Generalized Approximate Amenability of Direct Sum of Banach Algebras

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Abstract. In the present paper for two \(A\)-module Banach algebras \(A\) and \(B\), we investigate relations between \(\phi\)-\(A\)-module approximate amenability of \(A\), \(\psi\)-\(A\)-module approximate amenability of \(B\), and \(\phi \oplus \psi\)-\(A\)-module approximate amenability of \(A \oplus B\) (\(l^1\)-direct sum of \(A\) and \(B\)), where \(\phi \in \text{Hom}_A(A)\) and \(\psi \in \text{Hom}_A(B)\).

Keywords: Banach algebra, Module derivation, Module approximate amenability.


1. Introduction

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if \(A\) and \(B\) are approximately amenable Banach algebras and one of \(A\) or \(B\) has a bounded approximate identity, then \(A \oplus B\) is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have...
studied $\varphi$-derivations, and $\varphi$-amenability of Banach algebra $A$, whenever $\varphi$ is a continuous homomorphism on $A$ (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.

The organization of this paper is as follows:

Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for $\mathcal{A}$-module Banach algebras $A$ and $B$ where each has a bounded approximate identity we show that $A$ is $\varphi$-$\mathcal{A}$#$-module approximately amenable and $B$ is $\psi$-$\mathcal{A}$#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathcal{A}$#$-module approximately amenable.

In section 4 we show that if $A$ has a bounded approximately identity and $A_J A$, $B_J B$, $A$ are unital, then $A$ is $\varphi$-$\mathcal{A}$-module approximately amenable and $B$ is $\psi$-$\mathcal{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathcal{A}$-module approximately amenable.

2. Preliminaries

Let $\mathcal{A}$ and $A$ be Banach algebras such that $A$ is a Banach $\mathcal{A}$-bimodule with compatible actions given by

$$\alpha.(ab) = (\alpha.a)b, \ (ab).\alpha = a(b.\alpha) \quad (a, b \in A, \alpha \in \mathcal{A}).$$

Let $X$ be a Banach $A$-bimodule and a Banach $\mathcal{A}$-bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \ a.(\alpha.x) = (a.\alpha).x, \ (a.x).a = \alpha.(x.a)$$

$$\quad (a \in A, \alpha \in \mathcal{A}, x \in X), \quad (2.1)$$

and similar for the right or two-sided actions. Then we say that $X$ is a Banach $A$-$\mathcal{A}$-module. A Banach $A$-$\mathcal{A}$-module $X$ is called commutative $A$-$\mathcal{A}$-module, if $\alpha.x = x.\alpha$ ($\alpha \in \mathcal{A}, x \in X$). Note that in general, $A$ does not satisfy the compatibility condition $a.(\alpha.b) = (a.\alpha).b$ ($a, b \in A, \alpha \in \mathcal{A}$).

If $X$ is a commutative Banach $A$-$\mathcal{A}$-module, then so is $X^*$, where the actions of $A$ and $\mathcal{A}$ on $X^*$ are defined as follows

$$\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \ \langle a.f, x \rangle = \langle f, x.a \rangle \ (a \in A, \alpha \in \mathcal{A}, x \in X, f \in X^*),$$

and similar for the right actions.

Let $A$ and $B$ be Banach $\mathcal{A}$-bimodules. Then a $\mathcal{A}$-module morphism from $A$ to $B$ is a norm continuous map $h : A \rightarrow B$ with $h(a \pm b) = h(a) \pm h(b)$ which is multiplicative, that is

$$h(\alpha.a) = a.h(\alpha), \ h(a.\alpha) = h(a).\alpha, \ h(ab) = h(a)h(b) \quad (a \in A, b \in B, \alpha \in \mathcal{A}).$$
We denote by $\text{Hom}_A(A,B)$, the space of all such morphism and denote $\text{Hom}_C(A,A)$ by $\text{Hom}_C(A)$. In the case that $\mathfrak{A} = \mathbb{C}$, we denote $\text{Hom}_C(A,B)$ by $\text{Hom}(A,B)$ and denote $\text{Hom}_C(A,A)$ by $\text{Hom}(A)$.

Let $X$ be a Banach $A$-bimodule and let $\varphi \in \text{Hom}_A(A)$. A bounded map $D : A \rightarrow X$ is called a $\varphi$-$\mathfrak{A}$-module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \quad (a, b \in A), \quad (2.2)$$

and

$$D(a\alpha) = \alpha D(a), \quad D(a,\alpha) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}). \quad (2.3)$$

Although $D$ in general is not linear, but still its boundedness implies its norm continuity.

Let $X$ be a commutative Banach $A$-$\mathfrak{A}$-module. For every $x \in X$ define $ad^x_\mathfrak{A}$ by $ad^x_\mathfrak{A}(a) = \varphi(a).x - x.\varphi(a) \quad (a \in A)$. It is easily seen that $ad^x_\mathfrak{A}$ is a $\varphi$-$\mathfrak{A}$-module derivation. A $\varphi$-$\mathfrak{A}$-module derivation $D$ is called $\varphi$-inner if there is $x \in X$ such that $D(a) = ad^x_\mathfrak{A}(a) \quad (a \in A)$ and is called approximately $\varphi$-inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha ad^x_\alpha(a) \quad (a \in A)$. A Banach algebra $A$ is called $\varphi$-$\mathfrak{A}$-module amenable if for any commutative Banach $A$-$\mathfrak{A}$-module $X$, each $\varphi$-$\mathfrak{A}$-module derivation $D : A \rightarrow X^*$ is $\varphi$-inner, and $A$ is called $\varphi$-$\mathfrak{A}$-module approximately amenable if each $\varphi$-$\mathfrak{A}$-module derivation $D : A \rightarrow X^*$ is approximately $\varphi$-inner (see [1, 15]).

In the case that $\mathfrak{A} = \mathbb{C}$, $\varphi$-$\mathfrak{A}$-module derivations (resp. $\varphi$-$\mathfrak{A}$-module amenable Banach algebras, $\varphi$-$\mathfrak{A}$-module approximately amenable Banach algebras) are called $\varphi$-derivation (resp. $\varphi$-amenable, $\varphi$-approximately amenable) (see [9, 10]).

3. $\varphi \oplus \psi$-Module Approximate Amenability of the Direct Sum of Banach Algebras

We commence this section with the following remark from [1]:

Remark 3.1. Assume that $A$ has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_\mathfrak{A}(A)$ denotes the algebra of $\mathfrak{A}$-multipliers of $A$, that is $M_\mathfrak{A}(A) = \{(T_1, T_2) : T_1, T_2 \in L_\mathfrak{A}(A) : T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b) \quad (a, b \in A)\}$, where $L_\mathfrak{A}(A)$ is the space of all $\mathfrak{A}$-module morphisms on $A$. Then $M_\mathfrak{A}(A)$ is an $A$-$\mathfrak{A}$-module and $A$ embeds in $M_\mathfrak{A}(A)$ via $a \mapsto (L_a, R_a)$, where $L_a(b) = ab, R_a(b) = ba \quad (a, b \in A)$. For any element $T = (T_1, T_2)$ of $M_\mathfrak{A}(A)$ it is easy to see that $\| T_1 \| = \| T_2 \|$ and if we put $\| T \|$ equal to this common value, then $M_\mathfrak{A}(A)$ becomes a Banach $A$-$\mathfrak{A}$-module, and $A$ is dense in $M_\mathfrak{A}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \text{Hom}_\mathfrak{A}(A)$, then by continuity of $\varphi$ in the strict topology, it can be extended to an $\mathfrak{A}$-homomorphism $\tilde{\varphi} : M_\mathfrak{A}(A) \rightarrow M_\mathfrak{A}(A)$ defined by $\tilde{\varphi}(L_a, R_a) = (L_{\varphi(a)}, R_{\varphi(a)})$. 

Proposition 3.2. Let $A$ be an $\mathfrak{A}$-module Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable if and only if $M_\mathfrak{A}(A)$ is $\varphi$-$\mathfrak{A}$-module approximately amenable.

Proof. Let $M_\mathfrak{A}(A)$ be $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable and let $D : A \longrightarrow X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. Then by the following actions

$$T.x = \lim_{\alpha} T_1(e_\alpha).x, \quad x.T = \lim_{\alpha} x.T_2(e_\alpha) \quad (x \in X, T = (T_1, T_2) \in M_\mathfrak{A}(A)),$$

$X$ is a commutative Banach $M_\mathfrak{A}(A)$-$\mathfrak{A}$-module and by continuity of $D$ in the strict topology, it can be extended to a bounded $\tilde{\varphi}$-$\mathfrak{A}$-derivation $\tilde{D} : M_\mathfrak{A}(A) \longrightarrow X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\tilde{\varphi}$-$\mathfrak{A}$-module approximate amenability of $M_\mathfrak{A}(A)$, it follows that there exists a net $(x^*_\beta)_{\beta} \subset X^*$ such that

$$\tilde{D}(T) = \lim_{\beta} (\tilde{\varphi}(T).x^*_\beta - x^*_\beta.\tilde{\varphi}(T)).$$

Hence for every $a \in A$ we have

$$D(a) = \tilde{D}(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x^*_\beta - x^*_\beta.\tilde{\varphi}(L_a, R_a))$$

$$= \lim_{\beta} \left( (L_{\varphi(a)}, R_{\varphi(a)}).x^*_\beta - x^*_\beta.(L_{\varphi(a)}, R_{\varphi(a)}) \right)$$

$$= \lim_{\beta} \left( \lim_{\alpha} L_{\varphi(a)}(e_\alpha).x^*_\beta - \lim_{\alpha} x^*_\beta. R_{\varphi(a)}(e_\alpha) \right)$$

$$= \lim_{\beta} (\varphi(a).x^*_\beta - x^*_\beta.\varphi(a)).$$

This means that $D$ is approximately $\varphi$-inner and so $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable.

Conversely, Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. Let $X$ be a commutative Banach $M_\mathfrak{A}(A)$-$\mathfrak{A}$-module and let $D : M_\mathfrak{A}(A) \longrightarrow X^*$ be a $\tilde{\varphi}$-$\mathfrak{A}$-module derivation. We consider the module actions of $A$ on $X$ by

$$a.x = (L_a, R_a).x, \quad x.a = x.(L_a, R_a) \quad (a \in A, x \in X). \quad (3.1)$$

Thus $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Define $\tilde{D} : A \longrightarrow X^*$ by

$$\tilde{D}(a) = D(L_a, R_a) \quad (a \in A).$$

It is easy to see that $\tilde{D}$ is a $\varphi$-$\mathfrak{A}$-module derivation and from the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x^*_\beta)_{\beta} \subset X^*$ such that

$$\tilde{D}(a) = \lim_{\beta} (\varphi(a).x^*_\beta - x^*_\beta.\varphi(a)) \quad (a \in A).$$

Then $D(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x^*_\beta - x^*_\beta.\tilde{\varphi}(L_a, R_a)).$ Now by the continuity of $D$ and $\tilde{\varphi}$, and density of $A$ in $M_\mathfrak{A}(A)$ in the strict topology, we conclude that

$$D(T) = \lim_{\beta} (\tilde{\varphi}(T).x^*_\beta - x^*_\beta.\tilde{\varphi}(T)) \quad (T \in M_\mathfrak{A}(A)).$$

So $D$ is a approximately $\tilde{\varphi}$-inner. Therefore $M_\mathfrak{A}(A)$ is $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable. \hfill \square
Let $I$ be a closed ideal of a Banach algebra $A$ with a bounded approximate identity $(e_{\alpha})_n$, and let $X$ be a commutative Banach $I$-$\mathfrak{A}$-module. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi |_I \subset I$, then $X$ is a commutative Banach $A$-$\mathfrak{A}$-module with the following actions

$$a.x = \lim_{\alpha} \varphi(e_{\alpha})a.x, \quad x.a = \lim_{\alpha} x.\varphi(e_{\alpha})a \quad (a \in A, x \in X).$$  \hspace{1cm} (3.2)

**Proposition 3.3.** Let $I$ be a closed ideal of an $\mathfrak{A}$-module Banach algebra $A$ which has a bounded approximate identity $\{e_{\alpha}\}$, and let $I$ be $\mathfrak{A}$-invariant, i.e. $\mathfrak{A}.I \subseteq I$. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi|_I \subset I$. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable, then $I$ is $\varphi|_I$-$\mathfrak{A}$-module approximately amenable.

**Proof.** Let $X$ be a commutative Banach $M_{\mathfrak{A}}(I)$-$\mathfrak{A}$-module, and $D : M_{\mathfrak{A}}(I) \rightarrow X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation. By the same actions as (3.1), we can consider $X$ as a commutative Banach $I$-$\mathfrak{A}$-module. So, by (3.2), $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. By definition of $M_{\mathfrak{A}}(I)$, there is an $\mathfrak{A}$-module morphism $h : A \rightarrow M_{\mathfrak{A}}(I)$ and $Doh$ is a module derivation on $A$, so it is approximately $\varphi$-inner. Hence $D$ is approximately $\varphi$-inner. Since $I$ has a bounded approximate identity, by Proposition 3.2, $I$ is $\varphi|_I$-$\mathfrak{A}$-module approximately amenable. □

Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras. It is well known that $A \oplus B$, the $l^1$-direct sum of $A$ and $B$, is a Banach algebra with respect to the canonical multiplication defined by $(a,b)(c,d) := (ac,bd)$, and is a Banach $\mathfrak{A}$-bimodule by the following actions

$$\alpha.(a,b) := (\alpha.a,\alpha.b), \quad (a,b).\alpha := (a.\alpha,b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, then $\varphi \oplus \psi : A \oplus B \rightarrow A \oplus B$ defined by $\varphi \oplus \psi(a,b) = (\varphi(a),\psi(b))$ is an $\mathfrak{A}$-morphism on $A \oplus B$.

**Lemma 3.4.** Let $A$ be a unital $\mathfrak{A}$-module Banach algebra, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, and let $D : A \rightarrow X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. If the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero, then $D$ is $\varphi$-inner.

**Proof.** Let $e_A$ be the identity of $A$ and let the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero. We can easily show that $D = ad_{D(e)}(\varphi)$ (resp. $D = ad_{D(e)}^\varphi$, $D = 0$). So $D$ is $\varphi$-inner. □

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

**Proposition 3.5.** Let $A$ and $B$ be unital $\mathfrak{A}$-module Banach algebras with identities $e_A$ and $e_B$, respectively, and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ such that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$, and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ $(\alpha \in \mathfrak{A})$. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.
Proof. Let $X$ be a commutative Banach $A \oplus B$-module and let $D : A \oplus B \rightarrow X^*$ be a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. Write $Y_1 = \varphi(e_A).X^.*.\varphi(e_A)$, $Y_2 = \psi(e_B).X^.*.\psi(e_B)$, $Y_3 = \varphi(e_A).X^.*.\psi(e_B)$, $Y_4 = \psi(e_B).X^.*.\varphi(e_A)$, $Y_5 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^.*.\varphi(e_A)$, $Y_6 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^.*.\psi(e_B)$, $Y_7 = \varphi(e_A).X^.*.(1 - \varphi(e_A))(1 - \psi(e_B))$, $Y_8 = \psi(e_B).X^.*.(1 - \varphi(e_A))(1 - \psi(e_B))$, $Y_9 = (1 - \varphi(e_A))(1 - \psi(e_B))$ and let $\pi_j : X^* \rightarrow Y_j$ be the associated projections. Thus $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$.

Consider the derivations $D_j = \pi_j \circ D$, so $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$. From the fact that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$ ($\alpha \in \mathfrak{A}$), and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$), one can easily check that $Y_j$ for $j = 1, ..., 9$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Since the action of $\varphi(A) \oplus \psi(B)$ on (at least) one side on $Y_5$ (resp. $Y_6, Y_7, Y_8, Y_9$) is zero, by Lemma 3.4, we conclude that $D_5$ (resp. $D_6, D_7, D_8, D_9$ ) is approximately $\varphi \oplus \psi$-inner.

From the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $A \oplus 0 \rightarrow \varphi(e_A).X^.*.\varphi(e_A)$ is approximately $\varphi \oplus \psi$-inner and since the action of $0 \oplus \psi(B)$ on $\varphi(e_A).X^.*.\varphi(e_A)$ is zero, we conclude that $D_1$ is approximately $\varphi \oplus \psi$-inner. Similarly, the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_2 : A \oplus B \rightarrow \psi(e_B).X^.*.\psi(e_B)$ is approximately $\varphi \oplus \psi$-inner.

The right action of $\varphi(A) \oplus 0$ on $\varphi(e_A).X^.*.\psi(e_B)$ is zero. Hence, by Lemma 3.4, $D_3 |_{A \oplus 0}$ is $\varphi \oplus \psi$-inner. So there exists $\xi \in \varphi(e_A).X^.*.\psi(e_B)$ such that

$$D_3 |_{A \oplus 0} (a, 0) = \varphi(a).\xi - \xi.\varphi(a) = (\varphi(a), \psi(b)).\varphi(e_A).\xi.\psi(e_B),$$

for every $a \in A$ and $b \in B$. Similarly, there exists $\eta \in \varphi(e_A).X^.*.\psi(e_B)$ such that

$$D_3 |_{0 \oplus B} (0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)),$$

for every $a \in A$ and $b \in B$. Hence

$$D_3(a, b) = (\varphi(a), \psi(b)).\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)).$$

Since $D_3(e_A, e_B) = 0$, it follows that

$$0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B).$$

Then for every $a \in A$ and $b \in B$, we have

$$D_3(a, b) = (\varphi(a), \psi(b)).\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\xi.\psi(e_B)(\varphi(a), \psi(b)).$$

Thus $D_3$ is $\varphi \oplus \psi$-inner. The same argument holds for the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_4 : A \oplus B \rightarrow \psi(e_B).X^.*.\varphi(e_A)$. Therefore $D$ is approximately $\varphi \oplus \psi$-inner, and so $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. □

Lemma 3.6. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. If there is a $h$ in $\text{Hom}_{\mathfrak{A}}(A, B)$ such that $h \circ \varphi = \psi \circ h$ and the range of $h$ is a dense subset of $B$, then $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$ implies $\psi$-$\mathfrak{A}$-module approximate amenability of $B$. 

Proof. Let $D : B \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $B$-$\mathfrak{A}$-module $X$. Then by the following actions

$$a \cdot x = h(a).x, \quad x \cdot a = x.h(a) \quad (a \in A, x \in X),$$

$X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Let $\tilde{D} = D \circ h : A \to X^*$. One can easily prove that $D$ is a $\varphi$-$\mathfrak{A}$-module derivation. From the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x_\alpha^*)$ in $X^*$ such that $\tilde{D}(a) = \lim_{\alpha} (\varphi(a) \cdot x_\alpha^* - x_\alpha^* \cdot \varphi(a))$ ($a \in A$). Now continuity and density of $h(A)$ in $B$, imply that $D$ is approximately $\psi$-inner. Therefore $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable.

**Proposition 3.7.** Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. If $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable or $B$ is not $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Proof. Suppose that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. The projection map $\pi : A \oplus B \to A$ determines an $\mathfrak{A}$-module epimorphism of $A \oplus B$ onto $A$ such that $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable, then by Lemma 3.6, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. This contradicts the fact that $A$ is not $\varphi$-$\mathfrak{A}$-module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Similarly, we can prove the result for $B$. □

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$, the unitization of $\mathfrak{A}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let $A$ be a Banach $\mathfrak{A}$-bimodule. Then $A$ is a Banach $\mathfrak{A}^\#$-module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, \quad a.(\alpha, \lambda) = a.\alpha + \lambda a \quad (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let $A^\# = (A \oplus \mathfrak{A}^\#, \bullet)$, where the multiplication $\bullet$ is defined through

$$(a, u) \bullet (b, v) = (ab + a.v + u.b, uv) \quad (a, b \in A, u, v \in \mathfrak{A}^\#).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv), \quad (a, v).u = (a.u, vu) \quad (a \in A, u, v \in \mathfrak{A}^\#),$$

$A^\#$ is a unital $\mathfrak{A}^\#$-module Banach algebra with the identity $1_A = (0, 1_{\mathfrak{A}^\#})$ (see [4]).

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$, one defines $\varphi^\#: A^\# \to A^\#$ by $\varphi^\#(a, u) = (\varphi(a), u)$ $((a, u) \in A^\#$), then $\varphi^\# \in \text{Hom}_{\mathfrak{A}^\#}(A^\#)$.

The following proposition generalizes Proposition 2.7 of [5].

**Theorem 3.8.** Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}^\#}(B)$. Then $A$
is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable.

Proof. Suppose that $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable. By Proposition 12 of [13], $A^2$ is $\varphi^\#$-$\mathfrak{A}^\#$-module approximately amenable and $B^2$ is $\psi^\#$-$\mathfrak{A}^\#$-module approximately amenable, so by Proposition 3.5, $A^2 \oplus B^2$ is $\varphi^\# \oplus \psi^\#$-$\mathfrak{A}^\#$-module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^\#$-invariant ideal in $A^2 \oplus B^2$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^\#$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}^\#$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}^\#$-module approximately amenable. □

4. $\varphi \oplus \psi$-Module Approximate Amenability and $\varphi \oplus \psi$-Amenability of Direct Sum of Banach Algebras

We start this section with the following definition:

Definition 4.1. We say the Banach algebra $\mathfrak{A}$ acts trivially on $A$ from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where $f$ is a multiplicative linear functional on $\mathfrak{A}$.

We assume that $J_{A,\mathfrak{A}}$ is the closed linear span of

$$\{(a.\alpha) - b.a \mid \alpha \in \mathfrak{A}, a, b \in A\},$$

in $A$. It follows immediately that $J_{A,\mathfrak{A}}$ is both $A$- submodule and $\mathfrak{A}$-submodule of $A$. So $\frac{A}{J_{A,\mathfrak{A}}}$ is both Banach $A$-module and $\mathfrak{A}$-module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

Lemma 4.2. Let $A$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and $J_0$ be a closed ideal of $A$ such that $J_{A,\mathfrak{A}} \subseteq J_0$. If $\frac{A}{J_0}$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $a.\alpha - \alpha.a \in J_0$, i.e., $\frac{A}{J_0}$ is commutative Banach $\mathfrak{A}$-module.

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_A(A)$, one defines $\overline{\varphi} : \frac{A}{J_{A,\mathfrak{A}}} \to \frac{A}{J_{B,\mathfrak{A}}}$ by $\overline{\varphi}(a + J_{A,\mathfrak{A}}) = \varphi(a) + J_{B,\mathfrak{A}}$, then $\overline{\varphi} \in \text{Hom}_A(\frac{A}{J_{A,\mathfrak{A}}})$.

Theorem 4.3. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and let $\varphi \in \text{Hom}_A(A)$ and $\psi \in \text{Hom}_B(B)$. Then the following statements are valid:

(i) $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable) if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable).

(ii) Let $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left by $f \in \text{Hom}_C(\mathfrak{A})$. Suppose that $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable), then $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \psi$-amenable (resp. $\varphi \oplus \psi$-approximately amenable).
(iii) Let $\mathfrak{A}$ have a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \overline{\psi}$-amenable (resp. $\varphi \oplus \overline{\psi}$-approximately amenable). Then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable).

Proof. (i) Let $A \oplus B$ be $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable, and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \to X^*$ be $\varphi \oplus \overline{\psi}$-$\mathfrak{A}$-module derivation for some commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module $X$. Then $X$ becomes a $A \oplus B$-bimodule through the following actions

\[
(a, b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \quad (a \in A, b \in B, x \in X),
\]

and

\[
x.(a, b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B, x \in X).
\]

Hence $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define $\tilde{D} : A \oplus B \to X^*$ by

\[
\tilde{D}(a, b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B).
\]

It is easy to check that, $\tilde{D}$ is a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. From the $\varphi \oplus \psi$-$\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

\[
\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).
\]

Thus

\[
D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* - x^*.\varphi \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).
\]

This means that $D$ is $\varphi \oplus \overline{\psi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable.

Conversely, suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\varphi \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable. Let $\tilde{D} : A \oplus B \to X^*$ be a $\varphi \oplus \overline{\psi}$-$\mathfrak{A}$-module derivation for some commutative Banach $A \oplus B$-$\mathfrak{A}$-module $X$. We consider the following module actions of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ on $X$,

\[
(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a, b).x, \quad x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a, b),
\]

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of $X$, we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$. Thus $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}) = 0$. So $X$ is a commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module.

Define $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \to X^*$ by

\[
\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a, b) \quad (a \in A, b \in B).
\]

Also using (2.2) and (2.3) we see that $D$ vanishes on $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. Hence $\tilde{D}$ is well defined. One can easily check that $\tilde{D}$ is a $\varphi \oplus \overline{\psi}$-$\mathfrak{A}$-module derivation.
Now from the $\varphi \oplus \psi - \mathfrak{A}$-module amenability of $\frac{A}{J_{A,\varphi}} \oplus \frac{B}{J_{B,\psi}}$, it follows that there exists $x^* \in X^*$ such that

$$
\tilde{D}(a + J_{A,\varphi}, b + J_{B,\psi}) = \varphi \oplus \psi(a + J_{A,\varphi}, b + J_{B,\psi}).x^* \\
- x^*.\varphi \oplus \psi(a + J_{A,\varphi}, b + J_{B,\psi}) \quad (a \in A, b \in B).
$$

It follows that

$$
D(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).
$$

Thus $D$ is $\varphi \oplus \psi$-inner. So $A \oplus B$ is $\varphi \oplus \psi - \mathfrak{A}$-module amenable.

Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi - \mathfrak{A}$-module approximately amenable if and only if $\frac{A}{I} \oplus \frac{B}{I}$ is $\varphi \oplus \psi - \mathfrak{A}$-module approximately amenable.

(ii) Let $A \oplus B$ be $\varphi \oplus \psi - \mathfrak{A}$-module amenable and let $D : \frac{A}{J_{A,\varphi}} \oplus \frac{B}{J_{B,\psi}} \to X^*$ be a derivation for some Banach $\frac{A}{J_{A,\varphi}} \oplus \frac{B}{J_{B,\psi}}$-bimodule $X$. Then $X$ becomes an $A \oplus B$-bimodule through the actions as (4.1) and (4.2). Also $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is

$$
\alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, x \in X).
$$

Then $X$ is a commutative Banach $A \oplus B - \mathfrak{A}$-module. Define

$$
\Gamma : \frac{A \oplus B}{I} \to \frac{A}{J_{A,\varphi}} \oplus \frac{B}{J_{B,\psi}}, \quad (a, b) + I \mapsto (a + J_{A,\varphi}, b + J_{B,\psi}),
$$

where $I = J_{A,\varphi} \oplus J_{B,\psi}$. It is routinely checked that $\Gamma$ defines an $\mathfrak{A}$-bimodule morphism. Let $\Pi : A \oplus B \to \frac{A \oplus B}{I}$ be the quotient map, and let $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \to X^*$. For every $(a, b), (a', b') \in A \oplus B$, we may easily prove that

$$
\tilde{D}((a, b)(a', b')) = \tilde{D}(a, b).\varphi \oplus \psi(a', b') + \varphi \oplus \psi(a, b).\tilde{D}(a', b'),
$$

and for every $(a, b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$ , we have

$$
\tilde{D}(\alpha.(a, b)) = \tilde{D}((\alpha.a, \alpha.b)) = \tilde{D}((f(\alpha)a, f(\alpha)b)) \\
= D\left(f(\alpha)a + J_{A,\varphi}, f(\alpha)b + J_{B,\psi}\right) \\
= D\left(f(\alpha)(a + J_{A,\varphi}, b + J_{B,\psi})\right) \\
= D\left(f(\alpha)(a + J_{A,\varphi}, b + J_{B,\psi})\right) \\
= f(\alpha)D\left((a + J_{A,\varphi}, b + J_{B,\psi})\right) \\
= \alpha.D\left((a + J_{A,\varphi}, b + J_{B,\psi})\right) \\
= \alpha.\tilde{D}(a, b),
$$
and using Lemma 4.2, we have
\[
\tilde{D}((a, b).\alpha) = \tilde{D}(a.\alpha, b.\alpha) = D((a.\alpha + J_{A,\mathfrak{A}}, b.\alpha + J_{B,\mathfrak{A}})) = D\left((a.a + J_{A,\mathfrak{A}}, b.a + J_{B,\mathfrak{A}})\right) = D\left(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) = f(\alpha)D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) = D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right). \alpha = \tilde{D}(a, b).\alpha.
\]

Thus \(\tilde{D}\) is a \(\varphi+\psi\mathfrak{A}\)-module derivation and from the \(\varphi+\psi\mathfrak{A}\)-module amenability of \(A \oplus B\), it follows that there exists \(x^* \in X^*\) such that
\[
\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).
\]

It follows that
\[
D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* - x^*.\varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).
\]

So \(D\) is \(\varphi \oplus \overline{\psi}\)-inner. Therefore \(\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}\) is \(\varphi \oplus \overline{\psi}\)-amenable.

(iii) Suppose that \(\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}\) is \(\varphi \oplus \overline{\psi}\)-amenable. Since \(\mathfrak{A}\) has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that \(\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}\) is \(\varphi \oplus \overline{\psi}\)-\(\mathfrak{A}\)-module amenable. So by (i), \(A \oplus B\) is \(\varphi \oplus \psi\mathfrak{A}\)-module amenable.

Similar relations can be obtained between the \(\varphi \oplus \psi\mathfrak{A}\)-module approximate amenability of \(A \oplus B\) and \(\varphi \oplus \overline{\psi}\)-approximate amenability of \(\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}\). □

**Proposition 4.4.** Let \(A\) be an \(\mathfrak{A}\)-module Banach algebra, where \(\mathfrak{A}\) acts on \(A\) trivially from the left by \(f \in \text{Hom}_\mathbb{C}(\mathfrak{A})\). Let \(\varphi \in \text{Hom}_\mathfrak{A}(A)\) and \(\frac{A}{J_{A,\mathfrak{A}}}\) be unital. If \(A\) is \(\varphi\mathfrak{A}\)-module approximately amenable, then \(\frac{A}{J_{A,\mathfrak{A}}}\) is \(\varphi\)-approximately amenable.

**Proof.** Let \(X\) be a Banach \(\frac{A}{J_{A,\mathfrak{A}}}\)-bimodule and \(D : \frac{A}{J_{A,\mathfrak{A}}} \rightarrow X^*\) be a \(\varphi\)-derivation. Then \(X\) becomes a \(\mathfrak{A}\)-bimodule through the following actions
\[
a.x = (a + J_{A,\mathfrak{A}}).x, \quad x.a = x.(a + J_{A,\mathfrak{A}}) \quad (a \in A, x \in X),
\]

and \(X\) is an \(\mathfrak{A}\)-bimodule with \(f\)-trivial actions, that is \(\alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, x \in X)\). By Lemma 4.2, \(f(\alpha)a - a.\alpha \in J_{A,\mathfrak{A}} \quad (\alpha \in \mathfrak{A}, a \in A)\). So, \(f(\alpha)a + J_{A,\mathfrak{A}} = a.\alpha + J_{A,\mathfrak{A}} \quad (\alpha \in \mathfrak{A}, a \in A)\), and the actions of \(\mathfrak{A}\) and \(A\) on \(X\) are compatible. Thus \(X\) is a commutative Banach \(A-\mathfrak{A}\)-module. Let \(\tilde{D} : A \rightarrow X^*\) be defined by \(\tilde{D}(a) = D(a + J_{A,\mathfrak{A}}) \quad (a \in A)\). A similar argument as in the proof of Theorem 3.2 of [2], shows that \(\tilde{D}\) is approximately \(\varphi\)-inner. So, \(D\) is approximately \(\varphi\)-inner. Therefore \(\frac{A}{J_{A,\mathfrak{A}}}\) is \(\varphi\)-approximately amenable. □
Theorem 4.5. Let \( A \) have a bounded approximate identity, and let \( A \) and \( B \) be \( A \)-module Banach algebras, where \( A \) acts on \( A \) and \( B \) trivially from the left. Let \( \varphi \in \text{Hom}_A(A) \), \( \psi \in \text{Hom}_A(B) \), and let \( \frac{A}{J_{A,A}} \) and \( \frac{B}{J_{B,B}} \) be unital. Then \( A \) is \( \varphi \)-\( A \)-module approximately amenable and \( B \) is \( \psi \)-\( A \)-module approximately amenable if and only if \( A \oplus B \) is \( \varphi \oplus \psi \)-\( A \)-module approximately amenable.

Proof. Suppose that \( A \) is \( \varphi \)-\( A \)-module approximately amenable and \( B \) is \( \psi \)-\( A \)-module approximately amenable. By Proposition 4.4, \( \frac{A}{J_{A,A}} \) and \( \frac{B}{J_{B,B}} \) are \( \varphi \)-approximately amenable and \( \psi \)-approximately amenable, respectively. Now by using Proposition 3.5 for \( A = \mathbb{C} \), we conclude that \( \frac{A}{J_{A,A}} \oplus \frac{B}{J_{B,B}} \) is \( \varphi \oplus \psi \)-approximately amenable. So, Theorem 4.3, implies that \( A \oplus B \) is \( \varphi \oplus \psi \)-\( A \)-module approximately amenable.

Conversely, suppose that \( A \oplus B \) is \( \varphi \oplus \psi \)-\( A \)-module approximately amenable. Then by Proposition 3.7, \( A \) is \( \varphi \)-\( A \)-module approximately amenable and \( B \) is \( \psi \)-\( A \)-module approximately amenable.

\( \square \)

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References


