# Generalized Approximate Amenability of Direct Sum of Banach Algebras 

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#### Abstract

In the present paper for two $\mathfrak{A}$-module Banach algebras $A$ and $B$, we investigate relations between $\varphi$ - $\mathfrak{A}$-module approximate amenability of $A, \psi$ - $\mathfrak{A}$-module approximate amenability of $B$, and $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximate amenability of $A \oplus B\left(l^{1}\right.$-direct sum of $A$ and $\left.B\right)$, where $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \operatorname{Hom}_{\mathfrak{A}}(B)$.


Keywords: Banach algebra, Module derivation, Module approximate amenability.

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## 1. Introduction

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if A and B are approximately amenable Banach algebras and one of A or B has a bounded approximate identity, then $A \oplus B$ is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have
studied $\varphi$-derivations, and $\varphi$-amenability of Banach algebra $A$, whenever $\varphi$ is a continuous homomorphism on $A$ (see $[8,9,10,11,12]$ ).

The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.
The organization of this paper is as follows:
Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for $\mathfrak{A}$-module Banach algebras $A$ and $B$ where each has a bounded approximate identity we show that $A$ is $\varphi$ - $\mathfrak{A}^{\#}$-module approximately amenable and $B$ is $\psi$ - $\mathfrak{A}^{\#}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}^{\#}$-module approximately amenable.

In section 4 we show that if $\mathfrak{A}$ has a bounded approximately identity and $\frac{A}{J_{A, \mathfrak{A}}}$ and $\frac{B}{J_{B, \mathfrak{A l}}}$ are unital, then $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable and $B$ is $\psi$ - $\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable.

## 2. Preliminaries

Let $\mathfrak{A}$ and $A$ be Banach algebras such that $A$ is a Banach $\mathfrak{A}$-bimodule with compatible actions given by

$$
\alpha \cdot(a b)=(\alpha \cdot a) b, \quad(a b) \cdot \alpha=a(b \cdot \alpha) \quad(a, b \in A, \alpha \in \mathfrak{A})
$$

Let $X$ be a Banach $A$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible left actions defined by

$$
\begin{gather*}
\alpha \cdot(a \cdot x)=(\alpha \cdot a) \cdot x, a \cdot(\alpha \cdot x)=(a \cdot \alpha) \cdot x, \quad(\alpha \cdot x) \cdot a=\alpha \cdot(x \cdot a) \\
(a \in A, \alpha \in \mathfrak{A}, x \in X) \tag{2.1}
\end{gather*}
$$

and similar for the right or two-sided actions. Then we say that $X$ is a Banach $A$ - $\mathfrak{A}$-module. A Banach $A$ - $\mathfrak{A}$-module $X$ is called commutative $A$ - $\mathfrak{A}$-module, if $\alpha . x=x . \alpha(\alpha \in \mathfrak{A}, x \in X)$. Note that in general, $A$ dose not satisfy the compatibility condition $a .(\alpha . b)=(a . \alpha) . b \quad(a, b \in A, \alpha \in \mathfrak{A})$.

If $X$ is a commutative Banach $A$ - $\mathfrak{A}$-module, then so is $X^{*}$, where the actions of $A$ and $\mathfrak{A}$ on $X^{*}$ are defined as follows

$$
\langle\alpha . f, x\rangle=\langle f, x . \alpha\rangle,\langle a . f, x\rangle=\langle f, x . a\rangle\left(a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^{*}\right)
$$

and similar for the right actions.
Let $A$ and $B$ be Banach $\mathfrak{A}$-bimodules. Then a $\mathfrak{A}$-module morphism from $A$ to $B$ is a norm continuous map $h: A \longrightarrow B$ with $h(a \pm b)=h(a) \pm h(b)$ which is multiplicative, that is

$$
h(\alpha \cdot a)=\alpha \cdot h(a), \quad h(a . \alpha)=h(a) . \alpha, \quad h(a b)=h(a) h(b) \quad(a \in A, b \in B, \alpha \in \mathfrak{A}) .
$$

We denote by $\operatorname{Hom}_{\mathfrak{A}}(A, B)$, the space of all such morphism and denote $\operatorname{Hom}_{\mathfrak{A}}(A, A)$ by $\operatorname{Hom}_{\mathfrak{A}}(A)$. In the case that $\mathfrak{A}=\mathbb{C}$, we denote $\operatorname{Hom}_{\mathbb{C}}(A, B)$ by $\operatorname{Hom}(A, B)$ and denote $\operatorname{Hom}_{\mathbb{C}}(A, A)$ by $\operatorname{Hom}(A)$.

Let $X$ be a Banach $A$-bimodule and let $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$. A bounded map $D: A \longrightarrow X$ is called a $\varphi-\mathfrak{A}$-module derivation if

$$
\begin{equation*}
D(a \pm b)=D(a) \pm D(b), D(a b)=D(a) \cdot \varphi(b)+\varphi(a) \cdot D(b)(a, b \in A) \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
D(\alpha \cdot a)=\alpha \cdot D(a), D(a \cdot \alpha)=D(a) \cdot \alpha(a \in A, \alpha \in \mathfrak{A}) . \tag{2.3}
\end{equation*}
$$

Although $D$ in general is not linear, but still its boundedness implies its norm continuity.

Let $X$ be a commutative Banach $A$ - $\mathfrak{A}$-module. For every $x \in X$ define $a d_{x}^{\varphi}$ by $a d_{x}^{\varphi}(a)=\varphi(a) \cdot x-x \cdot \varphi(a)(a \in A)$. It is easily seen that $a d_{x}^{\varphi}$ is a $\varphi$ - $\mathfrak{A}$-module derivation. A $\varphi$ - $\mathfrak{A}$-module derivation $D$ is called $\varphi$-inner if there is $x \in X$ such that $D(a)=a d_{x}^{\varphi}(a)(a \in A)$ and is called approximately $\varphi$-inner if there exists a net $\left(x_{\alpha}\right)_{\alpha} \subseteq X$ such that $D(a)=\lim _{\alpha} a d_{x_{\alpha}}^{\varphi}(a)(a \in A)$. A Banach algebra $A$ is called $\varphi$ - $\mathfrak{A}$-module amenable if for any commutative Banach $A$ - $\mathfrak{A}$-module $X$, each $\varphi$ - $\mathfrak{A}$-module derivation $D: A \longrightarrow X^{*}$ is $\varphi$-inner, and A is called $\varphi$ - $\mathfrak{A}$ module approximately amenable if each $\varphi$ - $\mathfrak{A}$-module derivation $D: A \longrightarrow X^{*}$ is approximately $\varphi$-inner (see $[1,15]$ ).

In the case that $\mathfrak{A}=\mathbb{C}, \varphi$ - $\mathfrak{A}$-module derivations (resp. $\varphi$ - $\mathfrak{A}$-module amenable Banach algebras, $\varphi$ - $\mathfrak{A}$-module approximately amenable Banach algebras) are called $\varphi$-derivation (resp. $\varphi$-amenable, $\varphi$-approximately amenable) (see $[9,10]$ ).

## 3. $\varphi \oplus \psi$-Module Approximate Amenability of the Direct Sum of Banach Algebras

We commence this section with the following remark from [1]:
Remark 3.1. Assume that $A$ has a bounded approximate identity $\left(e_{\alpha}\right)_{\alpha}$, and let $M_{\mathfrak{A}}(A)$ denotes the algebra of $\mathfrak{A}$-multipliers of $A$, that is $M_{\mathfrak{A}}(A)=\left\{\left(T_{1}, T_{2}\right)\right.$ : $\left.T_{1}, T_{2} \in L_{\mathfrak{A}}(A): T_{1}(a b)=T_{1}(a) b, T_{2}(a b)=a T_{2}(b)(a, b \in A)\right\}$, where $L_{\mathfrak{A}}(A)$ is the space of all $\mathfrak{A}$-module morphisms on $A$. Then $M_{\mathfrak{A}}(A)$ is an $A$ - $\mathfrak{A}$-module and $A$ embeds in $M_{\mathfrak{A}}(A)$ via $a \longmapsto\left(L_{a}, R_{a}\right)$, where $L_{a}(b)=a b, R_{a}(b)=b a \quad(a, b \in$ $A)$. For any element $T=\left(T_{1}, T_{2}\right)$ of $M_{\mathfrak{A}}(A)$ it is easy to see that $\left\|T_{1}\right\|=\left\|T_{2}\right\|$ and if we put $\|T\|$ equal to this common value, then $M_{\mathfrak{A}}(A)$ becomes a Banach A- $\mathfrak{A}$-module, and $A$ is dense in $M_{\mathfrak{A}}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$, then by continuity of $\varphi$ in the strict topology, it can be extended to an $\mathfrak{A}$-homomorphism $\tilde{\varphi}: M_{\mathfrak{A}}(A) \longrightarrow M_{\mathfrak{A}}(A)$ defined by $\tilde{\varphi}\left(L_{a}, R_{a}\right)=\left(L_{\varphi(a)}, R_{\varphi(a)}\right)$.

Proposition 3.2. Let $A$ be an $\mathfrak{A}$-module Banach algebra with a bounded approximate identity $\left(e_{\alpha}\right)_{\alpha}$, and let $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$. Then $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable if and only if $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$ - $A$-module approximately amenable.

Proof. Let $M_{\mathfrak{A}}(A)$ be $\tilde{\varphi}$ - $\mathfrak{A}$-module approximately amenable and let $D: A \longrightarrow$ $X^{*}$ be a $\varphi$ - $\mathfrak{A}$-module derivation for some commutative Banach $A$ - $\mathfrak{A}$-module $X$. Then by the following actions

$$
T \cdot x=\lim _{\alpha} T_{1}\left(e_{\alpha}\right) \cdot x, x \cdot T=\lim _{\alpha} x \cdot T_{2}\left(e_{\alpha}\right)\left(x \in X, T=\left(T_{1}, T_{2}\right) \in M_{\mathfrak{A}}(A)\right)
$$

$X$ is a commutative Banach $M_{\mathfrak{A}}(A)-\mathfrak{A}$-module and by continuity of $D$ in the strict topology, it can be extended to a bounded $\tilde{\varphi}$ - $A$-derivation $\tilde{D}: M_{\mathfrak{A}}(A) \longrightarrow$ $X^{*}$, defined by $\tilde{D}\left(L_{a}, R_{a}\right)=D(a)$. From the $\tilde{\varphi}$ - $\mathfrak{A}$-module approximate amenability of $M_{\mathfrak{A}}(A)$, it follows that there exists a net $\left(x_{\beta}^{*}\right)_{\beta} \subset X^{*}$ such that

$$
\tilde{D}(T)=\lim _{\beta}\left(\tilde{\varphi}(T) \cdot x_{\beta}^{*}-x_{\beta}^{*} \cdot \tilde{\varphi}(T)\right)
$$

Hence for every $a \in A$ we have

$$
\begin{aligned}
D(a)=\tilde{D}\left(L_{a}, R_{a}\right) & =\lim _{\beta}\left(\tilde{\varphi}\left(L_{a}, R_{a}\right) \cdot x_{\beta}^{*}-x_{\beta}^{*} \cdot \tilde{\varphi}\left(L_{a}, R_{a}\right)\right) \\
& =\lim _{\beta}\left(\left(L_{\varphi(a)}, R_{\varphi(a)}\right) \cdot x_{\beta}^{*}-x_{\beta}^{*} \cdot\left(L_{\varphi(a)}, R_{\varphi(a)}\right)\right) \\
& =\lim _{\beta}\left(\lim _{\alpha} L_{\varphi(a)}\left(e_{\alpha}\right) \cdot x_{\beta}^{*}-\lim _{\alpha} x_{\beta}^{*} \cdot R_{\varphi(a)}\left(e_{\alpha}\right)\right) \\
& =\lim _{\beta}\left(\varphi(a) \cdot x_{\beta}^{*}-x_{\beta}^{*} \cdot \varphi(a)\right)
\end{aligned}
$$

This means that $D$ is approximately $\varphi$-inner and so $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable.
Conversely, Suppose that $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable. Let $X$ be a commutative Banach $M_{\mathfrak{A}}(A)-\mathfrak{A}$-module and let $D: M_{\mathfrak{A}}(A) \longrightarrow X^{*}$ be a $\tilde{\varphi}$ - $\mathfrak{A}$-module derivation. We consider the module actions of $A$ on $X$ by

$$
\begin{equation*}
a \cdot x=\left(L_{a}, R_{a}\right) \cdot x, x \cdot a=x \cdot\left(L_{a}, R_{a}\right)(a \in A, x \in X) \tag{3.1}
\end{equation*}
$$

Thus $X$ is a commutative Banach $A$ - $A$-module. Define $\tilde{D}: A \longrightarrow X^{*}$ by $\tilde{D}(a)=D\left(L_{a}, R_{a}\right)(a \in A)$. It is easy to see that $\tilde{D}$ is a $\varphi$ - $\mathcal{A}$-module derivation and from the $\varphi$ - $\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $\left(x_{\beta}^{*}\right)_{\beta} \subset X^{*}$ such that

$$
\tilde{D}(a)=\lim _{\beta}\left(\varphi(a) \cdot x_{\beta}^{*}-x_{\beta}^{*} \cdot \varphi(a)\right)(a \in A)
$$

Then $D\left(L_{a}, R_{a}\right)=\lim _{\beta}\left(\tilde{\varphi}\left(L_{a}, R_{a}\right) \cdot x_{\beta}^{*}-x_{\beta}^{*} \cdot \tilde{\varphi}\left(L_{a}, R_{a}\right)\right)$. Now by the continuity of $D$ and $\tilde{\varphi}$, and density of A in $M_{\mathfrak{A}}(A)$ in the strict topology, we conclude that

$$
D(T)=\lim _{\beta}\left(\tilde{\varphi}(T) \cdot x_{\beta}^{*}-x_{\beta}^{*} \cdot \tilde{\varphi}(T)\right)\left(T \in M_{\mathfrak{A}}(A)\right)
$$

So $D$ is a approximately $\tilde{\varphi}$-inner. Therefore $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$ - $\mathfrak{A}$-module approximately amenable.

Let $I$ be a closed ideal of a Banach algebra $A$ with a bounded approximate identity $\left(e_{\alpha}\right)_{\alpha}$, and let $X$ be a commutative Banach $I$ - $\mathfrak{A}$-module. Let $\varphi \in$ $\operatorname{Hom}_{\mathfrak{A}}(A)$ be such that $\left.\varphi\right|_{I} \subset I$, then $X$ is a commutative Banach $A$ - $\mathfrak{A}$-module with the following actions

$$
\begin{equation*}
a \cdot x=\lim _{\alpha} \varphi\left(e_{\alpha}\right) a \cdot x, \quad x \cdot a=\lim _{\alpha} x \cdot \varphi\left(e_{\alpha}\right) a \quad(a \in A, x \in X) . \tag{3.2}
\end{equation*}
$$

Proposition 3.3. Let $I$ be a closed ideal of an $\mathfrak{A}$-module Banach algebra $A$ which has a bounded approximate identity $\left\{e_{\alpha}\right\}$, and let $I$ be $\mathfrak{A}$-invariant, i.e. $\mathfrak{A} . I \subseteq I$. Let $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ be such that $\left.\varphi\right|_{I} \subset I$. If $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable, then $I$ is $\left.\varphi\right|_{I-\mathfrak{A}-m o d u l e ~ a p p r o x i m a t e l y ~ a m e n a b l e . ~} ^{\text {a }}$

Proof. Let $X$ be a commutative Banach $M_{\mathfrak{A}}(I)$ - $\mathfrak{A}$-module, and $D: M_{\mathfrak{A}}(I) \longrightarrow$ $X^{*}$ be a $\tilde{\varphi}$ - $\mathfrak{A}$-module derivation. By the same actions as (3.1), we can consider $X$ as a commutative Banach $I$ - $\mathfrak{A}$-module. So, by (3.2), $X$ is a commutative Banach $A$ - $\mathfrak{A}$-module. By definition of $M_{\mathfrak{A}}(I)$, there is an $\mathfrak{A}$-module morphism $h: A \longrightarrow M_{\mathfrak{A}}(I)$ and $D \circ h$ is a module derivation on $A$, so it is approximately $\varphi$ inner. Hence $D$ is approximately $\tilde{\varphi}$-inner. Since $I$ has a bounded approximate identity, by Proposition $3.2, I$ is $\left.\varphi\right|_{I^{-} \mathfrak{A} \text {-module approximately amenable. }}$

Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras. It is well known that $A \oplus B$, the $l^{1}$-direct sum of $A$ and $B$, is a Banach algebra with respect to the canonical multiplication defined by $(a, b)(c, d):=(a c, b d)$, and is a Banach $\mathfrak{A}$-bimodule by the following actions

$$
\alpha .(a, b):=(\alpha . a, \alpha . b),(a, b) \cdot \alpha:=(a . \alpha, b . \alpha)(\alpha \in \mathfrak{A}, a \in A, b \in B) .
$$

We note that if $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \operatorname{Hom}_{\mathfrak{A}}(B)$, then $\varphi \oplus \psi: A \oplus B \longrightarrow A \oplus B$ defined by $\varphi \oplus \psi(a, b)=(\varphi(a), \psi(b))$ is an $\mathfrak{A}$-morphism on $A \oplus B$.

Lemma 3.4. Let $A$ be a unital $\mathfrak{A}$-module Banach algebra, $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$, and let $D: A \longrightarrow X^{*}$ be a $\varphi$ - $\mathfrak{A}$-module derivation for some commutative Banach $A$ - $\mathfrak{A}$-module $X$. If the left (resp. right, two-sided) action of $\varphi(A)$ on $X^{*}$ is zero, then $D$ is $\varphi$-inner.

Proof. Let $e_{A}$ be the identity of $A$ and let the left (resp. right, two-sided) action of $\varphi(A)$ on $X^{*}$ is zero. We can easily show that $D=a d_{-D(e)}^{\varphi}$ (resp. $\left.D=a d_{D(e)}^{\varphi}, D=0\right)$. So $D$ is $\varphi$-inner.

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

Proposition 3.5. Let $A$ and $B$ be unital $\mathfrak{A}$-module Banach algebras with identities $e_{A}$ and $e_{B}$, respectively, and let $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \operatorname{Hom}_{\mathfrak{A}}(B)$ such that $\varphi\left(e_{A}\right) \cdot \alpha=\alpha \cdot \varphi\left(e_{A}\right)$, and $\psi\left(e_{B}\right) \cdot \alpha=\alpha \cdot \psi\left(e_{B}\right)(\alpha \in \mathfrak{A})$. If $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable and $B$ is $\psi$ - $\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable.

Proof. Let $X$ be a commutative Banach $A \oplus B$ - $\mathfrak{A}$-module and let $D: A \oplus$ $B \longrightarrow X^{*}$ be a $\varphi \oplus \psi$ - $\mathfrak{A}$-module derivation. Write $Y_{1}=\varphi\left(e_{A}\right) \cdot X^{*} \cdot \varphi\left(e_{A}\right), Y_{2}=$ $\psi\left(e_{B}\right) \cdot X^{*} \cdot \psi\left(e_{B}\right), Y_{3}=\varphi\left(e_{A}\right) \cdot X^{*} \cdot \psi\left(e_{B}\right), Y_{4}=\psi\left(e_{B}\right) \cdot X^{*} \cdot \varphi\left(e_{A}\right), Y_{5}=(1-$ $\left.\varphi\left(e_{A}\right)\right)\left(1-\psi\left(e_{B}\right)\right) \cdot X^{*} \cdot \varphi\left(e_{A}\right), Y_{6}=\left(1-\varphi\left(e_{A}\right)\right)\left(1-\psi\left(e_{B}\right)\right) \cdot X^{*} \cdot \psi\left(e_{B}\right), Y_{7}=$ $\varphi\left(e_{A}\right) \cdot X^{*} \cdot\left(1-\varphi\left(e_{A}\right)\right)\left(1-\psi\left(e_{B}\right)\right), Y_{8}=\psi\left(e_{B}\right) \cdot X^{*} .\left(1-\varphi\left(e_{A}\right)\right)\left(1-\psi\left(e_{B}\right)\right), Y_{9}=$ $\left(1-\varphi\left(e_{A}\right)\right)\left(1-\psi\left(e_{B}\right)\right) \cdot X^{*} .\left(1-\varphi\left(e_{A}\right)\right)\left(1-\psi\left(e_{B}\right)\right)$ and let $\pi_{j}: X^{*} \longrightarrow Y_{j}$ be the associated projections. Thus $X^{*}=Y_{1} \oplus Y_{2} \oplus Y_{3} \oplus Y_{4} \oplus Y_{5} \oplus Y_{6} \oplus Y_{7} \oplus Y_{8} \oplus Y_{9}$. Consider the derivations $D_{j}=\pi_{j} \circ D$, so $D=D_{1}+D_{2}+D_{3}+D_{4}+D_{5}+$ $D_{6}+D_{7}+D_{8}+D_{9}$. From the fact that $\varphi\left(e_{A}\right) \cdot \alpha=\alpha \cdot \varphi\left(e_{A}\right)(\alpha \in \mathfrak{A})$, and $\psi\left(e_{B}\right) \cdot \alpha=\alpha \cdot \psi\left(e_{B}\right)(\alpha \in \mathfrak{A})$, one can easily check that $Y_{j}$ for $j=1, \ldots, 9$ is a commutative Banach $A \oplus B$ - $A$-module. Since the action of $\varphi(A) \oplus \psi(B)$ on (at least) one side on $Y_{5}$ (resp. $\left.Y_{6}, Y_{7}, Y_{8}, Y_{9}\right)$ is zero, by Lemma 3.4, we conclude that $D_{5}$ (resp. $D_{6}, D_{7}, D_{8}, D_{9}$ ) is approximately $\varphi \oplus \psi$-inner.

From the $\varphi$ - $\mathfrak{A}$-module approximate amenability of $A$, it follows that the $\varphi \oplus \psi$ - $\mathfrak{A}$-module derivation $A \oplus 0 \longrightarrow \varphi\left(e_{A}\right) \cdot X^{*} \cdot \varphi\left(e_{A}\right)$ is approximately $\varphi \oplus \psi$ inner and since the action of $0 \oplus \psi(B)$ on $\varphi\left(e_{A}\right) \cdot X^{*} \cdot \varphi\left(e_{A}\right)$ is zero, we conclude that $D_{1}$ is approximately $\varphi \oplus \psi$-inner. Similarly, the $\varphi \oplus \psi$ - $\mathfrak{A}$-module derivation $D_{2}: A \oplus B \longrightarrow \psi\left(e_{B}\right) \cdot X^{*} . \psi\left(e_{B}\right)$ is approximately $\varphi \oplus \psi$-inner.

The right action of $\varphi(A) \oplus 0$ on $\varphi\left(e_{A}\right) \cdot X^{*} \cdot \psi\left(e_{B}\right)$ is zero. Hence, by Lemma 3.4, $\left.D_{3}\right|_{A \oplus 0}$ is $\varphi \oplus \psi$-inner. So there exists $\xi \in \varphi\left(e_{A}\right) \cdot X^{*} \cdot \psi\left(e_{B}\right)$ such that

$$
\left.D_{3}\right|_{A \oplus 0}(a, 0)=\varphi(a) \cdot \xi-\xi \cdot \varphi(a)=(\varphi(a), \psi(b)) \varphi\left(e_{A}\right) \cdot \xi \cdot \psi\left(e_{B}\right)
$$

for every $a \in A$ and $b \in B$. Similarly, there exists $\eta \in \varphi\left(e_{A}\right) \cdot X^{*} \cdot \psi\left(e_{B}\right)$ such that

$$
\left.D_{3}\right|_{0 \oplus B}(0, b)=\psi(b) \cdot \eta-\eta \cdot \psi(b)=-\varphi\left(e_{A}\right) \cdot \eta \cdot \psi\left(e_{B}\right)(\varphi(a), \psi(b)),
$$

for every $a \in A$ and $b \in B$. Hence

$$
D_{3}(a, b)=(\varphi(a), \psi(b)) \varphi\left(e_{A}\right) \cdot \xi \cdot \psi\left(e_{B}\right)-\varphi\left(e_{A}\right) \cdot \eta \cdot \psi\left(e_{B}\right)(\varphi(a), \psi(b))
$$

Since $D_{3}\left(e_{A}, e_{B}\right)=0$, it follows that

$$
0=D_{3}\left(e_{A}, e_{B}\right)=\varphi\left(e_{A}\right) \cdot \xi \cdot \psi\left(e_{B}\right)-\varphi\left(e_{A}\right) \cdot \eta \cdot \psi\left(e_{B}\right)
$$

Then for every $a \in A$ and $b \in B$, we have

$$
D_{3}(a, b)=(\varphi(a), \psi(b)) \varphi\left(e_{A}\right) \cdot \xi \cdot \psi\left(e_{B}\right)-\varphi\left(e_{A}\right) \cdot \xi \cdot \psi\left(e_{B}\right)(\varphi(a), \psi(b))
$$

Thus $D_{3}$ is $\varphi \oplus \psi$-inner. The same argument holds for the $\varphi \oplus \psi$ - $\mathfrak{A}$-module derivation $D_{4}: A \oplus B \longrightarrow \psi\left(e_{B}\right) \cdot X^{*} \cdot \varphi\left(e_{A}\right)$. Therefore $D$ is approximately $\varphi \oplus \psi$-inner, and so $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable.

Lemma 3.6. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \operatorname{Hom}_{\mathfrak{A}}(B)$. If there is a $h$ in $\operatorname{Hom}_{\mathfrak{A}}(A, B)$ such that $h \circ \varphi=\psi \circ h$ and the range of $h$ is a dense subset of $B$, then $\varphi$ - $A$-module approximate amenability of $A$ implies $\psi$ - $\mathfrak{A}$-module approximate amenability of $B$.

Proof. Let $D: B \longrightarrow X^{*}$ be a $\psi$ - $\mathfrak{A}$-module derivation for some commutative Banach $B$ - $\mathfrak{A}$-module $X$. Then by the following actions

$$
a \bullet x=h(a) . x, x \bullet a=x . h(a) \quad(a \in A, x \in X)
$$

$X$ is a commutative Banach $A$ - $\mathfrak{A}$-module. Let $\tilde{D}=D \circ h: A \longrightarrow X^{*}$. One can easily prove that $D$ is a $\varphi$ - $\mathfrak{A}$-module derivation. From the $\varphi$ - $\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $\left(x_{\alpha}^{*}\right)_{\alpha}$ in $X^{*}$ such that $\tilde{D}(a)=\lim _{\alpha}\left(\varphi(a) \bullet x_{\alpha}^{*}-x_{\alpha}^{*} \bullet \varphi(a)\right)(a \in A)$. Now continuity and density of $h(A)$ in $B$, imply that $D$ is approximately $\psi$-inner. Therefore $B$ is $\psi$ - $\mathfrak{A}$-module approximately amenable.

Proposition 3.7. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \operatorname{Hom}_{\mathfrak{A}}(B)$. If $A$ is not $\varphi$ - $\mathfrak{A}$-module approximately amenable or $B$ is not $\psi$ - $\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable.

Proof. Suppose that $A$ is not $\varphi$ - $\mathfrak{A}$-module approximately amenable. The projection map $\pi: A \oplus B \longrightarrow A$ determines an $\mathfrak{A}$-module epimorphism of $A \oplus B$ onto $A$ such that $\pi \circ(\varphi \oplus \psi)=\varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable, then by Lemma $3.6, A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable. This contradicts the fact that $A$ is not $\varphi$ - $\mathfrak{A}$-module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable. Similarly, we can prove the result for $B$.

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then $\mathfrak{A}^{\#}=\mathfrak{A} \oplus \mathbb{C}$, the unitization of $\mathfrak{A}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let $A$ be a Banach $\mathfrak{A}$-bimodule. Then $A$ is a Banach $\mathfrak{A}^{\#}$-module with the following module actions:

$$
(\alpha, \lambda) \cdot a=\alpha \cdot a+\lambda a, a \cdot(\alpha, \lambda)=a \cdot \alpha+\lambda a(\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A) .
$$

Let $A^{\sharp}=\left(A \oplus \mathfrak{A}^{\#}, \bullet\right)$, where the multiplication $\bullet$ is defined through

$$
(a, u) \bullet(b, v)=(a b+a . v+u . b, u v)\left(a, b \in A, u, v \in \mathfrak{A}^{\#}\right)
$$

Then with the actions defined by

$$
u \cdot(a, v)=(u \cdot a, u v),(a, v) \cdot u=(a \cdot u, v u)\left(a \in A, u, v \in \mathfrak{A}^{\#}\right),
$$

$A^{\sharp}$ is a unital $\mathfrak{A}^{\#}$-module Banach algebra with the identity $1_{A^{\sharp}}=\left(0,1_{\mathfrak{A}}\right.$ \# $)$ (see [4] ).

Before we turn to our next result we note that if for every $\varphi \in \operatorname{Hom}_{\mathfrak{A} \#}(A)$, one defines $\varphi^{\sharp}: A^{\sharp} \longrightarrow A^{\sharp}$ by $\varphi^{\sharp}(a, u)=(\varphi(a), u) \quad\left((a, u) \in A^{\sharp}\right)$, then $\varphi^{\sharp} \in$ $\operatorname{Hom}_{\mathfrak{A}} \#\left(A^{\sharp}\right)$.

The following proposition generalizes Proposition 2.7 of [5].
Theorem 3.8. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \operatorname{Hom}_{\mathfrak{A} \#}(A)$ and $\psi \in \operatorname{Hom}_{\mathfrak{A} \#}(B)$. Then $A$
is $\varphi$ - $\mathfrak{A}^{\#}$-module approximately amenable and $B$ is $\psi-\mathfrak{A}^{\#}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi-\mathfrak{A}^{\#}$-module approximately amenable.

Proof. Suppose that $A$ is $\varphi$ - $\mathfrak{A}^{\#}$-module approximately amenable and $B$ is $\psi$ -$\mathfrak{A}^{\#}$-module approximately amenable. By Proposition 12 of [13], $A^{\sharp}$ is $\varphi^{\sharp}$ -$\mathfrak{A}^{\#}$-module approximately amenable and $B^{\sharp}$ is $\psi^{\sharp}-\mathfrak{A} \#$-module approximately amenable, so by Proposition 3.5, $A^{\sharp} \oplus B^{\sharp}$ is $\varphi^{\sharp} \oplus \psi^{\sharp}-\mathfrak{A}^{\#}$-module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^{\#}$-invariant ideal in $A^{\sharp} \oplus B^{\sharp}$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}^{\#}$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi-\mathfrak{A}^{\#}$-module approximately amenable and $B$ is $\psi$ - $\mathfrak{A}^{\#}$-module approximately amenable.
4. $\varphi \oplus \psi$-Module Approximate Amenability and $\varphi \oplus \psi$-Amenability of Direct Sum of Banach Algebras

We start this section with the following definition:
Definition 4.1. We say the Banach algebra $\mathfrak{A}$ acts trivially on $A$ from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A, \alpha . a=f(\alpha) a$ (resp. $a . \alpha=f(\alpha) a$ ), where $f$ is a multiplicative linear functional on $\mathfrak{A}$.

We assume that $J_{A, \mathfrak{A}}$ is the closed linear span of

$$
\{(a . \alpha) b-a(\alpha . b) \mid \alpha \in \mathfrak{A}, a, b \in A\}
$$

in $A$. It follows immediately that $J_{A, \mathfrak{A}}$ is both $A$-submodule and $\mathfrak{A}$-submodule of $A$. So $\frac{A}{J_{A, \mathfrak{A}}}$ is both Banach $A$-module and $\mathfrak{A}$-module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].
Lemma 4.2. Let $A$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and $J_{0}$ be a closed ideal of $A$ such that $J_{A, \mathfrak{A}} \subseteq J_{0}$. If $\frac{A}{J_{0}}$ has a left or right identity $e+J_{0}$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $a . \alpha-\alpha . a \in J_{0}$, i.e, $\frac{A}{J_{0}}$ is commutative Banach $\mathfrak{A}$-module.

Before we turn to our next result we note that if for every $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$, one defines $\bar{\varphi}: \frac{A}{J_{A, \mathfrak{A}}} \longrightarrow \frac{A}{J_{A, \mathfrak{A}}}$ by $\bar{\varphi}\left(a+J_{A, \mathfrak{A}}\right)=\varphi(a)+J_{A, \mathfrak{A}}$, then $\bar{\varphi} \in \operatorname{Hom}_{\mathfrak{A}}\left(\frac{A}{J_{A, \mathfrak{A}}}\right)$.
Theorem 4.3. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras and let $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \operatorname{Hom}_{\mathfrak{A}}(B)$. Then the following statements are valid:
(i) $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable) if and only if $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}-\mathfrak{A}$-module amenable (resp. $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$-module approximately amenable).
(ii) Let $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left by $f \in \operatorname{Hom}_{\mathbb{C}}(\mathfrak{A})$. Suppose that $\frac{A}{J_{A, \mathfrak{2}}}$ and $\frac{B}{J_{B, \mathfrak{2}}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi-\mathfrak{A}-$ module amenable (resp. $\varphi \oplus \psi$ - $A$-module approximately amenable), then $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$-amenable (resp. $\bar{\varphi} \oplus \bar{\psi}$-approximately amenable).
(iii) Let $\mathfrak{A}$ have a bounded approximately identity and $\frac{A}{J_{A, \mathfrak{2}}} \oplus \frac{B}{J_{B, \mathfrak{2}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ amenable (resp. $\bar{\varphi} \oplus \bar{\psi}$-approximately amenable). Then $A \oplus B$ is $\varphi \oplus \psi$ -$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable).

Proof. (i) Let $A \oplus B$ be $\varphi \oplus \psi$ - $\mathfrak{A}$-module amenable, and let $D: \frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}} \longrightarrow$ $X^{*}$ be $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$-module derivation for some commutative Banach $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$ -$\mathfrak{A}$-module $X$. Then $X$ becomes a $A \oplus B$-bimodule through the following actions

$$
\begin{equation*}
(a, b) \cdot x:=\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) \cdot x \quad(a \in A, b \in B, x \in X) \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
x .(a, b):=x .\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) \quad(a \in A, b \in B, x \in X) . \tag{4.2}
\end{equation*}
$$

Hence $X$ is a commutative Banach $A \oplus B$ - $\mathfrak{A}$-module. Define $\tilde{D}: A \oplus B \longrightarrow X^{*}$ by

$$
\tilde{D}(a, b)=D\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)(a \in A, b \in B)
$$

It is easy to check that, $\tilde{D}$ is a $\varphi \oplus \psi$ - $\mathfrak{A}$-module derivation. From the $\varphi \oplus \psi$ -$\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^{*} \in X^{*}$ such that

$$
\tilde{D}(a, b)=\varphi \oplus \psi(a, b) \cdot x^{*}-x^{*} \cdot \varphi \oplus \psi(a, b) \quad(a \in A, b \in B)
$$

Thus

$$
\begin{aligned}
D\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)= & \bar{\varphi} \oplus \bar{\psi}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) \cdot x^{*} \\
& -x^{*} \cdot \bar{\varphi} \oplus \bar{\psi}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) .
\end{aligned}
$$

This means that $D$ is $\bar{\varphi} \oplus \bar{\psi}$-inner. Therefore $\frac{A}{J_{A, \mathfrak{l}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$-module amenable.

Conversely, suppose that $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$-module amenable. Let $D: A \oplus B \longrightarrow X^{*}$ be a $\varphi \oplus \psi$ - $\mathfrak{A}$-module derivation for some commutative Banach $A \oplus B$-\{-module $X$. We consider the following module actions of $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$ on $X$,

$$
\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) \cdot x:=(a, b) \cdot x, \quad x \cdot\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right):=x \cdot(a, b),
$$

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of $X$, we have $J_{A, \mathfrak{A}} X=J_{B, \mathfrak{A}} X=X J_{A, \mathfrak{A}}=X J_{B, \mathfrak{A}}=0$. Thus $\left(J_{A, \mathfrak{A}} \oplus J_{B, \mathfrak{A}}\right) X=$ $X\left(J_{A, \mathfrak{A}} \oplus J_{B, \mathfrak{A}}\right)=0$. So $X$ is a commutative Banach $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}-\mathfrak{A}$-module. Define $\tilde{D}: \frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}} \longrightarrow X^{*}$ by

$$
\tilde{D}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)=D(a, b) \quad(a \in A, b \in B)
$$

Also using (2.2) and (2.3) we see that $D$ vanishes on $J_{A, \mathfrak{A}} \oplus J_{B, \mathfrak{A}}$. Hence $\tilde{D}$ is well defined. One can easily check that $\tilde{D}$ is a $\bar{\varphi} \oplus \bar{\psi}-\mathfrak{A}$-module derivation.

Now from the $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$-module amenability of $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{L}}}$, it follows that there exists $x^{*} \in X^{*}$ such that

$$
\begin{aligned}
\tilde{D}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)= & \bar{\varphi} \oplus \bar{\psi}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) \cdot x^{*} \\
& -x^{*} \cdot \bar{\varphi} \oplus \bar{\psi}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)(a \in A, b \in B)
\end{aligned}
$$

It follows that

$$
D(a, b)=\varphi \oplus \psi(a, b) \cdot x^{*}-x^{*} \cdot \varphi \oplus \psi(a, b)(a \in A, b \in B)
$$

Thus $D$ is $\varphi \oplus \psi$-inner. So $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module amenable.
Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable if and only if $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$-module approximately amenable.
(ii) Let $A \oplus B$ be $\varphi \oplus \psi$ - $\mathfrak{A}$-module amenable and let $D: \frac{A}{J_{A, \mathfrak{q}}} \oplus \frac{B}{J_{B, \mathfrak{q}}} \longrightarrow X^{*}$ be a derivation for some Banach $\frac{A}{J_{A, \mathfrak{l}}} \oplus \frac{B}{J_{B, \mathfrak{2}}}$-bimodule $X$. Then $X$ becomes a $A \oplus B$-bimodule through the actions as (4.1) and (4.2). Also $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is

$$
\alpha . x=x . \alpha=f(\alpha) x(\alpha \in \mathfrak{A}, x \in X) .
$$

Then $X$ is a commutative Banach $A \oplus B-\mathfrak{A}$-module. Define

$$
\Gamma: \frac{A \oplus B}{I} \longrightarrow \frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A}}},(a, b)+I \longmapsto\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right),
$$

where $I=J_{A, \mathfrak{A}} \oplus J_{B, \mathfrak{A}}$. It is routinely checked that $\Gamma$ defines an $\mathfrak{A}$-bimodule morphism. Let $\Pi: A \oplus B \longrightarrow \frac{A \oplus B}{I}$ be the quotient map, and let $\tilde{D}:=D \circ \Gamma \circ \Pi$ : $A \oplus B \longrightarrow X^{*}$. For every $(a, b),\left(a^{\prime}, b^{\prime}\right) \in A \oplus B$, we may easily prove that

$$
\tilde{D}\left((a, b)\left(a^{\prime}, b^{\prime}\right)\right)=\tilde{D}(a, b) \cdot \varphi \oplus \psi\left(a^{\prime}, b^{\prime}\right)+\varphi \oplus \psi(a, b) \cdot \tilde{D}\left(a^{\prime}, b^{\prime}\right)
$$

and for every $(a, b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$, we have

$$
\begin{aligned}
\tilde{D}(\alpha .(a, b)) & =\tilde{D}((\alpha \cdot a, \alpha \cdot b))=\tilde{D}((f(\alpha) a, f(\alpha) b)) \\
& =D\left(\left(f(\alpha) a+J_{A, \mathfrak{A}}, f(\alpha) b+J_{B, \mathfrak{A}}\right)\right) \\
& =D\left(f(\alpha)\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)\right) \\
& =f(\alpha) D\left(\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)\right) \\
& =\alpha \cdot D\left(\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)\right) \\
& =\alpha \cdot \tilde{D}(a, b),
\end{aligned}
$$

and using Lemma 4.2, we have

$$
\begin{aligned}
\tilde{D}((a, b) \cdot \alpha) & =\tilde{D}((a \cdot \alpha, b \cdot \alpha))=D\left(\left(a \cdot \alpha+J_{A, \mathfrak{A}}, b \cdot \alpha+J_{B, \mathfrak{A}}\right)\right) \\
& =D\left(\left(\alpha \cdot a+J_{A, \mathfrak{A}}, \alpha \cdot b+J_{B, \mathfrak{A}}\right)\right) \\
& =D\left(f(\alpha)\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)\right) \\
& =f(\alpha) D\left(\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)\right) \\
& =D\left(\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)\right) \cdot \alpha \\
& =\tilde{D}(a, b) \cdot \alpha
\end{aligned}
$$

Thus $\tilde{D}$ is a $\varphi \oplus \psi$ - $\mathfrak{A}$-module derivation and from the $\varphi \oplus \psi$ - $\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^{*} \in X^{*}$ such that

$$
\tilde{D}(a, b)=\varphi \oplus \psi(a, b) \cdot x^{*}-x^{*} \cdot \varphi \oplus \psi(a, b) \quad(a \in A, b \in B)
$$

It follows that

$$
\begin{aligned}
D\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right)= & \bar{\varphi} \oplus \bar{\psi}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) \cdot x^{*} \\
& -x^{*} \cdot \bar{\varphi} \oplus \bar{\psi}\left(a+J_{A, \mathfrak{A}}, b+J_{B, \mathfrak{A}}\right) .
\end{aligned}
$$

So $D$ is $\bar{\varphi} \oplus \bar{\psi}$-inner. Therefore $\frac{A}{J_{A, \mathfrak{q}}} \oplus \frac{B}{J_{B, \mathfrak{q}}}$ is $\bar{\varphi} \oplus \bar{\psi}$-amenable.
(iii) Suppose that $\frac{A}{J_{A, \mathfrak{l}}} \oplus \frac{B}{J_{B, \mathfrak{l}}}$ is $\bar{\varphi} \oplus \bar{\psi}$-amenable. Since $\mathfrak{A}$ has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that $\frac{A}{J_{A, \mathfrak{L}}} \oplus \frac{B}{J_{B, \mathfrak{a}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - $\mathfrak{A}$-module amenable. So by (i), $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module amenable.

Similar relations can be obtained between the $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximate amenability of $A \oplus B$ and $\bar{\varphi} \oplus \bar{\psi}$-approximate amenability of $\frac{A}{J_{A, \mathfrak{l}}} \oplus \frac{B}{J_{B, \mathfrak{A}}}$.

Proposition 4.4. Let $A$ be an $\mathfrak{A}$-module Banach algebra, where $\mathfrak{A}$ acts on $A$ trivially from the left by $f \in \operatorname{Hom}_{\mathbb{C}}(\mathfrak{A})$. Let $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A)$ and $\frac{A}{J_{A, \mathfrak{A}}}$ be unital. If $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable, then $\frac{A}{J_{A, \mathfrak{A}}}$ is $\bar{\varphi}$-approximately amenable.

Proof. Let $X$ be a Banach $\frac{A}{J_{A, \mathfrak{A}}}$-bimodule and $D: \frac{A}{J_{A, \mathfrak{a}}} \longrightarrow X^{*}$ be a $\bar{\varphi}$ derivation. Then $X$ becomes a $A$-bimodule through the following actions

$$
a \cdot x=\left(a+J_{A, \mathfrak{A}}\right) \cdot x, x \cdot a=x \cdot\left(a+J_{A, \mathfrak{A}}\right) \quad(a \in A, x \in X),
$$

and $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is $\alpha . x=x . \alpha=f(\alpha) x(\alpha \in$ $\mathfrak{A}, x \in X)$. By Lemma 4.2, $f(\alpha) a-a . \alpha \in J_{A, \mathfrak{A}}(\alpha \in \mathfrak{A}, a \in A)$. So, $f(\alpha) a+$ $J_{A, \mathfrak{A}}=a . \alpha+J_{A, \mathfrak{A}}(\alpha \in \mathfrak{A}, a \in A)$, and the actions of $\mathfrak{A}$ and $A$ on $X$ are compatible. Thus $X$ is a commutative Banach $A$ - $\mathfrak{A}$-module. Let $\tilde{D}: A \longrightarrow X^{*}$ be defined by $\tilde{D}(a)=D\left(a+J_{A, \mathfrak{A}}\right)(a \in A)$. A similar argument as in the proof of Theorem 3.2 of [2], shows that $\tilde{D}$ is approximately $\varphi$-inner. So, $D$ is approximately $\bar{\varphi}$-inner. Therefore $\frac{A}{J_{A, 2}}$ is $\bar{\varphi}$-approximately amenable.

Theorem 4.5. Let $\mathfrak{A}$ have a bounded approximate identity, and let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, where $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left. Let $\varphi \in \operatorname{Hom}_{\mathfrak{A}}(A), \psi \in \operatorname{Hom}_{\mathfrak{A}}(B)$, and let $\frac{A}{J_{A, \mathfrak{A}}}$ and $\frac{B}{J_{B, \mathfrak{A}}}$ be unital. Then $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable and $B$ is $\psi$ - $\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - - -module approximately amenable.

Proof. Suppose that $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable and $B$ is $\psi$ -$\mathfrak{A}$-module approximately amenable. By Proposition $4.4, \frac{A}{J_{A, \mathfrak{A}}}$ and $\frac{B}{J_{B, \mathfrak{l}}}$ are $\bar{\varphi}$-approximately amenable and $\bar{\psi}$-approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A}=\mathbb{C}$, we conclude that $\frac{A}{J_{A, \mathfrak{A}}} \oplus \frac{B}{J_{B, \mathfrak{A l}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$ module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$ - $\mathfrak{A}$-module approximately amenable and $B$ is $\psi$ - $\mathfrak{A}$-module approximately amenable.

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