Abstract. In the present paper for two $\mathfrak{A}$-module Banach algebras $A$ and $B$, we investigate relations between $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, $\psi$-$\mathfrak{A}$-module approximate amenability of $B$, and $\varphi \oplus \psi$-$\mathfrak{A}$-module approximate amenability of $A \oplus B$ ($L^1$-direct sum of $A$ and $B$), where $\varphi \in \text{Hom}_A(A)$ and $\psi \in \text{Hom}_A(B)$.

Keywords: Banach algebra, Module derivation, Module approximate amenability.


1. Introduction

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if $A$ and $B$ are approximately amenable Banach algebras and one of $A$ or $B$ has a bounded approximate identity, then $A \oplus B$ is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have
studied \( \varphi \)-derivations, and \( \varphi \)-amenability of Banach algebra \( A \), whenever \( \varphi \) is a continuous homomorphism on \( A \) (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of \( A \oplus B \).

The organization of this paper is as follows:

Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for \( A \)-module Banach algebras \( A \) and \( B \) where each has a bounded approximate identity we show that \( A \) is \( \varphi \)-\( A \# \)-module approximately amenable and \( B \) is \( \psi \)-\( A \# \)-module approximately amenable if and only if \( A \oplus B \) is \( \varphi \oplus \psi \)-\( A \# \)-module approximately amenable.

In section 4 we show that if \( A \) has a bounded approximately identity and \( A, B \) and \( A \) are unital, then \( A \) is \( \varphi \)-\( A \# \)-module approximately amenable and \( B \) is \( \psi \)-\( A \# \)-module approximately amenable if and only if \( A \oplus B \) is \( \varphi \oplus \psi \)-\( A \# \)-module approximately amenable.

2. Preliminaries

Let \( \mathfrak{A} \) and \( A \) be Banach algebras such that \( A \) is a Banach \( \mathfrak{A} \)-bimodule with compatible actions given by

\[
\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \quad (a, b \in A, \alpha \in \mathfrak{A}).
\]

Let \( X \) be a Banach \( A \)-bimodule and a Banach \( \mathfrak{A} \)-bimodule with compatible left actions defined by

\[
\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x, \quad (a.x).a = \alpha.(x.a)
\]

\[(a \in A, \alpha \in \mathfrak{A}, x \in X), \quad (2.1)\]

and similar for the right or two-sided actions. Then we say that \( X \) is a Banach \( A\mathfrak{A} \)-module. A Banach \( A\mathfrak{A} \)-module \( X \) is called commutative \( A\mathfrak{A} \)-module, if \( \alpha.x = x.\alpha \) (\( \alpha \in \mathfrak{A}, x \in X \)). Note that in general, \( A \) does not satisfy the compatibility condition \( a.(\alpha.b) = (a.\alpha)b \) (\( a, b \in A, \alpha \in \mathfrak{A} \)).

If \( X \) is a commutative Banach \( A\mathfrak{A} \)-module, then so is \( X^* \), where the actions of \( A \) and \( \mathfrak{A} \) on \( X^* \) are defined as follows

\[
\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \quad \langle a.f, x \rangle = \langle f, x.a \rangle \quad (a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*),
\]

and similar for the right actions.

Let \( A \) and \( B \) be Banach \( \mathfrak{A} \)-bimodules. Then a \( \mathfrak{A} \)-module morphism from \( A \) to \( B \) is a norm continuous map \( h : A \rightarrow B \) with \( h(a \pm b) = h(a) \pm h(b) \) which is multiplicative, that is

\[
h(\alpha.a) = \alpha.h(a), \quad h(a.\alpha) = h(a).\alpha, \quad h(ab) = h(a)h(b) \quad (a \in A, b \in B, \alpha \in \mathfrak{A}).
\]
We denote by $\text{Hom}_A(A,B)$, the space of all such morphism and denote $\text{Hom}_A(A,A)$ by $\text{Hom}_A(A)$. In the case that $\mathfrak{A} = \mathbb{C}$, we denote $\text{Hom}_C(A,B)$ by $\text{Hom}(A,B)$ and denote $\text{Hom}_C(A,A)$ by $\text{Hom}(A)$.

Let $X$ be a Banach $A$-bimodule and let $\varphi \in \text{Hom}_A(A)$. A bounded map $D : A \to X$ is called a $\varphi$-$A$-module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \quad (a,b \in A),$$

(2.2)

and

$$D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}).$$

(2.3)

Although $D$ in general is not linear, but still its boundedness implies its norm continuity.

Let $X$ be a commutative Banach $A$-$\mathfrak{A}$-module. For every $x \in X$ define $ad^x_\varphi$ by $ad^x_\varphi(a) = \varphi(a).x - x.\varphi(a) \quad (a \in A)$. It is easily seen that $ad^x_\varphi$ is a $\varphi$-$\mathfrak{A}$-module derivation. A $\varphi$-$\mathfrak{A}$-module derivation $D$ is called $\varphi$-inner if there is $x \in X$ such that $D(a) = ad^x_\varphi(a) \quad (a \in A)$ and is called approximately $\varphi$-inner if there exists a net $(x_\alpha)_{\alpha} \subseteq X$ such that $D(a) = \lim_\alpha ad^x_{\alpha}(a) \quad (a \in A)$. A Banach algebra $A$ is called $\varphi$-$\mathfrak{A}$-module amenable if for any commutative Banach $A$-$\mathfrak{A}$-module $X$, each $\varphi$-$\mathfrak{A}$-module derivation $D : A \to X^*$ is $\varphi$-inner, and $A$ is called $\varphi$-$\mathfrak{A}$-module approximately amenable if each $\varphi$-$\mathfrak{A}$-module derivation $D : A \to X^*$ is approximately $\varphi$-inner (see [1, 15]).

In the case that $\mathfrak{A} = \mathbb{C}$, $\varphi$-$\mathfrak{A}$-module derivations (resp. $\varphi$-$\mathfrak{A}$-module amenable Banach algebras, $\varphi$-$\mathfrak{A}$-module approximately amenable Banach algebras) are called $\varphi$-derivation (resp. $\varphi$-amenable, $\varphi$-approximately amenable) (see [9, 10]).

3. **$\varphi \oplus \psi$-Module Approximate Amenability of the Direct Sum of Banach Algebras**

We commence this section with the following remark from [1]:

**Remark 3.1.** Assume that $A$ has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_\mathfrak{A}(A)$ denotes the algebra of $\mathfrak{A}$-multipliers of $A$, that is $M_\mathfrak{A}(A) = \{(T_1,T_2) : T_1,T_2 \in L_\mathfrak{A}(A) : T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b), a,b \in A\}$, where $L_\mathfrak{A}(A)$ is the space of all $\mathfrak{A}$-module morphisms on $A$. Then $M_\mathfrak{A}(A)$ is an $A$-$\mathfrak{A}$-module and $A$ embeds in $M_\mathfrak{A}(A)$ via $a \mapsto (L_a,R_a)$, where $L_a(b) = ab, R_a(b) = ba \quad (a,b \in A)$. For any element $T = (T_1,T_2)$ of $M_\mathfrak{A}(A)$ it is easy to see that $\|T_1\| = \|T_2\|$ and if we put $\|T\|$ equal to this common value, then $M_\mathfrak{A}(A)$ becomes a Banach $A$-$\mathfrak{A}$-module, and $A$ is dense in $M_\mathfrak{A}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \text{Hom}_\mathfrak{A}(A)$, then by continuity of $\varphi$ in the strict topology, it can be extended to an $\mathfrak{A}$-homomorphism $\tilde{\varphi} : M_\mathfrak{A}(A) \to M_\mathfrak{A}(A)$ defined by $\tilde{\varphi}(L_a,R_a) = (L_{\varphi(a)},R_{\varphi(a)})$. 


Proposition 3.2. Let $A$ be an $\mathcal{A}$-module Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and let $\varphi \in \text{Hom}_\mathcal{A}(A)$. Then $A$ is $\varphi$-$\mathcal{A}$-module approximately amenable if and only if $M_\mathcal{A}(A)$ is $\varphi$-$\mathcal{A}$-module approximately amenable.

Proof. Let $M_\mathcal{A}(A)$ be $\varphi$-$\mathcal{A}$-module approximately amenable and let $D : A \rightarrow X^*$ be a $\varphi$-$\mathcal{A}$-module derivation for some commutative Banach $A$-$\mathcal{A}$-module $X$. Then by the following actions

$T.x = \lim_\alpha T_1(e_\alpha).x, \quad x.T = \lim_\alpha x.T_2(e_\alpha) \quad (x \in X, T = (T_1, T_2) \in M_\mathcal{A}(A)),$

$X$ is a commutative Banach $M_\mathcal{A}(A)$-$\mathcal{A}$-module and by continuity of $D$ in the strict topology, it can be extended to a bounded $\varphi$-$\mathcal{A}$-derivation $\tilde{D} : M_\mathcal{A}(A) \rightarrow X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\varphi$-$\mathcal{A}$-module approximate amenability of $M_\mathcal{A}(A)$, it follows that there exists a net $(x^*_\beta)_\beta \subset X^*$ such that

$\tilde{D}(T) = \lim_\beta (\varphi(T).x^*_\beta - x^*_\beta.\varphi(T)).$

Hence for every $a \in A$ we have

$D(a) = \tilde{D}(L_a, R_a) = \lim_\beta \left( \varphi(L_a, R_a).x^*_\beta - x^*_\beta.\varphi(L_a, R_a) \right) = \lim_\beta \left( (L_{\varphi(a)}, R_{\varphi(a)}).x^*_\beta - x^*_\beta.(L_{\varphi(a)}, R_{\varphi(a)}) \right) = \lim_\beta \left( \varphi(a).x^*_\beta - \lim_\alpha x^*_\beta.(\varphi(a)) \right) = \lim_\beta \left( \varphi(a).x^*_\beta - x^*_\beta.\varphi(a) \right).$

This means that $D$ is approximately $\varphi$-inner and so $A$ is $\varphi$-$\mathcal{A}$-module approximately amenable.

Conversely, Suppose that $A$ is $\varphi$-$\mathcal{A}$-module approximately amenable. Let $X$ be a commutative Banach $M_\mathcal{A}(A)$-$\mathcal{A}$-module and let $D : M_\mathcal{A}(A) \rightarrow X^*$ be a $\varphi$-$\mathcal{A}$-module derivation. We consider the module actions of $A$ on $X$ by

$a.x = (L_a, R_a).x, \quad x.a = x.(L_a, R_a) \quad (a \in A, x \in X). \quad (3.1)$

Thus $X$ is a commutative Banach $A$-$\mathcal{A}$-module. Define $\tilde{D} : A \rightarrow X^*$ by $\tilde{D}(a) = D(L_a, R_a) \quad (a \in A)$. It is easy to see that $\tilde{D}$ is a $\varphi$-$\mathcal{A}$-module derivation and from the $\varphi$-$\mathcal{A}$-module approximate amenability of $A$, it follows that there exists a net $(x^*_\beta)_\beta \subset X^*$ such that

$\tilde{D}(a) = \lim_\beta \left( \varphi(a).x^*_\beta - x^*_\beta.\varphi(a) \right) \quad (a \in A).$

Then $D(L_a, R_a) = \lim_\beta \left( \varphi(L_a, R_a).x^*_\beta - x^*_\beta.\varphi(L_a, R_a) \right)$. Now by the continuity of $D$ and $\varphi$, and density of $A$ in $M_\mathcal{A}(A)$ in the strict topology, we conclude that

$D(T) = \lim_\beta \left( \varphi(T).x^*_\beta - x^*_\beta.\varphi(T) \right) \quad (T \in M_\mathcal{A}(A)).$

So $D$ is a approximately $\varphi$-inner. Therefore $M_\mathcal{A}(A)$ is $\varphi$-$\mathcal{A}$-module approximately amenable.
Let $I$ be a closed ideal of a Banach algebra $A$ with a bounded approximate identity $(e_\alpha)_\alpha$, and let $X$ be a commutative Banach $I$-$\mathfrak{A}$-module. Let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ be such that $\varphi|_I \subseteq I$, then $X$ is a commutative Banach $A$-$\mathfrak{A}$-module with the following actions

$$a.x = \lim_\alpha \varphi(e_\alpha)a.x, \quad x.a = \lim_\alpha x.\varphi(e_\alpha)a \quad (a \in A, x \in X). \quad (3.2)$$

**Proposition 3.3.** Let $I$ be a closed ideal of an $\mathfrak{A}$-module Banach algebra $A$ which has a bounded approximate identity $(e_\alpha)_\alpha$, and let $I$ be $\mathfrak{A}$-invariant, i.e. $\mathfrak{A}.I \subseteq I$. Let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ be such that $\varphi|_I \subseteq I$. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

**Proof.** Let $X$ be a commutative Banach $M_\mathfrak{A}(I)$-$\mathfrak{A}$-module, and $D: M_\mathfrak{A}(I) \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation. By the same actions as (3.1), we can consider $X$ as a commutative Banach $I$-$\mathfrak{A}$-module. So, by (3.2), $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. By definition of $M_\mathfrak{A}(I)$, there is an $\mathfrak{A}$-module morphism $h: A \to M_\mathfrak{A}(I)$ and $Doh$ is a module derivation on $A$, so it is approximately $\varphi$-inner. Hence $D$ is approximately $\varphi$-inner. Since $I$ has a bounded approximate identity, by Proposition 3.2, $I$ is $\varphi|_I$-$\mathfrak{A}$-module approximately amenable. \qed

Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras. It is well known that $A \oplus B$, the $l^1$-direct sum of $A$ and $B$, is a Banach algebra with respect to the canonical multiplication defined by $(a,b)(c,d) := (ac,bd)$, and is a Banach $\mathfrak{A}$-bimodule by the following actions

$$\alpha.(a,b) := (\alpha.a, \alpha.b), \quad (a,b).\alpha := (a.\alpha, b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$, then $\varphi \oplus \psi : A \oplus B \to A \oplus B$ defined by $\varphi \oplus \psi(a,b) = (\varphi(a), \psi(b))$ is an $\mathfrak{A}$-morphism on $A \oplus B$.

**Lemma 3.4.** Let $A$ be a unital $\mathfrak{A}$-module Banach algebra, $\varphi \in \text{Hom}_\mathfrak{A}(A)$, and let $D : A \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. If the left (resp. right, two-sided) action of $\varphi$ on $X^*$ is zero, then $D$ is $\varphi$-inner.

**Proof.** Let $e_A$ be the identity of $A$ and let the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero. We can easily show that $D = ad\tilde{\varphi}_{-D(e)}$ (resp. $D = ad\tilde{\varphi}_{D(e)}$, $D = 0$). So $D$ is $\varphi$-inner. \qed

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

**Proposition 3.5.** Let $A$ and $B$ be unital $\mathfrak{A}$-module Banach algebras with identities $e_A$ and $e_B$, respectively, and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$ such that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$, and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$). If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.
Proof. Let \( X \) be a commutative Banach \( A \oplus B - \mathfrak{A} \)-module and let \( D : A \oplus B \to X^* \) be a \( \varphi \oplus \psi - \mathfrak{A} \)-module derivation. Write \( Y_1 = \varphi(e_A).X^* \varphi(e_A) \), \( Y_2 = \psi(e_B).X^* \psi(e_B) \), \( Y_3 = \varphi(e_A).X^* \psi(e_B) \), \( Y_4 = \psi(e_B).X^* \varphi(e_A) \), \( Y_5 = (1 - \varphi(e_A))(1 - \psi(e_B)) \), \( Y_6 = (1 - \varphi(e_A)) \), \( Y_7 = \varphi(e_A).X^* \psi(e_B) \), \( Y_8 = \psi(e_B).X^* \varphi(e_A) \), \( Y_9 = (1 - \varphi(e_A)) \), \( Y_{10} = (1 - \psi(e_B)) \). Let \( Y_j \to Y_{j+1} \) be the associated projections. Thus \( X^* = \bigoplus Y_1 \oplus \bigoplus Y_2 \oplus \bigoplus Y_3 \oplus \bigoplus Y_4 \oplus \bigoplus Y_5 \oplus \bigoplus Y_6 \oplus \bigoplus Y_7 \oplus \bigoplus Y_8 \oplus \bigoplus Y_9 \).

Consider the derivations \( D_j = \pi_j \circ D \), so \( D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9 \). From the fact that \( \varphi(e_A) \cdot \alpha = \alpha \cdot \varphi(e_A) \) \( (\alpha \in \mathfrak{A}) \), and \( \psi(e_B) \cdot \alpha = \alpha \cdot \psi(e_B) \) \( (\alpha \in \mathfrak{A}) \), one can easily check that \( Y_j \) for \( j = 1, \ldots, 9 \) is a commutative Banach \( A \oplus B - \mathfrak{A} \)-module. Since the action of \( \varphi(A) \oplus \psi(B) \) on (at least) one side on \( Y_5 \) (resp. \( Y_6, Y_7, Y_8, Y_9 \)) is zero, by Lemma 3.4, we conclude that \( D_5 \) (resp. \( D_6, D_7, D_8, D_9 \)) is approximately \( \varphi \oplus \psi \)-inner.

From the \( \varphi \oplus \psi - \mathfrak{A} \)-module approximate amenability of \( A \), it follows that the \( \varphi \oplus \psi - \mathfrak{A} \)-module derivation \( A \oplus 0 \to \varphi(e_A).X^* \varphi(e_A) \) is approximately \( \varphi \oplus \psi \)-inner and since the action of \( 0 \oplus \psi(B) \) on \( \varphi(e_A).X^* \varphi(e_A) \) is zero, we conclude that \( D_1 \) is approximately \( \varphi \oplus \psi \)-inner. Similarly, the \( \varphi \oplus \psi - \mathfrak{A} \)-module derivation \( D_2 : A \oplus B \to \psi(e_B).X^* \psi(e_B) \) is approximately \( \varphi \oplus \psi \)-inner.

The right action of \( \varphi(A) \oplus 0 \) on \( \varphi(e_A).X^* \psi(e_B) \) is zero. Hence, by Lemma 3.4, \( D_3 \mid_{A \oplus 0} \) is \( \varphi \oplus \psi \)-inner. So there exists \( \xi \in \varphi(e_A).X^* \psi(e_B) \) such that

\[
D_3 \mid_{A \oplus B} (a,0) = \varphi(a) \cdot \xi - \xi \varphi(a) = (\varphi(a), \psi(b)) \varphi(e_A) \cdot \xi \psi(e_B),
\]

for every \( a \in A \) and \( b \in B \). Similarly, there exists \( \eta \in \varphi(e_A).X^* \psi(e_B) \) such that

\[
D_3 \mid_{0 \oplus B} (0,b) = \psi(b) \cdot \eta - \eta \psi(b) = -\varphi(e_A) \cdot \eta \psi(e_B) \cdot (\varphi(a), \psi(b)),
\]

for every \( a \in A \) and \( b \in B \). Hence

\[
D_3(a,b) = (\varphi(a), \psi(b)) \varphi(e_A) \cdot \xi \psi(e_B) - \varphi(e_A) \cdot \eta \psi(e_B) \cdot (\varphi(a), \psi(b)).
\]

Since \( D_3(e_A,e_B) = 0 \), it follows that

\[
0 = D_3(e_A,e_B) = \varphi(e_A) \cdot \xi \psi(e_B) - \varphi(e_A) \cdot \eta \psi(e_B).
\]

Then for every \( a \in A \) and \( b \in B \), we have

\[
D_3(a,b) = (\varphi(a), \psi(b)) \varphi(e_A) \cdot \xi \psi(e_B) - \varphi(e_A) \cdot \xi \psi(e_B) \cdot (\varphi(a), \psi(b)).
\]

Thus \( D_3 \) is \( \varphi \oplus \psi \)-inner. The same argument holds for the \( \varphi \oplus \psi - \mathfrak{A} \)-module derivations \( D_4 : A \oplus B \to \psi(e_B).X^* \varphi(e_A) \). Therefore \( D \) is approximately \( \varphi \oplus \psi \)-inner, and so \( A \oplus B \) is \( \varphi \oplus \psi - \mathfrak{A} \)-module approximately amenable.

\[\Box\]

**Lemma 3.6.** Let \( A \) and \( B \) be \( \mathfrak{A} \)-module Banach algebras, \( \varphi \in \text{Hom}_\mathfrak{A}(A) \) and \( \psi \in \text{Hom}_\mathfrak{A}(B) \). If there is a \( h \) in \( \text{Hom}_\mathfrak{A}(A,B) \) such that \( h \circ \varphi = \psi \circ h \) and the range of \( h \) is a dense subset of \( B \), then \( \varphi \mathfrak{A} \)-module approximate amenability of \( A \) implies \( \psi \mathfrak{A} \)-module approximate amenability of \( B \).
Proof. Let $D : B \to X^*$ be a $\psi$-$\mathfrak{A}$-module derivation for some commutative Banach $B$-$\mathfrak{A}$-module $X$. Then by the following actions

$$a \cdot x = h(a)x, \quad x \cdot a = xh(a) \quad (a \in A, x \in X),$$

$X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Let $\tilde{D} = D \circ h : A \to X^*$. One can easily prove that $D$ is a $\varphi$-$\mathfrak{A}$-module derivation. From the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x_\alpha^*)_\alpha$ in $X^*$ such that $\tilde{D}(a) = \lim_\alpha (\varphi(a) \cdot x_\alpha^* - x_\alpha^* \cdot \varphi(a))$ ($a \in A$). Now continuity and density of $h(A)$ in $B$, imply that $D$ is approximately $\psi$-inner. Therefore $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable.

Proposition 3.7. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$. If $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable or $B$ is not $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Proof. Suppose that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. The projection map $\pi : A \oplus B \to A$ determines an $\mathfrak{A}$-module epimorphism of $A \oplus B$ onto $A$ such that $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable, then by Lemma 3.6, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. This contradicts the fact that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Similarly, we can prove the result for $B$. \qed

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$, the unitization of $\mathfrak{A}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let $A$ be a Banach $\mathfrak{A}$-bimodule. Then $A$ is a Banach $\mathfrak{A}^\#$-module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, \quad a.(\alpha, \lambda) = a.\alpha + \lambda a \quad (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let $A^\# = (A \oplus \mathfrak{A}^\#, \cdot)$, where the multiplication $\cdot$ is defined through

$$(a, u) \cdot (b, v) = (ab + a.v + u.b, uv) \quad (a, b \in A, u, v \in \mathfrak{A}^\#).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv), \quad (a, v).u = (a.u, vu) \quad (a \in A, u, v \in \mathfrak{A}^\#),$$

$A^\#$ is a unital $\mathfrak{A}^\#$-module Banach algebra with the identity $1_A = (0, 1_{\mathfrak{A}^\#})$ (see [4]).

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$, one defines $\varphi^\# : A^\# \to A^\#$ by $\varphi^\#(a, u) = (\varphi(a), u)$ ($(a, u) \in A^\#$), then $\varphi^\# \in \text{Hom}_{\mathfrak{A}^\#}(A^\#)$.

The following proposition generalizes Proposition 2.7 of [5].

Theorem 3.8. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}^\#}(B)$. Then $A$
is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable.

Proof. Suppose that $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable. By Proposition 12 of [13], $A^\sharp$ is $\varphi^\#$-$\mathfrak{A}$-module approximately amenable and $B^\sharp$ is $\psi^\#$-$\mathfrak{A}$-module approximately amenable, so by Proposition 3.5, $A^\sharp \oplus B^\sharp$ is $\varphi^\#$-$\mathfrak{A}$-$\mathfrak{A}^\#$-module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^\#$-invariant ideal in $A^\sharp \oplus B^\sharp$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable.

$\square$

4. $\varphi \oplus \psi$-Module Approximate Amenability and $\varphi \oplus \psi$-Amenability of Direct Sum of Banach Algebras

We start this section with the following definition:

Definition 4.1. We say the Banach algebra $\mathfrak{A}$ acts trivially on $A$ from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where $f$ is a multiplicative linear functional on $\mathfrak{A}$.

We assume that $J_{A,\mathfrak{A}}$ is the closed linear span of

$$\{(a.\alpha)b - a(\alpha.b) \mid \alpha \in \mathfrak{A}, a, b \in A\},$$

in $A$. It follows immediately that $J_{A,\mathfrak{A}}$ is both $A$-submodule and $\mathfrak{A}$-submodule of $A$. So $\frac{A}{J_{A,\mathfrak{A}}}$ is both Banach $A$-module and $\mathfrak{A}$-module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

Lemma 4.2. Let $A$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and $J_0$ be a closed ideal of $A$ such that $J_{A,\mathfrak{A}} \subseteq J_0$. If $\frac{A}{J_0}$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $a.\alpha - \alpha.a \in J_0$, i.e., $\frac{A}{J_0}$ is commutative Banach $\mathfrak{A}$-module.

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_\mathfrak{A}(A)$, one defines $\overline{\varphi} : \frac{A}{J_{A,\mathfrak{A}}} \to \frac{A}{J_{B,\mathfrak{A}}}$ by $\overline{\varphi}(a + J_{A,\mathfrak{A}}) = \varphi(a) + J_{B,\mathfrak{A}}$, then $\overline{\varphi} \in \text{Hom}_\mathfrak{B}(\frac{A}{J_{A,\mathfrak{A}}}).$

Theorem 4.3. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$. Then the following statements are valid:

(i) $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable) if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module approximately amenable).

(ii) Let $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left by $f \in \text{Hom}_C(\mathfrak{A})$. Suppose that $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable), then $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-approximately amenable).
(iii) Let $\mathfrak{A}$ have a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \varphi'$-amenable (resp. $\varphi \oplus \varphi'$-approximately amenable). Then $A \oplus B$ is $\varphi \oplus \varphi'$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \varphi'$-$\mathfrak{A}$-module approximately amenable).

Proof. (i) Let $A \oplus B$ be $\varphi \oplus \varphi'$-$\mathfrak{A}$-module amenable, and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ be $\varphi \oplus \varphi'$-$\mathfrak{A}$-module derivation for some commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module $X$. Then $X$ becomes a $A \oplus B$-bimodule through the following actions

$$
(a, b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \quad (a \in A, b \in B, x \in X),
$$

and

$$
x.(a, b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B, x \in X).
$$

Hence $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define $\tilde{D} : A \oplus B \rightarrow X^*$ by

$$
\tilde{D}(a, b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B).
$$

It is easy to check that $\tilde{D}$ is a $\varphi \oplus \varphi'$-$\mathfrak{A}$-module derivation. From the $\varphi \oplus \varphi'$-$\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$
\tilde{D}(a, b) = \varphi \oplus \varphi'(a, b).x^* - x^*.\varphi \oplus \varphi(a, b) \quad (a \in A, b \in B).
$$

Thus

$$
D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \varphi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* - x^*.\varphi \oplus \varphi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).
$$

This means that $D$ is $\varphi \oplus \varphi'$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \varphi'$-$\mathfrak{A}$-module amenable.

Conversely, suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \varphi'$-$\mathfrak{A}$-module amenable. Let $D : A \oplus B \rightarrow X^*$ be a $\varphi \oplus \varphi'$-$\mathfrak{A}$-module derivation for some commutative Banach $A \oplus B$-$\mathfrak{A}$-module $X$. We consider the following module actions of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ on $X$,

$$(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a, b).x, \quad x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a, b),$$

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of $X$, we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$. Thus $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}) = 0$. So $X$ is a commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module.

Define $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ by

$$
\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a, b) \quad (a \in A, b \in B).
$$

Also using (2.2) and (2.3) we see that $D$ vanishes on $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. Hence $\tilde{D}$ is well defined. One can easily check that $\tilde{D}$ is a $\varphi \oplus \varphi'$-$\mathfrak{A}$-module derivation.
Now from the $\varphi \oplus \psi\mathfrak{A}$-module amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

It follows that

$$D(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus $D$ is $\varphi \oplus \psi$-inner. So $A \oplus B$ is $\varphi \oplus \psi\mathfrak{A}$-module amenable.

Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi\mathfrak{A}$-module approximately amenable if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \psi\mathfrak{A}$-module approximately amenable.

(ii) Let $A \oplus B$ be $\varphi \oplus \psi\mathfrak{A}$-module amenable and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ be a derivation for some Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-bimodule $X$. Then $X$ becomes a $A \oplus B$-bimodule through the actions as (4.1) and (4.2). Also $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is

$$\alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, \ x \in X).$$

Then $X$ is a commutative Banach $A \oplus B\mathfrak{A}$-module. Define

$$\Gamma : A \oplus B \longrightarrow \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}, \quad (a, b) + I \longmapsto (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}),$$

where $I = J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. It is routinely checked that $\Gamma$ defines an $\mathfrak{A}$-bimodule morphism. Let $\Pi : A \oplus B \longrightarrow A \oplus B$ be the quotient map, and let $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \longrightarrow X^*$. For every $(a, b), (a', b') \in A \oplus B$, we may easily prove that

$$\tilde{D}((a, b)(a', b')) = \tilde{D}(a, b).\varphi \oplus \psi(a', b') + \varphi \oplus \psi(a, b) \tilde{D}(a', b'),$$

and for every $(a, b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$, we have

$$\tilde{D}(\alpha.(a, b)) = \tilde{D}((\alpha.a, \alpha.b)) = \tilde{D}\left((f(\alpha)a, f(\alpha)b)\right) = D\left((f(\alpha)a + J_{A,\mathfrak{A}}, f(\alpha)b + J_{B,\mathfrak{A}})\right) = D\left(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) = f(\alpha)D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) = \alpha.D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) = \alpha.\tilde{D}(a, b),$$

where $\tilde{D}(a, b) = f(\alpha)D(a, b) + \varphi \oplus \psi(a, b)\tilde{D}(a', b')$.
and using Lemma 4.2, we have
\[
\tilde{D}((a.\alpha,b.\alpha)) = D((a.\alpha + J_{A,\mathfrak{A}}, b.\alpha + J_{B,\mathfrak{A}})) \\
= D((\alpha.a + J_{A,\mathfrak{A}}, \alpha.b + J_{B,\mathfrak{A}})) \\
= D(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\
= f(\alpha)D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\
= D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})).\alpha \\
= \tilde{D}(a,b).\alpha.
\]
Thus \( \tilde{D} \) is a \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module derivation and from the \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module amenability of \( A \oplus B \), it follows that there exists \( x^* \in X^* \) such that
\[
\tilde{D}(a,b) = \varphi \oplus \psi(a,b).x^* - x^*.\varphi \oplus \psi(a,b) \quad (a \in A, b \in B).
\]
It follows that
\[
\begin{align*}
D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\
&\quad - x^*.\varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).
\end{align*}
\]
So \( D \) is \( \varphi \oplus \psi \)-inner. Therefore \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \) is \( \varphi \oplus \psi \)-amenable.

(iii) Suppose that \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \) is \( \varphi \oplus \psi \)-amenable. Since \( \mathfrak{A} \) has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \) is \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module amenable. So by (i), \( A \oplus B \) is \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module amenable.

Similar relations can be obtained between the \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module approximate amenability of \( A \oplus B \) and \( \varphi \oplus \psi \)-approximate amenability of \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \). \( \square \)

**Proposition 4.4.** Let \( A \) be an \( \mathfrak{A} \)-module Banach algebra, where \( \mathfrak{A} \) acts on \( A \) trivially from the left by \( f \in \text{Hom}_C(\mathfrak{A}) \). Let \( \varphi \in \text{Hom}_\mathfrak{A}(A) \) and \( \frac{A}{J_{A,\mathfrak{A}}} \) be unital. If \( A \) is \( \varphi \)-\( \mathfrak{A} \)-module approximately amenable, then \( \frac{A}{J_{A,\mathfrak{A}}} \) is \( \varphi \)-approximately amenable.

**Proof.** Let \( X \) be a Banach \( \frac{A}{J_{A,\mathfrak{A}}} \)-bimodule and \( D : \frac{A}{J_{A,\mathfrak{A}}} \to X^* \) be a \( \varphi \)-derivation. Then \( X \) becomes a \( \mathfrak{A} \)-bimodule through the following actions
\[
a.x = (a + J_{A,\mathfrak{A}}).x, \quad x.a = x.(a + J_{A,\mathfrak{A}}) \quad (a \in A, x \in X),
\]
and \( X \) is an \( \mathfrak{A} \)-bimodule with \( f \)-trivial actions, that is \( \alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, x \in X) \). By Lemma 4.2, \( f(\alpha)a - a.\alpha \in J_{A,\mathfrak{A}} \quad (\alpha \in \mathfrak{A}, a \in A) \). So, \( f(\alpha)a + J_{A,\mathfrak{A}} = a.a + J_{A,\mathfrak{A}} \quad (a \in \mathfrak{A}, a \in A) \), and the actions of \( \mathfrak{A} \) and \( A \) on \( X \) are compatible. Thus \( X \) is a commutative Banach \( A-\mathfrak{A} \)-module. Let \( \tilde{D} : A \to X^* \) be defined by \( \tilde{D}(a) = D(a + J_{A,\mathfrak{A}}) \quad (a \in A) \). A similar argument as in the proof of Theorem 3.2 of [2], shows that \( \tilde{D} \) is approximately \( \varphi \)-inner. So, \( D \) is approximately \( \varphi \)-inner. Therefore \( \frac{A}{J_{A,\mathfrak{A}}} \) is \( \varphi \)-approximately amenable. \( \square \)
Theorem 4.5. Let $\mathfrak{A}$ have a bounded approximate identity, and let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, where $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left. Let $\varphi \in \text{Hom}_A(A)$, $\psi \in \text{Hom}_A(B)$, and let $A_{\mathfrak{A}}$ and $B_{\mathfrak{A}}$ be unital. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Proof. Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable. By Proposition 4.4, $A_{\mathfrak{A}}$ and $B_{\mathfrak{A}}$ are $\varphi$-approximately amenable and $\psi$-approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A} = \mathbb{C}$, we conclude that $A_{\mathfrak{A}} \oplus B_{\mathfrak{A}}$ is $\varphi \oplus \psi$-approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable. □

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References


