Generalized Approximate Amenability of Direct Sum of Banach Algebras

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Abstract. In the present paper for two \( \mathfrak{A} \)-module Banach algebras \( A \) and \( B \), we investigate relations between \( \varphi \)-\( \mathfrak{A} \)-module approximate amenability of \( A \), \( \psi \)-\( \mathfrak{A} \)-module approximate amenability of \( B \), and \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module approximate amenability of \( A \oplus B \) (\( \ell^1 \)-direct sum of \( A \) and \( B \)), where \( \varphi \in \text{Hom}_A(A) \) and \( \psi \in \text{Hom}_A(B) \).

Keywords: Banach algebra, Module derivation, Module approximate amenability.


1. Introduction

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if \( A \) and \( B \) are approximately amenable Banach algebras and one of \( A \) or \( B \) has a bounded approximate identity, then \( A \oplus B \) is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have
studied $\varphi$-derivations, and $\varphi$-amenability of Banach algebra $A$, whenever $\varphi$ is a continuous homomorphism on $A$ (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.

The organization of this paper is as follows: Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for $A$-module Banach algebras $A$ and $B$ where each has a bounded approximate identity we show that $A$ is $\varphi$-$A^\#$-module approximately amenable and $B$ is $\psi$-$A^\#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$A^\#$-module approximately amenable.

In section 4 we show that if $A$ has a bounded approximately identity and $A$ and $B$ are unital, then $A$ is $\varphi$-$A$-module approximately amenable and $B$ is $\psi$-$A$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$A$-module approximately amenable.

2. Preliminaries

Let $A$ and $B$ be Banach algebras such that $A$ is a Banach $A$-bimodule with compatible actions given by

$$
\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \quad (a, b \in A, \alpha \in A).
$$

Let $X$ be a Banach $A$-bimodule and a Banach $A$-bimodule with compatible left actions defined by

$$
\alpha.(a.x) = (\alpha.a)x, \quad a.(\alpha.x) = (a.\alpha)x, \quad (a,x).\alpha = \alpha.(x.a)
$$

$$(a \in A, \alpha \in A, x \in X), \quad (2.1)$$

and similar for the right or two-sided actions. Then we say that $X$ is a Banach $A$-$A$-module. A Banach $A$-$A$-module $X$ is called commutative $A$-$A$-module, if $\alpha.x = x.\alpha (\alpha \in A, x \in X)$. Note that in general, $A$ dose not satisfy the compatibility condition $a.(\alpha.b) = (a.\alpha).b$ $(a, b \in A, \alpha \in A)$.

If $X$ is a commutative Banach $A$-$A$-module, then so is $X^*$, where the actions of $A$ and $A$ on $X^*$ are defined as follows

$$
\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \quad \langle a.f, x \rangle = \langle f, x.a \rangle \quad (a \in A, \alpha \in A, x \in X, f \in X^*),
$$

and similar for the right actions.

Let $A$ and $B$ be Banach $A$-bimodules. Then a $A$-module morphism from $A$ to $B$ is a norm continuous map $h : A \to B$ with $h(a \pm b) = h(a) \pm h(b)$ which is multiplicative, that is

$$
h(\alpha.a) = \alpha.h(a), \quad h(a.\alpha) = h(a).\alpha, \quad h(ab) = h(a)h(b) \quad (a \in A, b \in B, \alpha \in A).$$
We denote by $\text{Hom}_\mathcal{A}(A, B)$, the space of all such morphism and denote $\text{Hom}_\mathcal{B}(A, A)$ by $\text{Hom}_\mathcal{B}(A)$. In the case that $\mathcal{A} = \mathbb{C}$, we denote $\text{Hom}_\mathcal{C}(A, B)$ by $\text{Hom}(A, B)$ and denote $\text{Hom}_\mathcal{C}(A, A)$ by $\text{Hom}(A).

Let $X$ be a Banach $A$-bimodule and let $\varphi \in \text{Hom}_\mathcal{A}(A)$. A bounded map $D : A \rightarrow X$ is called a $\varphi$-$\mathcal{A}$-module derivation if
\[
D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \quad (a, b \in A), \tag{2.2}
\]
and
\[
D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}). \tag{2.3}
\]

Although $D$ in general is not linear, but still its boundedness implies its norm continuity.

Let $X$ be a commutative Banach $A$-$\mathfrak{A}$-module. For every $x \in X$ define $ad^x_\varphi$ by $ad^x_\varphi(a) = \varphi(a).x - x.\varphi(a)$ ($a \in A$). It is easily seen that $ad^x_\varphi$ is a $\varphi$-$\mathfrak{A}$-module derivation. A $\varphi$-$\mathfrak{A}$-module derivation $D$ is called $\varphi$-inner if there is $x \in X$ such that $D(a) = ad^x_\varphi(a)$ ($a \in A$) and is called approximately $\varphi$-inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha ad^x_\alpha(a)$ ($a \in A$). A Banach algebra $A$ is called $\varphi$-$\mathfrak{A}$-module amenable if for any commutative Banach $A$-$\mathfrak{A}$-module $X$, each $\varphi$-$\mathfrak{A}$-module derivation $D : A \rightarrow X^*$ is $\varphi$-inner, and $A$ is called $\varphi$-$\mathfrak{A}$-module approximately amenable if each $\varphi$-$\mathfrak{A}$-module derivation $D : A \rightarrow X^*$ is approximately $\varphi$-inner (see [1, 15]).

In the case that $\mathcal{A} = \mathbb{C}$, $\varphi$-$\mathfrak{A}$-module derivations (resp. $\varphi$-$\mathfrak{A}$-module amenable Banach algebras, $\varphi$-$\mathfrak{A}$-module approximately amenable Banach algebras) are called $\varphi$-derivation (resp. $\varphi$-amenable, $\varphi$-approximately amenable) (see [9, 10]).

3. $\varphi \oplus \psi$-Module Approximate Amenability of the Direct Sum of Banach Algebras

We commence this section with the following remark from [1]:

Remark 3.1. Assume that $A$ has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_\mathcal{A}(A)$ denotes the algebra of $\mathcal{A}$-multipliers of $A$, that is $M_\mathcal{A}(A) = \{(T_1, T_2) : T_1, T_2 \in L_\mathcal{A}(A), T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b), a, b \in A\}$, where $L_\mathcal{A}(A)$ is the space of all $\mathcal{A}$-module morphisms on $A$. Then $M_\mathcal{A}(A)$ is an $A$-$\mathfrak{A}$-module and $A$ embeds in $M_\mathcal{A}(A)$ via $a \mapsto (L_a, R_a)$, where $L_a(b) = ab, R_a(b) = ba$ ($a, b \in A$). For any element $T = (T_1, T_2)$ of $M_\mathcal{A}(A)$ it is easy to see that $\|T_1\| = \|T_2\|$ and if we put $\|T\|$ equal to this common value, then $M_\mathcal{A}(A)$ becomes a Banach $A$-$\mathfrak{A}$-module, and $A$ is dense in $M_\mathcal{A}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \text{Hom}_\mathcal{A}(A)$, then by continuity of $\varphi$ in the strict topology, it can be extended to an $\mathfrak{A}$-homomorphism $\hat{\varphi} : M_\mathcal{A}(A) \rightarrow M_\mathcal{A}(A)$ defined by $\hat{\varphi}(L_a, R_a) = (L_{\varphi(a)}, R_{\varphi(a)})$. 
Proposition 3.2. Let $A$ be an $\mathfrak{A}$-module Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable if and only if $M_\mathfrak{A}(A)$ is $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable.

Proof. Let $M_\mathfrak{A}(A)$ be $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable and let $D : A \rightarrow X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. Then by the following actions

$$ T.x = \lim_\alpha T_1(e_\alpha).x, \quad x.T = \lim_\alpha x.T_2(e_\alpha) \quad (x \in X, T = (T_1, T_2) \in M_\mathfrak{A}(A)), $$

$X$ is a commutative Banach $M_\mathfrak{A}(A)$-$\mathfrak{A}$-module and by continuity of $D$ in the strict topology, it can be extended to a bounded $\tilde{\varphi}$-$\mathfrak{A}$-derivation $\tilde{D} : M_\mathfrak{A}(A) \rightarrow X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\tilde{\varphi}$-$\mathfrak{A}$-module approximate amenability of $M_\mathfrak{A}(A)$, it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$ \tilde{D}(T) = \lim_\beta (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)). $$

Hence for every $a \in A$ we have

$$ D(a) = \tilde{D}(L_a, R_a) = \lim_\beta (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a)) $n
$$
= \lim_\beta ((L_{\varphi(a)}, R_{\varphi(a)}).x_\beta^* - x_\beta^*.L_{\varphi(a)}(R_{\varphi(a)})) $n
$$
= \lim_\beta (\lim_\alpha L_{\varphi(a)}(e_\alpha).x_\beta^* - \lim_\alpha x_\beta^*.R_{\varphi(a)}(e_\alpha)) $n
$$
= \lim_\beta (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)).
$$

This means that $D$ is approximately $\varphi$-inner and so $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable.

Conversely, Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. Let $X$ be a commutative Banach $M_\mathfrak{A}(A)$-$\mathfrak{A}$-module and let $D : M_\mathfrak{A}(A) \rightarrow X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation. We consider the module actions of $A$ on $X$ by

$$ a.x = (L_a, R_a).x, \quad x.a = x.(L_a, R_a) \quad (a \in A, x \in X). \quad (3.1) $$

Thus $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Define $\tilde{D} : A \rightarrow X^*$ by $\tilde{D}(a) = D(L_a, R_a) \quad (a \in A)$. It is easy to see that $\tilde{D}$ is a $\varphi$-$\mathfrak{A}$-module derivation and from the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$ \tilde{D}(a) = \lim_\beta (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)) \quad (a \in A). $$

Then $D(L_a, R_a) = \lim_\beta (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a))$. Now by the continuity of $D$ and $\tilde{\varphi}$, and density of $A$ in $M_\mathfrak{A}(A)$ in the strict topology, we conclude that

$$ D(T) = \lim_\beta (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)) \quad (T \in M_\mathfrak{A}(A)). $$

So $D$ is a approximately $\tilde{\varphi}$-inner. Therefore $M_\mathfrak{A}(A)$ is $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable. \qed
Let $I$ be a closed ideal of a Banach algebra $A$ with a bounded approximate identity $(e_\alpha)_\alpha$, and let $X$ be a commutative Banach $I$-$\mathfrak{A}$-module. Let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ be such that $\varphi |_I \subset I$, then $X$ is a commutative Banach $A$-$\mathfrak{A}$-module with the following actions

\[ a.x = \lim_\alpha \varphi(e_\alpha)a.x, \quad x.a = \lim_\alpha x.\varphi(e_\alpha)a \quad (a \in A, x \in X). \tag{3.2} \]

**Proposition 3.3.** Let $I$ be a closed ideal of an $\mathfrak{A}$-module Banach algebra $A$ which has a bounded approximate identity $\{e_\alpha\}$, and let $I$ be $\mathfrak{A}$-invariant, i.e. $\mathfrak{A}I \subseteq I$. Let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ be such that $\varphi |_I \subset I$. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable, then $I$ is $\varphi |_I$-$\mathfrak{A}$-module approximately amenable.

**Proof.** Let $X$ be a commutative Banach $M_\mathfrak{A}(I)$-$\mathfrak{A}$-module, and $D : M_\mathfrak{A}(I) \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation. By the same actions as (3.1), we can consider $X$ as a commutative Banach $A$-$\mathfrak{A}$-module. So, by (3.2), $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. By definition of $M_\mathfrak{A}(I)$, there is an $\mathfrak{A}$-module morphism $h : A \to M_\mathfrak{A}(I)$ and $Doh$ is a module derivation on $A$, so it is approximately $\varphi$-inner. Hence $D$ is approximately $\varphi$-inner. Since $I$ has a bounded approximate identity, by Proposition 3.2, $I$ is $\varphi |_I$-$\mathfrak{A}$-module approximately amenable.

Let $A$ and $B$ be $\mathfrak{A}$-modules Banach algebras. It is well known that $A \oplus B$, the $l^1$-direct sum of $A$ and $B$, is a Banach algebra with respect to the canonical multiplication defined by $(a, b)(c, d) := (ac, bd)$, and is a Banach $\mathfrak{A}$-bimodule by the following actions

\[ \alpha.(a, b) := (\alpha.a, \alpha.b), \quad (a, b).\alpha := (a.\alpha, b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B). \]

We note that if $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$, then $\varphi \oplus \psi : A \oplus B \to A \oplus B$ defined by $\varphi \oplus \psi(a, b) = (\varphi(a), \psi(b))$ is an $\mathfrak{A}$-morphism on $A \oplus B$.

**Lemma 3.4.** Let $A$ be a unital $\mathfrak{A}$-module Banach algebra, $\varphi \in \text{Hom}_\mathfrak{A}(A)$, and let $D : A \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. If the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero, then $D$ is $\varphi$-inner.

**Proof.** Let $e_A$ be the identity of $A$ and let the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero. We can easily show that $D = ad_{-D(e)}^\varphi$ (resp. $D = ad_{D(e)}^\varphi$, $D = 0$). So $D$ is $\varphi$-inner.

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

**Proposition 3.5.** Let $A$ and $B$ be unital $\mathfrak{A}$-module Banach algebras with identities $e_A$ and $e_B$, respectively, and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$ such that $\varphi(e_A) = 1_A$, $\psi(e_B) = 1_B$, and $\psi(e_B) = \alpha \psi(e_B)$ ($\alpha \in \mathfrak{A}$). If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.
Proof. Let \( X \) be a commutative Banach \( A \oplus B \)-module and let \( D : A \oplus B \rightarrow X^* \) be a \( \varphi \oplus \psi \)-module derivation. Write \( Y_1 = \varphi(e_A).X^* \varphi(e_A), \ Y_2 = \psi(e_B).X^* \varphi(e_B), \ Y_3 = \varphi(e_A).X^* \psi(e_B), \ Y_4 = \psi(e_B).X^* \varphi(e_A), \ Y_5 = (1 - \varphi(e_A)).(1 - \psi(e_B)).X^* \varphi(e_A), \ Y_6 = (1 - \varphi(e_A)).(1 - \psi(e_B)).X^* \psi(e_B), \ Y_7 = \varphi(e_A).X^* (1 - \varphi(e_A)).(1 - \psi(e_B)), \ Y_8 = \psi(e_B).X^* (1 - \varphi(e_A)).(1 - \psi(e_B)), \ Y_9 = (1 - \varphi(e_A)).(1 - \psi(e_B)).X^* (1 - \varphi(e_A))(1 - \psi(e_B)) \) and let \( \pi_j : X^* \rightarrow Y_j \) be the associated projections. Thus \( X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9 \).

Consider the derivations \( D_j = \pi_j \circ D, \) so \( D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9. \) From the fact that \( \varphi(e_A).\alpha = \alpha.\varphi(e_A) (\alpha \in \mathfrak{A}), \) and \( \psi(e_B).\alpha = \alpha.\psi(e_B) (\alpha \in \mathfrak{A}), \) one can easily check that \( Y_j \) for \( j = 1, ..., 9 \) is a commutative Banach \( A \oplus B \)-module. Since the action of \( \varphi(A) \oplus \psi(B) \) on (at least) one side on \( Y_5 \) (resp. \( Y_6, Y_7, Y_8, Y_9 \)) is zero, by Lemma 3.4, we conclude that \( D_5 \) (resp. \( D_6, D_7, D_8, D_9 \)) is approximately \( \varphi \oplus \psi \)-inner.

From the \( \varphi \oplus \psi \)-module approximate amenability of \( A \), it follows that the \( \varphi \oplus \psi \)-module derivation \( A \oplus 0 \rightarrow \varphi(e_A).X^* \varphi(e_A) \) is approximately \( \varphi \oplus \psi \)-inner and since the action of \( 0 \oplus \psi(B) \) on \( \varphi(e_A).X^* \varphi(e_A) \) is zero, we conclude that \( D_1 \) is approximately \( \varphi \oplus \psi \)-inner. Similarly, the \( \varphi \oplus \psi \)-module derivation \( D_2 : A \oplus B \rightarrow \psi(e_B).X^* \psi(e_B) \) is approximately \( \varphi \oplus \psi \)-inner.

The right action of \( \varphi(A) \oplus 0 \) on \( \varphi(e_A).X^* \psi(e_B) \) is zero. Hence, by Lemma 3.4, \( D_3 |_{A \oplus 0} \) is \( \varphi \oplus \psi \)-inner. So there exists \( \xi \in \varphi(e_A).X^* \psi(e_B) \) such that
\[
D_3 |_{A \oplus 0} (a, 0) = \varphi(a).\xi - \xi.\varphi(a) = (\varphi(a), \psi(b)) \varphi(e_A).\xi.\psi(e_B),
\]
for every \( a \in A \) and \( b \in B. \) Similarly, there exists \( \eta \in \varphi(e_A).X^* \psi(e_B) \) such that
\[
D_3 |_{0 \oplus B} (0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)),
\]
for every \( a \in A \) and \( b \in B. \) Hence
\[
D_3(a, b) = (\varphi(a), \psi(b)) \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A) \eta.\psi(e_B)(\varphi(a), \psi(b)).
\]
Since \( D_3(e_A, e_B) = 0, \) it follows that
\[
0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A) \eta.\psi(e_B).
\]
Then for every \( a \in A \) and \( b \in B, \) we have
\[
D_3(a, b) = (\varphi(a), \psi(b)) \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A) \xi.\psi(e_B)(\varphi(a), \psi(b)).
\]
Thus \( D_3 \) is \( \varphi \oplus \psi \)-inner. The same argument holds for the \( \varphi \oplus \psi \)-module derivation \( D_4 : A \oplus B \rightarrow \psi(e_B).X^* \varphi(e_A). \) Therefore \( D \) is approximately \( \varphi \oplus \psi \)-inner, and so \( A \oplus B \) is \( \varphi \oplus \psi \)-module approximately amenable. \( \square \)

Lemma 3.6. Let \( A \) and \( B \) be \( \mathfrak{A} \)-module Banach algebras, \( \varphi \in \text{Hom}_\mathfrak{A}(A) \) and \( \psi \in \text{Hom}_\mathfrak{A}(B). \) If there is a \( h \) in \( \text{Hom}_\mathfrak{A}(A, B) \) such that \( h \circ \varphi = \psi \circ h \) and the range of \( h \) is a dense subset of \( B, \) then \( \varphi \)-module approximate amenability of \( A \) implies \( \psi \)-module approximate amenability of \( B. \)
Proof. Let $D : B \rightarrow X^*$ be a $\psi$-$\mathfrak{A}$-module derivation for some commutative Banach $B$-$\mathfrak{A}$-module $X$. Then by the following actions

$$a \cdot x = h(a).x, \quad x \cdot a = x.h(a) \quad (a \in A, x \in X),$$

$X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Let $\tilde{D} = D \circ h : A \rightarrow X^*$. One can easily prove that $D$ is a $\varphi$-$\mathfrak{A}$-module derivation. From the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x^*_\alpha)_\alpha$ in $X^*$ such that $\tilde{D}(a) = \lim_{\alpha} (\varphi(a) \cdot x^*_\alpha - x^*_\alpha \cdot \varphi(a))$ $(a \in A)$. Now continuity and density of $h(A)$ in $B$, imply that $D$ is approximately $\psi$-inner. Therefore $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable.

Proposition 3.7. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$. If $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable or $B$ is not $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Proof. Suppose that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. The projection map $\pi : A \oplus B \rightarrow A$ determines an $\mathfrak{A}$-module epimorphism of $A \oplus B$ onto $A$ such that $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable, then by Lemma 3.6, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. This contradicts the fact that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Similarly, we can prove the result for $B$.

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$, the unitization of $\mathfrak{A}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let $A$ be a Banach $\mathfrak{A}$-bimodule. Then $A$ is a Banach $\mathfrak{A}^\#$-module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, \quad a.(\alpha, \lambda) = a.\alpha + \lambda a \quad (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let $A^\sharp = (A \oplus \mathfrak{A}^\#, \bullet)$, where the multiplication $\bullet$ is defined through

$$(a, u) \bullet (b, v) = (ab + a.v + u.b, uv) \quad (a, b \in A, u, v \in \mathfrak{A}^\#).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv), \quad (a, v).u = (a.u, vu) \quad (a \in A, u, v \in \mathfrak{A}^\#),$$

$A^\sharp$ is a unital $\mathfrak{A}^\#$-module Banach algebra with the identity $1_{A^\sharp} = (0, 1_{\mathfrak{A}^\#})$ (see [4]).

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$, one defines $\varphi^\# : A^\sharp \rightarrow A^\sharp$ by $\varphi^\#(a, u) = (\varphi(a), u)$ ($(a, u) \in A^\sharp$), then $\varphi^\# \in \text{Hom}_{\mathfrak{A}^\#}(A^\sharp)$.

The following proposition generalizes Proposition 2.7 of [5].

Theorem 3.8. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}^\#}(B)$. Then $A$
is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable.

Proof. Suppose that $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable. By Proposition 12 of [13], $A^2$ is $\varphi^\#$-$\mathfrak{A}^\#$-module approximately amenable and $B^2$ is $\psi^\#$-$\mathfrak{A}^\#$-module approximately amenable, so by Proposition 3.5, $A^2 \oplus B^2$ is $\varphi^\# \oplus \psi^\#$-$\mathfrak{A}^\#$-module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^\#$-invariant ideal in $A^2 \oplus B^2$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable. $\square$

4. $\varphi \oplus \psi$-Module Approximate Amenability and $\varphi \oplus \psi$-Amenability of Direct Sum of Banach Algebras

We start this section with the following definition:

**Definition 4.1.** We say the Banach algebra $\mathfrak{A}$ acts trivially on $A$ from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where $f$ is a multiplicative linear functional on $\mathfrak{A}$.

We assume that $J_{\mathfrak{A},A}$ is the closed linear span of $\{(a.\alpha)b - a(\alpha.b) \mid \alpha \in \mathfrak{A}, a, b \in A\}$, in $A$. It follows immediately that $J_{\mathfrak{A},A}$ is both $A$-submodule and $\mathfrak{A}$-submodule of $A$. So $\frac{A}{J_{\mathfrak{A},A}}$ is both Banach $A$-module and $\mathfrak{A}$-module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

**Lemma 4.2.** Let $A$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and $J_0$ be a closed ideal of $A$ such that $J_{\mathfrak{A},A} \subseteq J_0$. If $\frac{A}{J_0}$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $\alpha.a - a.\alpha \in J_0$, i.e., $\frac{A}{J_0}$ is commutative Banach $\mathfrak{A}$-module.

Before we turn to our next result we note that if for every $\varphi \in \hom_{\mathfrak{A}}(A)$, one defines $\overline{\varphi} : \frac{A}{J_{\mathfrak{A},A}} \rightarrow \frac{J_{\mathfrak{A},A}}{J_{\mathfrak{A},A}}$ by $\overline{\varphi}(a + J_{\mathfrak{A},A}) = \varphi(a) + J_{\mathfrak{A},A}$, then $\overline{\varphi} \in \hom_{\mathfrak{A}}(\frac{A}{J_{\mathfrak{A},A}})$.

**Theorem 4.3.** Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and let $\varphi \in \hom_{\mathfrak{A}}(A)$ and $\psi \in \hom_{\mathfrak{A}}(B)$. Then the following statements are valid:

(i) $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable) if and only if $\frac{A}{J_{\mathfrak{A},A}} \oplus \frac{B}{J_{\mathfrak{B},B}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module approximately amenable).

(ii) Let $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left by $f \in \hom_{\mathfrak{A}}(\mathfrak{A})$. Suppose that $\frac{A}{J_{\mathfrak{A},A}}$ and $\frac{B}{J_{\mathfrak{B},B}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable), then $\frac{A}{J_{\mathfrak{A},A}} \oplus \frac{B}{J_{\mathfrak{B},B}}$ is $\overline{\varphi} \oplus \overline{\psi}$-amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-approximately amenable).
(iii) Let $\mathfrak{A}$ have a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \bar{\psi}$-amenable (resp. $\varphi \oplus \bar{\psi}$-approximately amenable). Then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable).

Proof. (i) Let $A \oplus B$ be $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable, and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ be $\varphi \oplus \bar{\psi}$-$\mathfrak{A}$-module derivation for some commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module $X$. Then $X$ becomes a $A \oplus B$-bimodule through the following actions

$$(a, b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \quad (a \in A, b \in B, x \in X),$$

and

$$x.(a, b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B, x \in X).$$

Hence $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define $\tilde{D} : A \oplus B \longrightarrow X^*$ by

$$\tilde{D}(a, b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B).$$

It is easy to check that, $\tilde{D}$ is a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. From the $\varphi \oplus \psi$-$\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus

$$D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^*$$

$$- x^*.\varphi \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).$$

This means that $D$ is $\varphi \oplus \bar{\psi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \bar{\psi}$-$\mathfrak{A}$-module amenable.

Conversely, suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \bar{\psi}$-$\mathfrak{A}$-module amenable. Let $D : A \oplus B \longrightarrow X^*$ be a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation for some commutative Banach $A \oplus B$-$\mathfrak{A}$-module $X$. We consider the following module actions of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ on $X$,

$$(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a, b).x, \quad x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a, b),$$

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of $X$, we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$. Thus $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}) = 0$. So $X$ is a commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module. Define $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ by

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a, b) \quad (a \in A, b \in B).$$

Also using (2.2) and (2.3) we see that $D$ vanishes on $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. Hence $\tilde{D}$ is well defined. One can easily check that $\tilde{D}$ is a $\varphi \oplus \bar{\psi}$-$\mathfrak{A}$-module derivation.
Now from the $\varphi \oplus \psi$-$\mathfrak{A}$-module amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \cdot x^*$$

$$- x^* \cdot \varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) (a \in A, b \in B).$$

It follows that

$$D(a, b) = \varphi \oplus \psi(a, b) \cdot x^* - x^* \cdot \varphi \oplus \psi(a, b) (a \in A, b \in B).$$

Thus $D$ is $\varphi \oplus \psi$-inner. So $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable.

Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

(ii) Let $A \oplus B$ be $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \to X^*$ be a derivation for some Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-bimodule $X$. Then $X$ becomes a $A \oplus B$-bimodule through the actions as (4.1) and (4.2). Also $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is

$$\alpha \cdot x = x \cdot \alpha = f(\alpha) x (\alpha \in \mathfrak{A}, x \in X).$$

Then $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define

$$\Gamma : A \oplus B \to \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}, (a, b) + I \mapsto (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}),$$

where $I = J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. It is routinely checked that $\Gamma$ defines an $\mathfrak{A}$-bimodule morphism. Let $\Pi : A \oplus B \to A \ominus B$ be the quotient map, and let $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \to X^*$. For every $(a, b), (a', b') \in A \oplus B$, we may easily prove that

$$\tilde{D}((a, b)(a', b')) = \tilde{D}(a, b) \cdot \varphi \oplus \psi(a', b') + \varphi \oplus \psi(a, b) \cdot \tilde{D}(a', b'),$$

and for every $(a, b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$, we have

$$\tilde{D}(\alpha(a, b)) = \tilde{D}((\alpha.a, \alpha.b)) = \tilde{D}((f(\alpha)a, f(\alpha)b))$$

$$= D((f(\alpha)a + J_{A,\mathfrak{A}}, f(\alpha)b + J_{B,\mathfrak{A}}))$$

$$= D(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}))$$

$$= f(\alpha) \tilde{D}((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}))$$

$$= \alpha \tilde{D}((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}))$$

$$= \alpha \tilde{D}(a, b),$$
and using Lemma 4.2, we have
\[
\tilde{D}(a, b) = \tilde{D}(a, \alpha) = D((a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}}) = D((a, a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}}))
\]
\[
= D\left((\alpha, a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}})\right)
\]
\[
= D\left(f(\alpha)(a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}})\right)
\]
\[
= f(\alpha)D((a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}})) = D((a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}}), \alpha)
\]
\[
= \tilde{D}(a, b)\alpha.
\]
Thus \(\tilde{D}\) is a \(\varphi \oplus \psi\)-module derivation and from the \(\varphi \oplus \psi\)-module amenability of \(A \oplus B\), it follows that there exists \(x^* \in X^*\) such that
\[
\tilde{D}(a, b) = \varphi \oplus \psi(a, b) x^* - x^* \varphi \psi(a, b) \quad (a \in A, b \in B).
\]
It follows that
\[
D(a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}}) = \overline{\varphi} \oplus \overline{\psi}(a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}}) x^* - x^* \overline{\varphi} \overline{\psi}(a + J_{A, \mathfrak{A}}, b + J_{B, \mathfrak{A}}).
\]
So \(D\) is \(\overline{\varphi} \oplus \overline{\psi}\)-inner. Therefore \(\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}\) is \(\overline{\varphi} \oplus \overline{\psi}\)-amenable.

(iii) Suppose that \(\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}\) is \(\overline{\varphi} \oplus \overline{\psi}\)-amenable. Since \(\mathfrak{A}\) has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that \(\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}\) is \(\varphi \oplus \psi\)-module amenable. So by (i), \(A \oplus B\) is \(\varphi \oplus \psi\)-module amenable.

Similar relations can be obtained between the \(\varphi \oplus \psi\)-module approximate amenability of \(A \oplus B\) and \(\varphi \oplus \overline{\psi}\)-approximate amenability of \(\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}\).

**Proposition 4.4.** Let \(A\) be an \(\mathfrak{A}\)-module Banach algebra, where \(\mathfrak{A}\) acts on \(A\) trivially from the left by \(f \in \text{Hom}_\mathbb{C}(\mathfrak{A})\). Let \(\varphi \in \text{Hom}_\mathbb{C}(A)\) and \(\frac{A}{J_{A, \mathfrak{A}}}\) be unital. If \(A\) is \(\varphi\)-\(\mathfrak{A}\)-module approximately amenable, then \(\frac{A}{J_{A, \mathfrak{A}}}\) is \(\overline{\varphi}\)-approximately amenable.

**Proof.** Let \(X\) be a Banach \(\frac{A}{J_{A, \mathfrak{A}}}\)-bimodule and \(D : \frac{A}{J_{A, \mathfrak{A}}} \rightarrow X^*\) be a \(\overline{\varphi}\)-derivation. Then \(X\) becomes an \(\mathfrak{A}\)-bimodule through the following actions
\[
a.x = (a + J_{A, \mathfrak{A}}) x, \quad x.a = x.(a + J_{A, \mathfrak{A}}) \quad (a \in A, x \in X),
\]
and \(X\) is an \(\mathfrak{A}\)-bimodule with \(f\)-trivial actions, that is \(\alpha.x = x.\alpha = f(\alpha)x\) \((\alpha \in \mathfrak{A}, x \in X)\). By Lemma 4.2, \(f(\alpha)a - a.\alpha \in J_{A, \mathfrak{A}}\) \((\alpha \in \mathfrak{A}, a \in A)\). So, \(f(\alpha)a + J_{A, \mathfrak{A}} = a.a + J_{A, \mathfrak{A}}\) \((\alpha \in \mathfrak{A}, a \in A)\), and the actions of \(\mathfrak{A}\) and \(A\) on \(X\) are compatible. Thus \(X\) is a commutative Banach \(A\)-\(\mathfrak{A}\)-module. Let \(\tilde{D} : A \rightarrow X^*\) be defined by \(\tilde{D}(a) = D(a + J_{A, \mathfrak{A}})\) \((a \in A)\). A similar argument as in the proof of Theorem 3.2 of [2], shows that \(\tilde{D}\) is approximately \(\varphi\)-inner. So, \(D\) is approximately \(\overline{\varphi}\)-inner. Therefore \(\frac{A}{J_{A, \mathfrak{A}}}\) is \(\overline{\varphi}\)-approximately amenable. □
Theorem 4.5. Let $\mathfrak{A}$ have a bounded approximate identity, and let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, where $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, and let $\frac{A}{J_{\mathfrak{A},A}}$ and $\frac{B}{J_{\mathfrak{B},A}}$ be unital. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Proof. Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable. By Proposition 4.4, $\frac{A}{J_{\mathfrak{A},A}}$ and $\frac{B}{J_{\mathfrak{B},A}}$ are $\varphi$-approximately amenable and $\psi$-approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A} = \mathbb{C}$, we conclude that $\frac{A}{J_{\mathfrak{A},A}} \oplus \frac{B}{J_{\mathfrak{B},A}}$ is $\varphi \oplus \psi$-approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable. □

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References


