cofinite for all $\mathfrak{p} \in \text{Supp} M$. If $J \subseteq \mathfrak{p}$, then R/\mathfrak{p} is J -torsion and then $H_{I,J}^d(R/\mathfrak{p}) \cong H_I^d(R/\mathfrak{p})$. Since $\dim R/\mathfrak{p} \leq d$, then, in view of [12, Proposition 5.1], $H_I^d(R/\mathfrak{p})$ is Artinian and I -cofinite. If $J \not\subseteq \mathfrak{p}$, then $\dim(R/\mathfrak{p})/J(R/\mathfrak{p}) < \dim(R/\mathfrak{p}) \leq d$ and so $H_{I,J}^d(R/\mathfrak{p}) = 0$ by [16, Theorem 4.3]. The proof is completed. \square

Theorem 2.13. *Let M be a Weakly Laskerian module. Then $H_{I,J}^i(R/\mathfrak{p})$ is in S for all $i > T_s^{I,J}(M)$ and $\mathfrak{p} \in \text{Supp}(M)$.*

Proof. We use descending induction on i . Now, assume that $i > T_s^{I,J}(M)$ and that the claim holds for $i + 1$. We want to show that $H_{I,J}^i(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \text{Supp}(M)$. Suppose the contrary. We set: $A = \{\mathfrak{p} \mid \mathfrak{p} \in \text{Supp}M, H_{I,J}^i(R/\mathfrak{p}) \text{ is not in } S\}$. Clearly $A \neq \emptyset$; it follows that the set A has a maximal element, let \mathfrak{p} be one such. Since $\mathfrak{p} \in \text{Supp}(M)$, there is a non-zero map $f : M \rightarrow R/\mathfrak{p}$. The exact sequence $0 \rightarrow \text{Ker}f \rightarrow M \rightarrow \text{Im}f \rightarrow 0$, yields the exact sequence

$$H_{I,J}^i(M) \rightarrow H_{I,J}^i(\text{Im}f) \rightarrow H_{I,J}^{i+1}(\text{Ker}f).$$

Since $\text{Supp}(\text{Ker}f) \subset \text{Supp}(M)$, it follows from the inductive hypothesis that the R -module $H_{I,J}^{i+1}(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \text{Supp}(\text{ker } f)$, so, that, in view of the Theorem 2.10, and the above exact sequence, the R -module $H_{I,J}^i(\text{Im}f)$ is in S . There is a filtration

$$0 = N_t \subset N_{t-1} \subset N_{t-2} \subset \dots \subset N_0 = \text{Coker}f$$

of submodules of $\text{Coker}f$, such that for each $0 \leq i \leq t$, $N_{i-1}/N_i \cong R/\mathfrak{q}_i$ where $\mathfrak{q}_i \in \text{Supp}(\text{Coker}f)$. Then by maximality of \mathfrak{p} , $H_{I,J}^i(R/\mathfrak{q}_i)$ is in S . Next the exact sequence $0 \rightarrow \text{Im}f \rightarrow R/\mathfrak{p} \rightarrow \text{Coker}f \rightarrow 0$, yields the exact sequence

$$H_{I,J}^i(\text{Im}f) \rightarrow H_{I,J}^i(R/\mathfrak{p}) \rightarrow H_{I,J}^i(\text{Coker}f).$$

It follows that $H_{I,J}^i(R/\mathfrak{p})$ is in S , which is a contradiction. \square

Lemma 2.14. *If N and M are Weakly Laskerian modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$, then $T_s^{I,J}(N) \leq T_s^{I,J}(M)$. In particular, if $\text{Supp}(N) = \text{Supp}(M)$ then $T_s^{I,J}(N) = T_s^{I,J}(M)$.*

Proof. It is enough to show that $H_{I,J}^i(N)$ is in S for all finite R -module N with $\text{Supp}N \subseteq \text{Supp}M$ and for all $i > T_s^{I,J}(M)$. In view of the previous theorem, $H_{I,J}^i(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \text{Supp}(M)$. Now, since $\text{Supp}(N) \subseteq \text{Supp}(M)$, the result follows by Theorem 2.10. \square

As an immediate result of Theorems (2.13) and (2.10), we have the following Corollary. This shows that the study of generalized local cohomology of Weakly Laskerian module M in a Serre subcategory in the lower range depends just on the support of module M .

Corollary 2.15. *Let M be a Weakly Laskerian module. Then*

$$T_s^{I,J}(M) = \text{Sup}\{T_s^{I,J}(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Supp}(M)\}.$$

Theorem 2.16. *Let $T_{I,J}^s(M) > 0$ and $\mathfrak{a} \in \widetilde{W}(I, J)$. If M has finite krull dimension, then $H_{I,J}^i(M)/\mathfrak{a}H_{I,J}^i(M)$ is in S for all $i \geq T_s^{I,J}(M) = t$.*

Proof. When $i > T_s^{I,J}(M)$, the result is clearly, it is enough to show that $H_{I,J}^i(M)/\mathfrak{a}H_{I,J}^i(M)$ is in S . We proceed by induction on $\dim M = n$. If $n = 0$, then M is \mathfrak{m} -torsion and there is nothing to prove. So let $n > 0$ and

suppose that the result has been proved for any finitely generated module N with $\dim(N) = n - 1$. Since $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$ for $i > 0$, we can assume that M is (I, J) -torsion. Thus, there is an element $x \in \mathfrak{a}$, such that x is a non-zero divisor on M . Now, one can complete the proof by using an argument similar to the proof of [7, Theorem 3.3]. \square

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