Generalized Approximate Amenability of Direct Sum of Banach Algebras

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Abstract. In the present paper for two \( A \)-module Banach algebras \( A \) and \( B \), we investigate relations between \( \varphi \)-\( A \)-module approximate amenability of \( A \), \( \psi \)-\( A \)-module approximate amenability of \( B \), and \( \varphi \oplus \psi \)-\( A \)-module approximate amenability of \( A \oplus B \) (\( l^1 \)-direct sum of \( A \) and \( B \)), where \( \varphi \in \text{Hom}_A(A) \) and \( \psi \in \text{Hom}_A(B) \).

Keywords: Banach algebra, Module derivation, Module approximate amenability.


1. Introduction

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if \( A \) and \( B \) are approximately amenable Banach algebras and one of \( A \) or \( B \) has a bounded approximate identity, then \( A \oplus B \) is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have
studied $\varphi$-derivations, and $\varphi$-amenability of Banach algebra $A$, whenever $\varphi$ is a continuous homomorphism on $A$ (see [8, 9, 10, 11, 12]). The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.

The organization of this paper is as follows:

Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for $A$-module Banach algebras $A$ and $B$ where each has a bounded approximate identity we show that $A$ is $\varphi$-$A^\#$-module approximately amenable and $B$ is $\psi$-$A^\#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$A^\#$-module approximately amenable.

In section 4 we show that if $A$ has a bounded approximately identity and $A_J A$, $A_J B$, $A$ are unital, then $A$ is $\varphi$-$A^\#$-module approximately amenable and $B$ is $\psi$-$A^\#$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$A^\#$-module approximately amenable.

2. Preliminaries

Let $A$ and $B$ be Banach algebras such that $A$ is a Banach $A$-bimodule with compatible actions given by

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \quad (a, b \in A, \alpha \in A).$$

Let $X$ be a Banach $A$-bimodule and a Banach $A$-bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x, \quad (\alpha.x).a = \alpha.(x.a)$$

$$\quad (a \in A, \alpha \in A, x \in X), \quad (2.1)$$

and similar for the right or two-sided actions. Then we say that $X$ is a Banach $A$-$A$-module. A Banach $A$-$A$-module $X$ is called commutative $A$-$A$-module, if $\alpha.x = x.\alpha$ ($\alpha \in A, x \in X$). Note that in general, $A$ does not satisfy the compatibility condition $a.(\alpha.b) = (a.\alpha).b$ ($a, b \in A, \alpha \in A$).

If $X$ is a commutative Banach $A$-$A$-module, then so is $X^*$, where the actions of $A$ and $A$ on $X^*$ are defined as follows

$$\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \quad \langle a.f, x \rangle = \langle f, x.a \rangle \quad (a \in A, \alpha \in A, x \in X, f \in X^*),$$

and similar for the right actions.

Let $A$ and $B$ be Banach $A$-bimodules. Then a $A$-module morphism from $A$ to $B$ is a norm continuous map $h : A \rightarrow B$ with $h(a \pm b) = h(a) \pm h(b)$ which is multiplicative, that is

$$h(\alpha.a) = a.h(\alpha), \quad h(a.\alpha) = h(a).\alpha, \quad h(ab) = h(a)h(b) \quad (a \in A, b \in B, \alpha \in A).$$
We denote by $\text{Hom}_A(A, B)$, the space of all such morphism and denote $\text{Hom}_A(A, A)$ by $\text{Hom}_A(A)$. In the case that $A = \mathbb{C}$, we denote $\text{Hom}_C(A, B)$ by $\text{Hom}(A, B)$ and denote $\text{Hom}_C(A, A)$ by $\text{Hom}(A)$.

Let $X$ be a Banach $A$-bimodule and let $\varphi \in \text{Hom}_A(A)$. A bounded map $D : A \rightarrow X$ is called a $\varphi$-$\mathfrak{A}$-module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a)\varphi(b) + \varphi(a).D(b) \quad (a, b \in A),$$

and

$$D(\alpha.a) = \alpha.D(a), \quad D(\alpha.a) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}).$$

Although $D$ in general is not linear, but still its boundedness implies its norm continuity.

Let $X$ be a commutative Banach $A$-$\mathfrak{A}$-module. For every $x \in X$ define $ad^x_\varphi$ by $ad^x_\varphi(a) = \varphi(a).x - x.\varphi(a) \quad (a \in A)$. It is easily seen that $ad^x_\varphi$ is a $\varphi$-$\mathfrak{A}$-module derivation. A $\varphi$-$\mathfrak{A}$-module derivation $D$ is called $\varphi$-inner if there is $x \in X$ such that $D(a) = ad^x_\varphi(a) \quad (a \in A)$ and is called approximately $\varphi$-inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha ad^x_\alpha(a) \quad (a \in A)$. A Banach algebra $A$ is called $\varphi$-$\mathfrak{A}$-module amenable if for any commutative Banach $A$-$\mathfrak{A}$-module $X$, each $\varphi$-$\mathfrak{A}$-module derivation $D : A \rightarrow X^*$ is $\varphi$-inner, and $A$ is called $\varphi$-$\mathfrak{A}$-module approximately amenable if each $\varphi$-$\mathfrak{A}$-module derivation $D : A \rightarrow X^*$ is approximately $\varphi$-inner (see $[1, 15]$).

In the case that $\mathfrak{A} = \mathbb{C}$, $\varphi$-$\mathfrak{A}$-module derivations (resp. $\varphi$-$\mathfrak{A}$-module amenable Banach algebras, $\varphi$-$\mathfrak{A}$-module approximately amenable Banach algebras) are called $\varphi$-derivation (resp. $\varphi$-amenable, $\varphi$-approximately amenable) (see $[9, 10]$).

3. $\varphi \oplus \psi$-Module Approximate Amenability of the Direct Sum of Banach Algebras

We commence this section with the following remark from $[1]$:

**Remark 3.1.** Assume that $A$ has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_\mathfrak{A}(A)$ denotes the algebra of $\mathfrak{A}$-multipliers of $A$, that is $M_\mathfrak{A}(A) = \{ (T_1, T_2) : T_1, T_2 \in L_\mathfrak{A}(A) : T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b) \quad (a, b \in A) \}$, where $L_\mathfrak{A}(A)$ is the space of all $\mathfrak{A}$-module morphisms on $A$. Then $M_\mathfrak{A}(A)$ is an $A$-$\mathfrak{A}$-module and $A$ embeds in $M_\mathfrak{A}(A)$ via $a \mapsto (L_a, R_a)$, where $L_a(b) = ab, R_a(b) = ba \quad (a, b \in A)$. For any element $T = (T_1, T_2)$ of $M_\mathfrak{A}(A)$ it is easy to see that $\| T_1 \| = \| T_2 \|$ and if we put $\| T \|$ equal to this common value, then $M_\mathfrak{A}(A)$ becomes a Banach $A$-$\mathfrak{A}$-module, and $A$ is dense in $M_\mathfrak{A}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \text{Hom}_\mathfrak{A}(A)$, then by continuity of $\varphi$ in the strict topology, it can be extended to an $\mathfrak{A}$-homomorphism $\hat{\varphi} : M_\mathfrak{A}(A) \rightarrow M_\mathfrak{A}(A)$ defined by $\hat{\varphi}(L_a, R_a) = (L_{\varphi(a)}, R_{\varphi(a)})$. 


Proposition 3.2. Let $A$ be an $\mathfrak{A}$-module Banach algebra with a bounded approximate identity $(e_\alpha)_{\alpha}$, and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable if and only if $M_\mathfrak{A}(A)$ is $\varphi$-$\mathfrak{A}$-module approximately amenable.

Proof. Let $M_\mathfrak{A}(A)$ be $\varphi$-$\mathfrak{A}$-module approximately amenable and let $D : A \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. Then by the following actions

$$T.x = \lim_{\alpha} T_1(e_\alpha).x, \quad x.T = \lim_{\alpha} x.T_2(e_\alpha) \quad (x \in X, T = (T_1, T_2) \in M_\mathfrak{A}(A)),$$

$X$ is a commutative Banach $M_\mathfrak{A}(A)$-$\mathfrak{A}$-module and by continuity of $D$ in the strict topology, it can be extended to a bounded $\tilde{\varphi}$-$\mathfrak{A}$-derivation $\tilde{D} : M_\mathfrak{A}(A) \to X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\tilde{\varphi}$-$\mathfrak{A}$-module approximate amenability of $M_\mathfrak{A}(A)$, it follows that there exists a net $(x_\beta^*)_{\beta} \subset X^*$ such that

$$\tilde{D}(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)).$$

Hence for every $a \in A$ we have

$$D(a) = \tilde{D}(L_a, R_a) = \lim_{\beta} \left( \tilde{\varphi}(L_a, R_a)x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a) \right)$$

$$= \lim_{\beta} \left( (L_{\varphi(a)}, R_{\varphi(a)}).x_\beta^* - x_\beta^*.L_{\varphi(a)}, R_{\varphi(a)}) \right)$$

$$= \lim_{\beta} \left( \lim_{\alpha} L_{\varphi(a)}(e_\alpha).x_\beta^* - \lim_{\alpha} x_\beta^*.R_{\varphi(a)}(e_\alpha) \right)$$

$$= \lim_{\beta} \left( \varphi(a).x_\beta^* - x_\beta^*.\varphi(a) \right).$$

This means that $D$ is approximately $\varphi$-inner and so $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable.

Conversely, Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. Let $X$ be a commutative Banach $M_\mathfrak{A}(A)$-$\mathfrak{A}$-module and let $D : M_\mathfrak{A}(A) \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation. We consider the module actions of $A$ on $X$ by

$$a.x = (L_a, R_a).x, \quad x.a = x.(L_a, R_a) \quad (a \in A, x \in X).$$

Thus $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. Define $\bar{D} : A \to X^*$ by $\bar{D}(a) = D(L_a, R_a) \quad (a \in A)$. It is easy to see that $\bar{D}$ is a $\varphi$-$\mathfrak{A}$-module derivation and from the $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that there exists a net $(x_\beta^*)_{\beta} \subset X^*$ such that

$$\bar{D}(a) = \lim_{\beta} (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)) \quad (a \in A).$$

Then $D(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a)x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a))$. Now by the continuity of $D$ and $\tilde{\varphi}$, and density of $A$ in $M_\mathfrak{A}(A)$ in the strict topology, we conclude that

$$D(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)) \quad (T \in M_\mathfrak{A}(A)).$$

So $D$ is a approximately $\tilde{\varphi}$-inner. Therefore $M_\mathfrak{A}(A)$ is $\varphi$-$\mathfrak{A}$-module approximately amenable. □
Let $I$ be a closed ideal of a Banach algebra $A$ with a bounded approximate identity $\{e_\alpha\}_\alpha$, and let $X$ be a commutative Banach $I$-$\mathfrak{A}$-module. Let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ be such that $\varphi |_I \subseteq I$, then $X$ is a commutative Banach $A$-$\mathfrak{A}$-module with the following actions

$$a.x = \lim_\alpha \varphi(e_\alpha)a.x, \quad x.a = \lim_\alpha x.\varphi(e_\alpha)a \quad (a \in A, x \in X). \quad (3.2)$$

**Proposition 3.3.** Let $I$ be a closed ideal of an $\mathfrak{A}$-module Banach algebra $A$ which has a bounded approximate identity $\{e_\alpha\}_\alpha$, and let $I$ be $\mathfrak{A}$-invariant, i.e. $\mathfrak{A}.I \subseteq I$. Let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ be such that $\varphi |_I \subseteq I$. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable, then $I$ is $\varphi |_I$-$\mathfrak{A}$-module approximately amenable.

**Proof.** Let $X$ be a commutative Banach $M_\mathfrak{A}(I)$-$\mathfrak{A}$-module, and $D : M_\mathfrak{A}(I) \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation. By the same actions as (3.1), we can consider $X$ as a commutative Banach $I$-$\mathfrak{A}$-module. So, by (3.2), $X$ is a commutative Banach $A$-$\mathfrak{A}$-module. By definition of $M_\mathfrak{A}(I)$, there is an $\mathfrak{A}$-module morphism $h : A \to M_\mathfrak{A}(I)$ and $Doh$ is a module derivation on $A$, so it is approximately $\varphi$-inner. Hence $D$ is approximately $\varphi$-inner. Since $I$ has a bounded approximate identity, by Proposition 3.2, $I$ is $\varphi |_I$-$\mathfrak{A}$-module approximately amenable. $\square$

Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras. It is well known that $A \oplus B$, the $l^1$-direct sum of $A$ and $B$, is a Banach algebra with respect to the canonical multiplication defined by $(a,b)(c,d) := (ac, bd)$, and is a Banach $\mathfrak{A}$-bimodule by the following actions

$$\alpha.(a,b) := (\alpha.a, \alpha.b), \quad (a,b).\alpha := (a.\alpha, b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$, then $\varphi \oplus \psi : A \oplus B \to A \oplus B$ defined by $\varphi \oplus \psi(a,b) = (\varphi(a), \psi(b))$ is an $\mathfrak{A}$-morphism on $A \oplus B$.

**Lemma 3.4.** Let $A$ be a unital $\mathfrak{A}$-module Banach algebra, $\varphi \in \text{Hom}_\mathfrak{A}(A)$, and let $D : A \to X^*$ be a $\varphi$-$\mathfrak{A}$-module derivation for some commutative Banach $A$-$\mathfrak{A}$-module $X$. If the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero, then $D$ is $\varphi$-inner.

**Proof.** Let $e_\mathfrak{A}$ be the identity of $A$ and let the left (resp. right, two-sided) action of $\varphi(A)$ on $X^*$ is zero. We can easily show that $D = ad_{D(e)}(\varphi(e))$ (resp. $D = ad_{D(e)}^\mathfrak{A}(\varphi(e))$, $D = 0$). So $D$ is $\varphi$-inner. $\square$

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

**Proposition 3.5.** Let $A$ and $B$ be unital $\mathfrak{A}$-module Banach algebras with identities $e_A$ and $e_B$, respectively, and let $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$ such that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$, and $\psi(e_B).\alpha = \alpha.\psi(e_B) \quad (\alpha \in \mathfrak{A})$. If $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.
Proof. Let $X$ be a commutative Banach $A \oplus B$-module and let $D : A \oplus B \rightarrow X^*$ be a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. Write $Y_1 = \varphi(e_A).X^*.\varphi(e_A)$, $Y_2 = \psi(e_B).X^*.\psi(e_B)$, $Y_3 = \varphi(e_A).X^*.\varphi(e_A)$, $Y_4 = \psi(e_B).X^*.\varphi(e_A)$, $Y_5 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\varphi(e_A)$, $Y_6 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\psi(e_B)$, $Y_7 = \varphi(e_A).X^*.\psi(e_B)(1 - \varphi(e_A)), Y_8 = \psi(e_B).X^*.\psi(e_B)(1 - \varphi(e_A)), Y_9 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\psi(e_B)$, and let $\pi_j : X^* \rightarrow Y_j$ be the associated projections. Thus $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$.

Consider the derivations $D_j = \pi_j \circ D$, so $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$. From the fact that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$ ($\alpha \in \mathfrak{A}$), and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$), one can easily check that $Y_j$ for $j = 1, \ldots, 9$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Since the action of $\varphi(A) \oplus \psi(B)$ on (at least) one side on $Y_5$ (resp. $Y_6, Y_7, Y_8, Y_9$) is zero, by Lemma 3.4, we conclude that $D_5$ (resp. $D_6, D_7, D_8, D_9$) is approximately $\varphi \oplus \psi$-inner.

From the $\varphi \oplus \psi$-$\mathfrak{A}$-module approximate amenability of $A$, it follows that the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $A \oplus 0 \rightarrow \varphi(e_A).X^*.\varphi(e_A)$ is approximately $\varphi \oplus \psi$-inner and since the action of $0 \oplus \psi(B)$ on $\varphi(e_A).X^*.\varphi(e_A)$ is zero, we conclude that $D_1$ is approximately $\varphi \oplus \psi$-inner. Similarly, the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_2 : A \oplus B \rightarrow \psi(e_B).X^*.\psi(e_B)$ is approximately $\varphi \oplus \psi$-inner.

The right action of $\varphi(B) \oplus 0$ on $\varphi(e_A).X^*.\psi(e_B)$ is zero. Hence, by Lemma 3.4, $D_3 |_{A \oplus 0} = \varphi \oplus \psi$-inner. So there exists $\xi \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3 \mid_{A \oplus 0} (a, 0) = \varphi(a).\xi - \xi.\varphi(a) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B),$$

for every $a \in A$ and $b \in B$. Similarly, there exists $\eta \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3 \mid_{0 \oplus B} (0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)),$$

for every $a \in A$ and $b \in B$. Hence

$$D_3(a, b) = \varphi(a), \psi(b)\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)).$$

Since $D_3(e_A, e_B) = 0$, it follows that

$$0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B).$$

Then for every $a \in A$ and $b \in B$, we have

$$D_3(a, b) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\xi.\psi(e_B).$$

Thus $D_3$ is $\varphi \oplus \psi$-inner. The same argument holds for the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_4 : A \oplus B \rightarrow \psi(e_B).X^*.\varphi(e_A)$. Therefore $D$ is approximately $\varphi \oplus \psi$-inner, and so $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. □

Lemma 3.6. Let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, $\varphi \in \text{Hom}_\mathfrak{A}(A)$ and $\psi \in \text{Hom}_\mathfrak{A}(B)$. If there is a $h$ in $\text{Hom}_\mathfrak{A}(A, B)$ such that $h \circ \varphi = \psi \circ h$ and the range of $h$ is a dense subset of $B$, then $\varphi$-$\mathfrak{A}$-module approximate amenability of $A$ implies $\psi$-$\mathfrak{A}$-module approximate amenability of $B$. 

Proof. Let \( D : B \to X^* \) be a \( \psi\text{-}\mathfrak{A}\text{-module derivation} \) for some commutative Banach \( B\text{-}\mathfrak{A}\text{-module} \) \( X \). Then by the following actions

\[
a \cdot x = h(a)x, \quad x \cdot a = xh(a) \quad (a \in A, x \in X),
\]

\( X \) is a commutative Banach \( A\text{-}\mathfrak{A}\text{-module} \). Let \( \bar{D} = D \circ h : A \to X^* \). One can easily prove that \( \bar{D} \) is a \( \varphi\text{-}\mathfrak{A}\text{-module derivation} \). From the \( \varphi\text{-}\mathfrak{A}\text{-module approximate amenability of} \) \( A \), it follows that there exists a net \( (x'_a)_{\alpha} \) in \( X^* \) such that \( \bar{D}(a) = \lim_{\alpha} (\varphi(a) \cdot x'_a - x'_a \cdot \varphi(a)) \quad (a \in A) \). Now continuity and density of \( h(A) \) in \( B \), imply that \( \bar{D} \) is approximately \( \psi\text{-inner} \). Therefore \( B \) is \( \psi\text{-}\mathfrak{A}\text{-module approximately amenable} \). \( \square \)

**Proposition 3.7.** Let \( A \) and \( B \) be \( \mathfrak{A}\text{-module Banach algebras} \), \( \varphi \in \text{Hom}_{\mathfrak{A}}(A) \) and \( \psi \in \text{Hom}_{\mathfrak{A}}(B) \). If \( A \) is not \( \varphi\text{-}\mathfrak{A}\text{-module approximately amenable} \) or \( B \) is not \( \psi\text{-}\mathfrak{A}\text{-module approximately amenable} \), then \( A \oplus B \) is not \( \varphi \oplus \psi\text{-}\mathfrak{A}\text{-module approximately amenable} \).

**Proof.** Suppose that \( A \) is not \( \varphi\text{-}\mathfrak{A}\text{-module approximately amenable} \). The projection map \( \pi : A \oplus B \to A \) determines an \( \mathfrak{A}\text{-module epimorphism} \) of \( A \oplus B \) onto \( A \) such that \( \pi \circ (\varphi \oplus \psi) = \varphi \circ \pi \). So, if \( A \oplus B \) is \( \varphi \oplus \psi\text{-}\mathfrak{A}\text{-module approximately amenable} \), then by Lemma 3.6, \( A \) is \( \varphi\text{-}\mathfrak{A}\text-module approximately amenable}. This contradicts the fact that \( A \) is not \( \varphi\text{-}\mathfrak{A}\text-module approximately amenable} \). Therefore \( A \oplus B \) is not \( \varphi \oplus \psi\text{-}\mathfrak{A}\text-module approximately amenable}. Similarly, we can prove the result for \( B \). \( \square \)

Let \( \mathfrak{A} \) be a non-unital Banach algebra. Then \( \mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C} \), the unitization of \( \mathfrak{A} \) is a unital Banach algebra which contains \( \mathfrak{A} \) as a closed ideal. Let \( A \) be a Banach \( \mathfrak{A}\text{-bimodule} \). Then \( A \) is a Banach \( \mathfrak{A}^\#\text{-module} \) with the following module actions:

\[
(\alpha, \lambda).a = \alpha a + \lambda a, \quad a.(\alpha, \lambda) = a\alpha + \lambda a \quad (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).
\]

Let \( A^\# = (A \oplus \mathfrak{A}^\#, \bullet) \), where the multiplication \( \bullet \) is defined through

\[
(a, u) \bullet (b, v) = (ab + a.v + u.b, uv) \quad (a, b \in A, u, v \in \mathfrak{A}^\#).
\]

Then with the actions defined by

\[
u. (a, v) = (u.a, uv), \quad (a, v).u = (a.u, vu) \quad (a \in A, u, v \in \mathfrak{A}^\#),
\]

\( A^\# \) is a unital \( \mathfrak{A}^\#\text{-module Banach algebra} \) with the identity \( 1_{A^\#} = (0, 1_{\mathfrak{A}^\#}) \) (see [4]).

Before we turn to our next result we note that if for every \( \varphi \in \text{Hom}_{\mathfrak{A}^\#}(A) \), one defines \( \varphi^\#: A^\# \to A^\# \) by \( \varphi^\#(a, u) = (\varphi(a), u) \quad ((a, u) \in A^\#) \), then \( \varphi^\# \in \text{Hom}_{\mathfrak{A}^\#}(A^\#) \).

The following proposition generalizes Proposition 2.7 of [5].

**Theorem 3.8.** Let \( A \) and \( B \) be \( \mathfrak{A}\text{-module Banach algebras} \) and each has a bounded approximate identity. Let \( \varphi \in \text{Hom}_{\mathfrak{A}^\#}(A) \) and \( \psi \in \text{Hom}_{\mathfrak{A}^\#}(B) \). Then \( A \)
is \( \varphi - \mathcal{A}#\)-module approximately amenable and \( B \) is \( \psi - \mathcal{A}#\)-module approximately amenable if and only if \( A \oplus B \) is \( \varphi \oplus \psi - \mathcal{A}#\)-module approximately amenable.

**Proof.** Suppose that \( A \) is \( \varphi - \mathcal{A}#\)-module approximately amenable and \( B \) is \( \psi - \mathcal{A}#\)-module approximately amenable. By Proposition 12 of [13], \( A^2 \) is \( \varphi^# - \mathcal{A}#\)-module approximately amenable and \( B^2 \) is \( \psi^# - \mathcal{A}#\)-module approximately amenable, so by Proposition 3.5, \( A^2 \oplus B^2 \) is \( \varphi^# \oplus \psi^# - \mathcal{A}#\)-module approximately amenable. Since \( A \oplus B \) is a closed \( \mathcal{A}#\)-invariant ideal in \( A^2 \oplus B^2 \), the result follows from Proposition 3.3.

For the converse, suppose that \( A \oplus B \) is \( \varphi \oplus \psi - \mathcal{A}#\)-module approximately amenable. Then by Proposition 3.7, \( A \) is \( \varphi - \mathcal{A}#\)-module approximately amenable and \( B \) is \( \psi - \mathcal{A}#\)-module approximately amenable. \( \square \)

4. \( \varphi \oplus \psi \)-Module Approximate Amenability and \( \varphi \oplus \psi \)-Amenability of Direct Sum of Banach Algebras

We start this section with the following definition:

**Definition 4.1.** We say the Banach algebra \( \mathcal{A} \) acts trivially on \( A \) from the left (right) if for every \( \alpha \in \mathcal{A} \) and \( a \in A \), \( \alpha.a = f(\alpha)a \) (resp. \( a.\alpha = f(\alpha)a \)), where \( f \) is a multiplicative linear functional on \( \mathcal{A} \).

We assume that \( J_{A,\mathcal{A}} \) is the closed linear span of \( \{(a.\alpha)a - a.\alpha.a | \alpha \in \mathcal{A}, a, b \in A\} \), in \( A \). It follows immediately that \( J_{A,\mathcal{A}} \) is both \( \mathcal{A} \)-submodule and \( \mathcal{A} \)-module of \( A \). So \( \frac{A}{J_{A,\mathcal{A}}} \) is both Banach \( \mathcal{A} \)-module and \( \mathcal{A} \)-module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

**Lemma 4.2.** Let \( A \) be a Banach algebra and Banach \( \mathcal{A} \)-module with compatible actions, and \( J_0 \) be a closed ideal of \( A \) such that \( J_{A,\mathcal{A}} \subseteq J_0 \). If \( \frac{A}{J_0} \) has a left or right identity \( e + J_0 \), then for each \( \alpha \in \mathcal{A} \) and \( a \in A \) we have \( a.\alpha - \alpha.a \in J_0 \), i.e, \( \frac{A}{J_0} \) is commutative Banach \( \mathcal{A} \)-module.

Before we turn to our next result we note that if for every \( \varphi \in \text{Hom}_\mathcal{A}(A) \), one defines \( \overline{\varphi} : \frac{A}{J_{A,\mathcal{A}}} \longrightarrow \frac{A}{J_{B,\mathcal{A}}} \) by \( \overline{\varphi}(a + J_{A,\mathcal{A}}) = \varphi(a) + J_{B,\mathcal{A}} \), then \( \overline{\varphi} \in \text{Hom}_\mathcal{A}(\frac{A}{J_{A,\mathcal{A}}}, \frac{B}{J_{B,\mathcal{A}}}) \).

**Theorem 4.3.** Let \( A \) and \( B \) be \( \mathcal{A} \)-module Banach algebras and let \( \varphi \in \text{Hom}_\mathcal{A}(A) \) and \( \psi \in \text{Hom}_\mathcal{A}(B) \). Then the following statements are valid:

(i) \( A \oplus B \) is \( \varphi \oplus \psi - \mathcal{A} \)-module amenable (resp. \( \varphi \oplus \psi - \mathcal{A} \)-module approximately amenable) if and only if \( \frac{A}{J_{A,\mathcal{A}}} \oplus \frac{B}{J_{B,\mathcal{A}}} \) is \( \varphi \oplus \psi - \mathcal{A} \)-module amenable (resp. \( \varphi \oplus \psi - \mathcal{A} \)-module approximately amenable).

(ii) Let \( \mathcal{A} \) acts on \( A \) and \( B \) trivially from the left by \( f \in \text{Hom}_\mathcal{C}(\mathcal{A}) \). Suppose that \( \frac{A}{J_{A,\mathcal{A}}} \) and \( \frac{B}{J_{B,\mathcal{A}}} \) are unital, and \( A \oplus B \) is \( \varphi \oplus \psi - \mathcal{A} \)-module amenable (resp. \( \varphi \oplus \psi - \mathcal{A} \)-module approximately amenable), then \( \frac{A}{J_{A,\mathcal{A}}} \oplus \frac{B}{J_{B,\mathcal{A}}} \) is \( \varphi \oplus \psi \)-amenable (resp. \( \varphi \oplus \psi \)-approximately amenable).
(iii) Let $\mathfrak{A}$ have a bounded approximately identity and $\frac{A_{\mathfrak{A}}}{J_{\mathfrak{A}}} \oplus \frac{B_{\mathfrak{A}}}{J_{\mathfrak{B}}} \oplus \mathfrak{A}$-amenable (resp. $\varphi \oplus \psi$-approximately amenable). Then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable).

Proof. (i) Let $A \oplus B$ be $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable, and let $D : \frac{A_{\mathfrak{A}}}{J_{\mathfrak{A}}} \oplus \frac{B_{\mathfrak{A}}}{J_{\mathfrak{B}}} \oplus \mathfrak{A}$-module derivation for some commutative Banach $X$. Then $X$ becomes a $A \oplus B$-bimodule through the following actions

$$
(a, b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \quad (a \in A, b \in B, x \in X),
$$

(4.1)

and

$$
x.(a, b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B, x \in X).
$$

(4.2)

Hence $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define $\tilde{D} : A \oplus B \to X^*$ by

$$
\tilde{D}(a, b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B).
$$

It is easy to check that, $\tilde{D}$ is a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. From the $\varphi \oplus \psi$-$\mathfrak{A}$-module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$
\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).
$$

Thus

$$
D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* - x^*.\varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).
$$

This means that $D$ is $\varphi \oplus \psi$-inner. Therefore $\frac{A_{\mathfrak{A}}}{J_{\mathfrak{A}}} \oplus \frac{B_{\mathfrak{B}}}{J_{\mathfrak{B}}} \oplus \mathfrak{A}$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable.

Conversely, suppose that $\frac{A_{\mathfrak{A}}}{J_{\mathfrak{A}}} \oplus \frac{B_{\mathfrak{B}}}{J_{\mathfrak{B}}} \oplus \mathfrak{A}$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable. Let $D : A \oplus B \to X^*$ be a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation for some commutative Banach $X$. We consider the following module actions of $\frac{A_{\mathfrak{A}}}{J_{\mathfrak{A}}} \oplus \frac{B_{\mathfrak{B}}}{J_{\mathfrak{B}}} \oplus \mathfrak{A}$ on $X$,

$$(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a, b).x, \quad x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a, b),$$

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of $X$, we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{B}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{B}} = 0$. Thus $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{B}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{B}}) = 0$. So $X$ is a commutative Banach $\frac{A_{\mathfrak{A}}}{J_{\mathfrak{A}}} \oplus \frac{B_{\mathfrak{B}}}{J_{\mathfrak{B}}} \oplus \mathfrak{A}$-module.

Define $\tilde{D} : \frac{A_{\mathfrak{A}}}{J_{\mathfrak{A}}} \oplus \frac{B_{\mathfrak{B}}}{J_{\mathfrak{B}}} \to X^*$ by

$$
\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a, b) \quad (a \in A, b \in B).
$$

Also using (2.2) and (2.3) we see that $D$ vanishes on $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{B}}$. Hence $\tilde{D}$ is well defined. One can easily check that $\tilde{D}$ is a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation.
Now from the $\varphi \oplus \psi$-$\mathfrak{A}$-module amenability of $\frac{A}{J_{A,\alpha}} \oplus \frac{B}{J_{B,\alpha}}$, it follows that there exists $x^* \in X^*$ such that

$$
\tilde{D}(a + J_{A,\alpha}, b + J_{B,\alpha}) = \varphi \oplus \psi(a + J_{A,\alpha}, b + J_{B,\alpha}).x^* - x^* \varphi \oplus \psi(a + J_{A,\alpha}, b + J_{B,\alpha}) \quad (a \in A, b \in B).
$$

It follows that

$$
D(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).
$$

Thus $D$ is $\varphi \oplus \psi$-inner. So $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable.

Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

(ii) Let $A \oplus B$ be $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable and let $D : \frac{A}{J_{A,\alpha}} \oplus \frac{B}{J_{B,\alpha}} \rightarrow X^*$ be a derivation for some Banach $\frac{A}{J_{A,\alpha}} \oplus \frac{B}{J_{B,\alpha}}$-bimodule $X$. Then $X$ becomes an $A \oplus B$-bimodule through the actions as (4.1) and (4.2). Also $X$ is an $\mathfrak{A}$-bimodule with $f$-trivial actions, that is

$$
\alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, \ x \in X).
$$

Then $X$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Define

$$
\Gamma : A \oplus B \rightarrow \frac{A}{J_{A,\alpha}} \oplus \frac{B}{J_{B,\alpha}}, \quad (a, b) + I \mapsto (a + J_{A,\alpha}, b + J_{B,\alpha}),
$$

where $I = J_{A,\alpha} \oplus J_{B,\alpha}$. It is routinely checked that $\Gamma$ defines an $\mathfrak{A}$-bimodule morphism. Let $\Pi : A \oplus B \rightarrow \frac{A \oplus B}{I}$ be the quotient map, and let $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \rightarrow X^*$. For every $(a, b), (a', b') \in A \oplus B$, we may easily prove that

$$
\tilde{D}((a, b)(a', b')) = \tilde{D}(a, b).\varphi \oplus \psi(a', b') + \varphi \oplus \psi(a, b).\tilde{D}(a', b'),
$$

and for every $(a, b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$, we have

$$
\tilde{D}(\alpha.(a, b)) = \tilde{D}((\alpha.a, \alpha.b)) = D((f(\alpha)a, f(\alpha)b)) = D((f(\alpha)a + J_{A,\alpha}, f(\alpha)b + J_{B,\alpha})) = D\left(f(\alpha)(a + J_{A,\alpha}, b + J_{B,\alpha})\right) = f(\alpha)D\left((a + J_{A,\alpha}, b + J_{B,\alpha})\right) = \alpha.D\left((a + J_{A,\alpha}, b + J_{B,\alpha})\right) = \alpha.D(a, b),
$$
and using Lemma 4.2, we have
\[ \tilde{D}(a,b) = \tilde{D}(a.a,b.a) = D(a.a + J_{A,\mathfrak{A}}, b.a + J_{B,\mathfrak{A}}) \]
\[ = D(a.a + J_{A,\mathfrak{A}}, a.b + J_{B,\mathfrak{A}}) \]
\[ = D(f(a) + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \]
\[ = f(a)D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \]
\[ = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}), \alpha \]
\[ = \tilde{D}(a,b).\alpha. \]

Thus \( \tilde{D} \) is a \( \varphi \oplus \psi \oplus \mathfrak{A} \)-module derivation and from the \( \varphi \oplus \psi \oplus \mathfrak{A} \)-module amenability of \( A \oplus B \), it follows that there exists \( x^* \in X^* \) such that
\[ \tilde{D}(a,b) = \varphi \oplus \psi(a,b).x^* - x^*.\varphi \oplus \psi(a,b) \ (a \in A, b \in B). \]

It follows that
\[ D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \]
\[ - x^*.\varphi \oplus \psi(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).\]

So \( D \) is \( \varphi \oplus \psi \)-inner. Therefore \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \) is \( \varphi \oplus \psi \)-amenable.

(iii) Suppose that \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \) is \( \varphi \oplus \psi \)-amenable. Since \( \mathfrak{A} \) has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \) is \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module amenable. So by (i), \( A \oplus B \) is \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module amenable.

Similar relations can be obtained between the \( \varphi \oplus \psi \)-\( \mathfrak{A} \)-module approximate amenability of \( A \oplus B \) and \( \varphi \oplus \psi \)-approximate amenability of \( \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \). \( \square \)

**Proposition 4.4.** Let \( A \) be an \( \mathfrak{A} \)-module Banach algebra, where \( \mathfrak{A} \) acts on \( A \) trivially from the left by \( f \in \text{Hom}_C(\mathfrak{A}) \). Let \( \varphi \in \text{Hom}_C(A) \) and \( \frac{A}{J_{A,\mathfrak{A}}} \) be unital. If \( A \) is \( \varphi \)-\( \mathfrak{A} \)-module approximately amenable, then \( \frac{A}{J_{A,\mathfrak{A}}} \) is \( \varphi \)-approximately amenable.

**Proof.** Let \( X \) be a Banach \( \frac{A}{J_{A,\mathfrak{A}}} \)-bimodule and \( D : \frac{A}{J_{A,\mathfrak{A}}} \to X^* \) be a \( \varphi \)-derivation. Then \( X \) becomes a \( \mathfrak{A} \)-bimodule through the following actions
\[ a.x = (a + J_{A,\mathfrak{A}}).x, \quad x.a = x.(a + J_{A,\mathfrak{A}}) \ (a \in A, x \in X), \]
and \( X \) is an \( \mathfrak{A} \)-bimodule with \( f \)-trivial actions, that is \( \alpha.x = x.\alpha = f(\alpha)x \ (\alpha \in \mathfrak{A}, x \in X) \). By Lemma 4.2, \( f(\alpha)a - a.\alpha \in J_{A,\mathfrak{A}} \ (\alpha \in \mathfrak{A}, a \in A) \). So, \( f(\alpha)a + J_{A,\mathfrak{A}} = a.\alpha + J_{A,\mathfrak{A}} \ (\alpha \in \mathfrak{A}, a \in A) \), and the actions of \( \mathfrak{A} \) and \( A \) on \( X \) are compatible. Thus \( X \) is a commutative Banach \( A-\mathfrak{A} \)-module. Let \( \tilde{D} : A \to X^* \) be defined by \( \tilde{D}(a) = D(a + J_{A,\mathfrak{A}}) \ (a \in A) \). A similar argument as in the proof of Theorem 3.2 of [2], shows that \( \tilde{D} \) is approximately \( \varphi \)-inner. So, \( D \) is approximately \( \varphi \)-inner. Therefore \( \frac{A}{J_{A,\mathfrak{A}}} \) is \( \varphi \)-approximately amenable. \( \square \)
Theorem 4.5. Let $\mathfrak{A}$ have a bounded approximate identity, and let $A$ and $B$ be $\mathfrak{A}$-module Banach algebras, where $\mathfrak{A}$ acts on $A$ and $B$ trivially from the left. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, and let $A$ and $B$ be unital. Then $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Proof. Suppose that $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable. By Proposition 4.4, $A$ and $B$ are $\varphi$-approximately amenable and $\psi$-approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A} = \mathbb{C}$, we conclude that $A \oplus B$ is $\varphi \oplus \psi$-approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Then by Proposition 3.7, $A$ is $\varphi$-$\mathfrak{A}$-module approximately amenable and $B$ is $\psi$-$\mathfrak{A}$-module approximately amenable.

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