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Generalized Approximate Amenability of Direct Sum of Banach Algebras

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Abstract. In the present paper for two $\mathfrak{A}$-module Banach algebras A and B, we investigate relations between φ-$\mathfrak{A}$-module approximate amenability of A, ψ-$\mathfrak{A}$-module approximate amenability of B, and $\varphi \oplus \psi$-$\mathfrak{A}$-module approximate amenability of $A \oplus B$ (l^1-direct sum of A and B), where $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathtt{Hom}_{\mathfrak{A}}(B)$.

Keywords: Banach algebra, Module derivation, Module approximate amenability.

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1. Introduction

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if A and B are approximately amenable Banach algebras and one of A or B has a bounded approximate identity, then $A \oplus B$ is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have

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studied φ-derivations, and φ-amenability of Banach algebra A, whenever φ is a continuous homomorphism on A (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.
The organization of this paper is as follows:
Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for $\mathfrak{A}$-module Banach algebras A and B where each has a bounded approximate identity we show that A is φ-$\mathfrak{A}^{\#}$-module approximately amenable and B is ψ-$\mathfrak{A}^{\#}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^{\#}$-module approximately amenable.

In section 4 we show that if $\mathfrak{A}$ has a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, then A is φ-$\mathfrak{A}$-module approximately amenable and B is ψ-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

2. Preliminaries

Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions given by

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \qquad (a, b \in A, \alpha \in \mathfrak{A}).$$

Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \; a.(\alpha.x) = (a.\alpha).x, \; (\alpha.x).a = \alpha.(x.a)$$

$$(a \in A, \alpha \in \mathfrak{A}, x \in X), \tag{2.1}$$

and similar for the right or two-sided actions. Then we say that X is a Banach A-$\mathfrak{A}$-module. A Banach A-$\mathfrak{A}$-module X is called commutative A-$\mathfrak{A}$-module, if $\alpha.x = x.\alpha$ $(\alpha \in \mathfrak{A}, x \in X)$. Note that in general, A dose not satisfy the compatibility condition $a.(\alpha.b) = (a.\alpha).b$ $(a, b \in A, \alpha \in \mathfrak{A})$.

If X is a commutative Banach A-$\mathfrak{A}$-module , then so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined as follows

$$\langle \alpha.f, x\rangle = \langle f, x.\alpha\rangle, \; \langle a.f, x\rangle = \langle f, x.a\rangle \; (a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*),$$

and similar for the right actions.

Let A and B be Banach $\mathfrak{A}$-bimodules. Then a $\mathfrak{A}$-module morphism from A to B is a norm continuous map $h : A \longrightarrow B$ with $h(a \pm b) = h(a) \pm h(b)$ which is multiplicative, that is

$$h(\alpha.a) = \alpha.h(a), \quad h(a.\alpha) = h(a).\alpha, \quad h(ab) = h(a)h(b) \;\; (a \in A, b \in B, \alpha \in \mathfrak{A}).$$

We denote by $\mathtt{Hom}_{\mathfrak{A}}(A,B)$, the space of all such morphism and denote $\mathtt{Hom}_{\mathfrak{A}}(A,A)$ by $\mathtt{Hom}_{\mathfrak{A}}(A)$. In the case that $\mathfrak{A}=\mathbb{C}$, we denote $\mathtt{Hom}_{\mathbb{C}}(A,B)$ by $\mathtt{Hom}(A,B)$ and denote $\mathtt{Hom}_{\mathbb{C}}(A,A)$ by $\mathtt{Hom}(A)$.

Let X be a Banach A-bimodule and let $\varphi\in\mathtt{Hom}_{\mathfrak{A}}(A)$. A bounded map $D:A\longrightarrow X$ is called a φ-$\mathfrak{A}$-module derivation if

$$D(a\pm b)=D(a)\pm D(b),\ D(ab)=D(a).\varphi(b)+\varphi(a).D(b)\ (a,b\in A), \quad (2.2)$$

and

$$D(\alpha.a)=\alpha.D(a),\ D(a.\alpha)=D(a).\alpha\ (a\in A,\alpha\in\mathfrak{A}). \quad (2.3)$$

Although D in general is not linear, but still its boundedness implies its norm continuity.

Let X be a commutative Banach A-$\mathfrak{A}$-module. For every $x\in X$ define ad_x^{φ} by $ad_x^{\varphi}(a)=\varphi(a).x-x.\varphi(a)$ $(a\in A)$. It is easily seen that ad_x^{φ} is a φ-$\mathfrak{A}$-module derivation. A φ-$\mathfrak{A}$-module derivation D is called φ-inner if there is $x\in X$ such that $D(a)=ad_x^{\varphi}(a)$ $(a\in A)$ and is called approximately φ-inner if there exists a net $(x_\alpha)_\alpha\subseteq X$ such that $D(a)=\lim_\alpha ad_{x_\alpha}^{\varphi}(a)$ $(a\in A)$. A Banach algebra A is called φ-$\mathfrak{A}$-module amenable if for any commutative Banach A-$\mathfrak{A}$-module X, each φ-$\mathfrak{A}$-module derivation $D:A\longrightarrow X^*$ is φ-inner, and A is called φ-$\mathfrak{A}$-module approximately amenable if each φ-$\mathfrak{A}$-module derivation $D:A\longrightarrow X^*$ is approximately φ-inner (see [1, 15]).

In the case that $\mathfrak{A}=\mathbb{C}$, φ-$\mathfrak{A}$-module derivations (resp. φ-$\mathfrak{A}$-module amenable Banach algebras, φ-$\mathfrak{A}$-module approximately amenable Banach algebras) are called φ-derivation (resp. φ-amenable, φ-approximately amenable) (see [9, 10]).

3. $\varphi\oplus\psi$-Module Approximate Amenability of the Direct Sum of Banach Algebras

We commence this section with the following remark from [1]:

Remark 3.1. Assume that A has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_{\mathfrak{A}}(A)$ denotes the algebra of $\mathfrak{A}$-multipliers of A, that is $M_{\mathfrak{A}}(A)=\{(T_1,T_2): T_1,T_2\in L_{\mathfrak{A}}(A): T_1(ab)=T_1(a)b, T_2(ab)=aT_2(b)(a,b\in A)\}$, where $L_{\mathfrak{A}}(A)$ is the space of all $\mathfrak{A}$-module morphisms on A. Then $M_{\mathfrak{A}}(A)$ is an A-$\mathfrak{A}$-module and A embeds in $M_{\mathfrak{A}}(A)$ via $a\longmapsto(L_a,R_a)$, where $L_a(b)=ab, R_a(b)=ba$ $(a,b\in A)$. For any element $T=(T_1,T_2)$ of $M_{\mathfrak{A}}(A)$ it is easy to see that $\parallel T_1\parallel=\parallel T_2\parallel$ and if we put $\parallel T\parallel$ equal to this common value, then $M_{\mathfrak{A}}(A)$ becomes a Banach A-$\mathfrak{A}$-module, and A is dense in $M_{\mathfrak{A}}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi\in\mathtt{Hom}_{\mathfrak{A}}(A)$, then by continuity of φ in the strict topology, it can be extended to an $\mathfrak{A}$-homomorphism $\tilde{\varphi}:M_{\mathfrak{A}}(A)\longrightarrow M_{\mathfrak{A}}(A)$ defined by $\tilde{\varphi}(L_a,R_a)=(L_{\varphi(a)},R_{\varphi(a)})$.

Proposition 3.2. *Let A be an $\mathfrak{A}$-module Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and let $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$. Then A is φ-$\mathfrak{A}$-module approximately amenable if and only if $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable.*

Proof. Let $M_{\mathfrak{A}}(A)$ be $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable and let $D : A \longrightarrow X^*$ be a φ-$\mathfrak{A}$-module derivation for some commutative Banach A-$\mathfrak{A}$-module X. Then by the following actions

$$T.x = \lim_\alpha T_1(e_\alpha).x,\ \ x.T = \lim_\alpha x.T_2(e_\alpha)\ (x \in X, T = (T_1, T_2) \in M_{\mathfrak{A}}(A)),$$

X is a commutative Banach $M_{\mathfrak{A}}(A)$-$\mathfrak{A}$-module and by continuity of D in the strict topology, it can be extended to a bounded $\tilde{\varphi}$-$\mathfrak{A}$-derivation $\tilde{D} : M_{\mathfrak{A}}(A) \longrightarrow X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\tilde{\varphi}$-$\mathfrak{A}$-module approximate amenability of $M_{\mathfrak{A}}(A)$, it follows that there exists a net $(x^*_\beta)_\beta \subset X^*$ such that

$$\tilde{D}(T) = \lim_\beta \big(\tilde{\varphi}(T).x^*_\beta - x^*_\beta.\tilde{\varphi}(T)\big).$$

Hence for every $a \in A$ we have

$$\begin{aligned} D(a) = \tilde{D}(L_a, R_a) &= \lim_\beta \big(\tilde{\varphi}(L_a, R_a).x^*_\beta - x^*_\beta.\tilde{\varphi}(L_a, R_a)\big) \\ &= \lim_\beta \big((L_{\varphi(a)}, R_{\varphi(a)}).x^*_\beta - x^*_\beta.(L_{\varphi(a)}, R_{\varphi(a)})\big) \\ &= \lim_\beta \big(\lim_\alpha L_{\varphi(a)}(e_\alpha).x^*_\beta - \lim_\alpha x^*_\beta.R_{\varphi(a)}(e_\alpha)\big) \\ &= \lim_\beta \big(\varphi(a).x^*_\beta - x^*_\beta.\varphi(a)\big). \end{aligned}$$

This means that D is approximately φ-inner and so A is φ-$\mathfrak{A}$-module approximately amenable.

Conversely, Suppose that A is φ-$\mathfrak{A}$-module approximately amenable. Let X be a commutative Banach $M_{\mathfrak{A}}(A)$-$\mathfrak{A}$-module and let $D : M_{\mathfrak{A}}(A) \longrightarrow X^*$ be a $\tilde{\varphi}$-$\mathfrak{A}$-module derivation. We consider the module actions of A on X by

$$a.x = (L_a, R_a).x,\ \ x.a = x.(L_a, R_a)\ (a \in A, x \in X). \tag{3.1}$$

Thus X is a commutative Banach A-$\mathfrak{A}$-module. Define $\tilde{D} : A \longrightarrow X^*$ by $\tilde{D}(a) = D(L_a, R_a)$ $(a \in A)$. It is easy to see that $\tilde{D}$ is a φ-$\mathfrak{A}$-module derivation and from the φ-$\mathfrak{A}$-module approximate amenability of A, it follows that there exists a net $(x^*_\beta)_\beta \subset X^*$ such that

$$\tilde{D}(a) = \lim_\beta \big(\varphi(a).x^*_\beta - x^*_\beta.\varphi(a)\big)\ (a \in A).$$

Then $D(L_a, R_a) = \lim_\beta \big(\tilde{\varphi}(L_a, R_a).x^*_\beta - x^*_\beta.\tilde{\varphi}(L_a, R_a)\big)$. Now by the continuity of D and $\tilde{\varphi}$, and density of A in $M_{\mathfrak{A}}(A)$ in the strict topology, we conclude that

$$D(T) = \lim_\beta \big(\tilde{\varphi}(T).x^*_\beta - x^*_\beta.\tilde{\varphi}(T)\big)\ (T \in M_{\mathfrak{A}}(A)).$$

So D is a approximately $\tilde{\varphi}$-inner. Therefore $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$-$\mathfrak{A}$-module approximately amenable. $\square$

Let I be a closed ideal of a Banach algebra A with a bounded approximate identity $(e_\alpha)_\alpha$, and let X be a commutative Banach I-$\mathfrak{A}$-module. Let $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi\mid_I \subset I$, then X is a commutative Banach A-$\mathfrak{A}$-module with the following actions

$$a.x = \lim_\alpha \varphi(e_\alpha)a.x, \quad x.a = \lim_\alpha x.\varphi(e_\alpha)a \qquad (a \in A, x \in X). \tag{3.2}$$

Proposition 3.3. *Let I be a closed ideal of an $\mathfrak{A}$-module Banach algebra A which has a bounded approximate identity $\{e_\alpha\}$, and let I be $\mathfrak{A}$-invariant, i.e. $\mathfrak{A}.I \subseteq I$. Let $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi\mid_I \subset I$. If A is φ-$\mathfrak{A}$-module approximately amenable, then I is $\varphi\mid_I$-$\mathfrak{A}$-module approximately amenable.*

Proof. Let X be a commutative Banach $M_{\mathfrak{A}}(I)$-$\mathfrak{A}$-module, and $D : M_{\mathfrak{A}}(I) \longrightarrow X^*$ be a $\tilde{\varphi}$-$\mathfrak{A}$-module derivation. By the same actions as (3.1), we can consider X as a commutative Banach I-$\mathfrak{A}$-module. So, by (3.2), X is a commutative Banach A-$\mathfrak{A}$-module. By definition of $M_{\mathfrak{A}}(I)$, there is an $\mathfrak{A}$-module morphism $h : A \longrightarrow M_{\mathfrak{A}}(I)$ and $D \circ h$ is a module derivation on A, so it is approximately φ-inner. Hence D is approximately $\tilde{\varphi}$-inner. Since I has a bounded approximate identity, by Proposition 3.2, I is $\varphi\mid_I$-$\mathfrak{A}$-module approximately amenable. $\square$

Let A and B be $\mathfrak{A}$-module Banach algebras. It is well known that $A \oplus B$, the l^1-direct sum of A and B, is a Banach algebra with respect to the canonical multiplication defined by $(a, b)(c, d) := (ac, bd)$, and is a Banach $\mathfrak{A}$-bimodule by the following actions

$$\alpha.(a, b) := (\alpha.a, \alpha.b), \ (a, b).\alpha := (a.\alpha, b.\alpha) \ (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathtt{Hom}_{\mathfrak{A}}(B)$, then $\varphi \oplus \psi : A \oplus B \longrightarrow A \oplus B$ defined by $\varphi \oplus \psi(a, b) = \big(\varphi(a), \psi(b)\big)$ is an $\mathfrak{A}$-morphism on $A \oplus B$.

Lemma 3.4. *Let A be a unital $\mathfrak{A}$-module Banach algebra, $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$, and let $D : A \longrightarrow X^*$ be a φ-$\mathfrak{A}$-module derivation for some commutative Banach A-$\mathfrak{A}$-module X. If the left (resp. right, two-sided) action of $\varphi(A)$ on X^* is zero, then D is φ-inner.*

Proof. Let e_A be the identity of A and let the left (resp. right, two-sided) action of $\varphi(A)$ on X^* is zero. We can easily show that $D = ad^{\varphi}_{-D(e)}$ (resp. $D = ad^{\varphi}_{D(e)}, D = 0$). So D is φ-inner. $\square$

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

Proposition 3.5. *Let A and B be unital $\mathfrak{A}$-module Banach algebras with identities e_A and e_B, respectively, and let $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathtt{Hom}_{\mathfrak{A}}(B)$ such that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$, and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ $(\alpha \in \mathfrak{A})$. If A is φ-$\mathfrak{A}$-module approximately amenable and B is ψ-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.*

Proof. Let X be a commutative Banach $A \oplus B$-$\mathfrak{A}$-module and let $D : A \oplus B \longrightarrow X^*$ be a $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation. Write $Y_1 = \varphi(e_A).X^*.\varphi(e_A)$, $Y_2 = \psi(e_B).X^*.\psi(e_B)$, $Y_3 = \varphi(e_A).X^*.\psi(e_B)$, $Y_4 = \psi(e_B).X^*.\varphi(e_A)$, $Y_5 = \big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big).X^*.\varphi(e_A)$, $Y_6 = \big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big).X^*.\psi(e_B)$, $Y_7 = \varphi(e_A).X^*.\big(1-\varphi(e_A)\big)\big(1-\psi(e_B)\big)$, $Y_8 = \psi(e_B).X^*.\big(1-\varphi(e_A)\big)\big(1-\psi(e_B)\big)$, $Y_9 = \big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big).X^*.\big(1 - \varphi(e_A)\big)\big(1 - \psi(e_B)\big)$ and let $\pi_j : X^* \longrightarrow Y_j$ be the associated projections. Thus $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$. Consider the derivations $D_j = \pi_j \circ D$, so $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$. From the fact that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$ $(\alpha \in \mathfrak{A})$, and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ $(\alpha \in \mathfrak{A})$, one can easily check that Y_j for $j = 1, ..., 9$ is a commutative Banach $A \oplus B$-$\mathfrak{A}$-module. Since the action of $\varphi(A) \oplus \psi(B)$ on (at least) one side on Y_5 (resp. Y_6, Y_7, Y_8, Y_9) is zero, by Lemma 3.4, we conclude that D_5 (resp. D_6, D_7, D_8, D_9) is approximately $\varphi \oplus \psi$-inner.

From the φ-$\mathfrak{A}$-module approximate amenability of A, it follows that the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $A \oplus 0 \longrightarrow \varphi(e_A).X^*.\varphi(e_A)$ is approximately $\varphi \oplus \psi$-inner and since the action of $0 \oplus \psi(B)$ on $\varphi(e_A).X^*.\varphi(e_A)$ is zero, we conclude that D_1 is approximately $\varphi \oplus \psi$-inner. Similarly, the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_2 : A \oplus B \longrightarrow \psi(e_B).X^*.\psi(e_B)$ is approximately $\varphi \oplus \psi$-inner.

The right action of $\varphi(A) \oplus 0$ on $\varphi(e_A).X^*.\psi(e_B)$ is zero. Hence, by Lemma 3.4, $D_3 |_{A\oplus 0}$ is $\varphi \oplus \psi$-inner. So there exists $\xi \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3 |_{A\oplus 0} (a, 0) = \varphi(a).\xi - \xi.\varphi(a) = \big(\varphi(a), \psi(b)\big)\varphi(e_A).\xi.\psi(e_B),$$

for every $a \in A$ and $b \in B$. Similarly, there exists $\eta \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3 |_{0\oplus B} (0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)\big(\varphi(a), \psi(b)\big),$$

for every $a \in A$ and $b \in B$. Hence

$$D_3(a, b) = \big(\varphi(a), \psi(b)\big)\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B)\big(\varphi(a), \psi(b)\big).$$

Since $D_3(e_A, e_B) = 0$, it follows that

$$0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B).$$

Then for every $a \in A$ and $b \in B$, we have

$$D_3(a, b) = \big(\varphi(a), \psi(b)\big)\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\xi.\psi(e_B)\big(\varphi(a), \psi(b)\big).$$

Thus D_3 is $\varphi \oplus \psi$-inner. The same argument holds for the $\varphi \oplus \psi$-$\mathfrak{A}$-module derivation $D_4 : A \oplus B \longrightarrow \psi(e_B).X^*.\varphi(e_A)$. Therefore D is approximately $\varphi \oplus \psi$-inner, and so $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. □

Lemma 3.6. *Let A and B be $\mathfrak{A}$-module Banach algebras, $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathtt{Hom}_{\mathfrak{A}}(B)$. If there is a h in $\mathtt{Hom}_{\mathfrak{A}}(A, B)$ such that $h \circ \varphi = \psi \circ h$ and the range of h is a dense subset of B, then φ-$\mathfrak{A}$-module approximate amenability of A implies ψ-$\mathfrak{A}$-module approximate amenability of B.*

Proof. Let $D : B \longrightarrow X^*$ be a ψ-$\mathfrak{A}$-module derivation for some commutative Banach B-$\mathfrak{A}$-module X. Then by the following actions

$$a \bullet x = h(a).x,\ x \bullet a = x.h(a) \qquad (a \in A, x \in X),$$

X is a commutative Banach A-$\mathfrak{A}$-module. Let $\tilde{D} = D \circ h : A \longrightarrow X^*$. One can easily prove that D is a φ-$\mathfrak{A}$-module derivation. From the φ-$\mathfrak{A}$-module approximate amenability of A, it follows that there exists a net $(x^*_\alpha)_\alpha$ in X^* such that $\tilde{D}(a) = \lim_\alpha \big(\varphi(a) \bullet x^*_\alpha - x^*_\alpha \bullet \varphi(a)\big)$ $(a \in A)$. Now continuity and density of $h(A)$ in B, imply that D is approximately ψ-inner. Therefore B is ψ-$\mathfrak{A}$-module approximately amenable. □

Proposition 3.7. *Let A and B be $\mathfrak{A}$-module Banach algebras, $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathtt{Hom}_{\mathfrak{A}}(B)$. If A is not φ-$\mathfrak{A}$-module approximately amenable or B is not ψ-$\mathfrak{A}$-module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.*

Proof. Suppose that A is not φ-$\mathfrak{A}$-module approximately amenable. The projection map $\pi : A \oplus B \longrightarrow A$ determines an $\mathfrak{A}$-module epimorphism of $A \oplus B$ onto A such that $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable, then by Lemma 3.6, A is φ-$\mathfrak{A}$-module approximately amenable. This contradicts the fact that A is not φ-$\mathfrak{A}$-module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.
Similarly, we can prove the result for B. □

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$, the unitization of $\mathfrak{A}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let A be a Banach $\mathfrak{A}$-bimodule. Then A is a Banach $\mathfrak{A}^\#$-module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a,\ a.(\alpha, \lambda) = a.\alpha + \lambda a\ (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let $A^\sharp = (A \oplus \mathfrak{A}^\#, \bullet)$, where the multiplication $\bullet$ is defined through

$$(a, u) \bullet (b, v) = (ab + a.v + u.b, uv)\ (a, b \in A, u, v \in \mathfrak{A}^\#).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv),\ (a, v).u = (a.u, vu)\ (a \in A, u, v \in \mathfrak{A}^\#),$$

$A^\sharp$ is a unital $\mathfrak{A}^\#$-module Banach algebra with the identity $1_{A^\sharp} = (0, 1_{\mathfrak{A}^\#})$ (see [4]).

Before we turn to our next result we note that if for every $\varphi \in \mathtt{Hom}_{\mathfrak{A}^\#}(A)$, one defines $\varphi^\sharp : A^\sharp \longrightarrow A^\sharp$ by $\varphi^\sharp(a, u) = \big(\varphi(a), u\big)$ $\big((a, u) \in A^\sharp\big)$, then $\varphi^\sharp \in \mathtt{Hom}_{\mathfrak{A}^\#}(A^\sharp)$.

The following proposition generalizes Proposition 2.7 of [5].

Theorem 3.8. *Let A and B be $\mathfrak{A}$-module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \mathtt{Hom}_{\mathfrak{A}^\#}(A)$ and $\psi \in \mathtt{Hom}_{\mathfrak{A}^\#}(B)$. Then A*

is φ-$\mathfrak{A}^{\#}$-module approximately amenable and B is ψ-$\mathfrak{A}^{\#}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^{\#}$-module approximately amenable.

Proof. Suppose that A is φ-$\mathfrak{A}^{\#}$-module approximately amenable and B is ψ-$\mathfrak{A}^{\#}$-module approximately amenable. By Proposition 12 of [13], $A^{\sharp}$ is $\varphi^{\sharp}$-$\mathfrak{A}^{\#}$-module approximately amenable and $B^{\sharp}$ is $\psi^{\sharp}$-$\mathfrak{A}^{\#}$-module approximately amenable, so by Proposition 3.5, $A^{\sharp} \oplus B^{\sharp}$ is $\varphi^{\sharp} \oplus \psi^{\sharp}$-$\mathfrak{A}^{\#}$-module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^{\#}$-invariant ideal in $A^{\sharp} \oplus B^{\sharp}$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}^{\#}$-module approximately amenable. Then by Proposition 3.7, A is φ-$\mathfrak{A}^{\#}$-module approximately amenable and B is ψ-$\mathfrak{A}^{\#}$-module approximately amenable. □

4. $\varphi \oplus \psi$-Module Approximate Amenability and $\varphi \oplus \psi$-Amenability of Direct Sum of Banach Algebras

We start this section with the following definition:

Definition 4.1. We say the Banach algebra $\mathfrak{A}$ acts trivially on A from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where f is a multiplicative linear functional on $\mathfrak{A}$.

We assume that $J_{A,\mathfrak{A}}$ is the closed linear span of

$$\{(a.\alpha)b - a(\alpha.b) \mid \alpha \in \mathfrak{A}, a, b \in A\},$$

in A. It follows immediately that $J_{A,\mathfrak{A}}$ is both A-submodule and $\mathfrak{A}$-submodule of A. So $\frac{A}{J_{A,\mathfrak{A}}}$ is both Banach A-module and $\mathfrak{A}$-module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

Lemma 4.2. *Let A be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and J_0 be a closed ideal of A such that $J_{A,\mathfrak{A}} \subseteq J_0$. If $\frac{A}{J_0}$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $a.\alpha - \alpha.a \in J_0$, i.e, $\frac{A}{J_0}$ is commutative Banach $\mathfrak{A}$-module.*

Before we turn to our next result we note that if for every $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$, one defines $\overline{\varphi} : \frac{A}{J_{A,\mathfrak{A}}} \longrightarrow \frac{A}{J_{A,\mathfrak{A}}}$ by $\overline{\varphi}(a + J_{A,\mathfrak{A}}) = \varphi(a) + J_{A,\mathfrak{A}}$, then $\overline{\varphi} \in \mathtt{Hom}_{\mathfrak{A}}(\frac{A}{J_{A,\mathfrak{A}}})$.

Theorem 4.3. *Let A and B be $\mathfrak{A}$-module Banach algebras and let $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \mathtt{Hom}_{\mathfrak{A}}(B)$. Then the following statements are valid:*

- (i) *$A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable) if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-$\mathfrak{A}$-module approximately amenable).*
- (ii) *Let $\mathfrak{A}$ acts on A and B trivially from the left by $f \in \mathtt{Hom}_{\mathbb{C}}(\mathfrak{A})$. Suppose that $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module amenable (resp. $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable), then $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$-approximately amenable).*

(iii) *Let $\mathfrak{A}$ have a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi}\oplus\overline{\psi}$-amenable (resp. $\overline{\varphi}\oplus\overline{\psi}$-approximately amenable). Then $A\oplus B$ is $\varphi\oplus\psi$-$\mathfrak{A}$-module amenable (resp. $\varphi\oplus\psi$-$\mathfrak{A}$-module approximately amenable).*

Proof. (i) Let $A\oplus B$ be $\varphi\oplus\psi$-$\mathfrak{A}$-module amenable, and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ be $\overline{\varphi}\oplus\overline{\psi}$-$\mathfrak{A}$-module derivation for some commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module X. Then X becomes a $A\oplus B$-bimodule through the following actions

$$(a,b).x := (a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}).x \ \ (a\in A, b\in B, x\in X), \tag{4.1}$$

and

$$x.(a,b) := x.(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}) \ \ (a\in A, b\in B, x\in X). \tag{4.2}$$

Hence X is a commutative Banach $A\oplus B$-$\mathfrak{A}$-module. Define $\tilde{D} : A\oplus B \longrightarrow X^*$ by

$$\tilde{D}(a,b) = D(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}) \ (a\in A, b\in B).$$

It is easy to check that, $\tilde{D}$ is a $\varphi\oplus\psi$-$\mathfrak{A}$-module derivation. From the $\varphi\oplus\psi$-$\mathfrak{A}$-module amenability of $A\oplus B$, it follows that there exists $x^*\in X^*$ such that

$$\tilde{D}(a,b) = \varphi\oplus\psi(a,b).x^* - x^*.\varphi\oplus\psi(a,b) \ \ (a\in A, b\in B).$$

Thus

$$\begin{aligned} D(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}) = {} & \overline{\varphi}\oplus\overline{\psi}(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}).x^* \\ & - x^*.\overline{\varphi}\oplus\overline{\psi}(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}). \end{aligned}$$

This means that D is $\overline{\varphi}\oplus\overline{\psi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi}\oplus\overline{\psi}$-$\mathfrak{A}$-module amenable.

Conversely, suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi}\oplus\overline{\psi}$-$\mathfrak{A}$-module amenable. Let $D : A\oplus B \longrightarrow X^*$ be a $\varphi\oplus\psi$-$\mathfrak{A}$-module derivation for some commutative Banach $A\oplus B$-$\mathfrak{A}$-module X. We consider the following module actions of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ on X,

$$(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}).x := (a,b).x, \ \ x.(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}) := x.(a,b),$$

for all $a\in A, b\in B$ and $x\in X$. Using (2.1) and the commutativity of X, we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$. Thus $(J_{A,\mathfrak{A}}\oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}}\oplus J_{B,\mathfrak{A}}) = 0$. So X is a commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$-$\mathfrak{A}$-module. Define $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \longrightarrow X^*$ by

$$\tilde{D}(a+J_{A,\mathfrak{A}}, b+J_{B,\mathfrak{A}}) = D(a,b) \quad (a\in A, b\in B).$$

Also using (2.2) and (2.3) we see that D vanishes on $J_{A,\mathfrak{A}}\oplus J_{B,\mathfrak{A}}$. Hence $\tilde{D}$ is well defined. One can easily check that $\tilde{D}$ is a $\overline{\varphi}\oplus\overline{\psi}$-$\mathfrak{A}$-module derivation.

Now from the $\overline{\varphi}\oplus\overline{\psi}$-$\mathfrak{A}$-module amenability of $\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}$, it follows that there exists $x^*\in X^*$ such that

$$\begin{aligned}\tilde{D}(a+J_{A,\mathfrak{A}},b+J_{B,\mathfrak{A}}) = {} & \overline{\varphi}\oplus\overline{\psi}(a+J_{A,\mathfrak{A}},b+J_{B,\mathfrak{A}}).x^* \\ & - x^*.\overline{\varphi}\oplus\overline{\psi}(a+J_{A,\mathfrak{A}},b+J_{B,\mathfrak{A}})\ (a\in A, b\in B).\end{aligned}$$

It follows that

$$D(a,b)=\varphi\oplus\psi(a,b).x^*-x^*.\varphi\oplus\psi(a,b)\ (a\in A, b\in B).$$

Thus D is $\varphi\oplus\psi$-inner. So $A\oplus B$ is $\varphi\oplus\psi$-$\mathfrak{A}$-module amenable.

Similarly, we can show that $A\oplus B$ is $\varphi\oplus\psi$-$\mathfrak{A}$-module approximately amenable if and only if $\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi}\oplus\overline{\psi}$-$\mathfrak{A}$-module approximately amenable.

(ii) Let $A\oplus B$ be $\varphi\oplus\psi$-$\mathfrak{A}$-module amenable and let $D:\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}\longrightarrow X^*$ be a derivation for some Banach $\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}$-bimodule X. Then X becomes a $A\oplus B$-bimodule through the actions as (4.1) and (4.2). Also X is an $\mathfrak{A}$-bimodule with f-trivial actions, that is

$$\alpha.x=x.\alpha=f(\alpha)x\ \big(\alpha\in\mathfrak{A},\ x\in X\big).$$

Then X is a commutative Banach $A\oplus B$-$\mathfrak{A}$-module. Define

$$\Gamma:\frac{A\oplus B}{I}\longrightarrow\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}},\ (a,b)+I\longmapsto(a+J_{A,\mathfrak{A}},b+J_{B,\mathfrak{A}}),$$

where $I=J_{A,\mathfrak{A}}\oplus J_{B,\mathfrak{A}}$. It is routinely checked that Γ defines an $\mathfrak{A}$-bimodule morphism. Let $\Pi:A\oplus B\longrightarrow\frac{A\oplus B}{I}$ be the quotient map, and let $\tilde{D}:=D\circ\Gamma\circ\Pi: A\oplus B\longrightarrow X^*$. For every $(a,b),(a',b')\in A\oplus B$, we may easily prove that

$$\tilde{D}\big((a,b)(a',b')\big)=\tilde{D}(a,b).\varphi\oplus\psi(a',b')+\varphi\oplus\psi(a,b).\tilde{D}(a',b'),$$

and for every $(a,b)\in A\oplus B$, and $\alpha\in\mathfrak{A}$, we have

$$\begin{aligned}\tilde{D}\Big(\alpha.(a,b)\Big) &= \tilde{D}\Big((\alpha.a,\alpha.b)\Big)=\tilde{D}\Big((f(\alpha)a,f(\alpha)b)\Big)\\ &= D\Big((f(\alpha)a+J_{A,\mathfrak{A}},f(\alpha)b+J_{B,\mathfrak{A}})\Big)\\ &= D\Big(f(\alpha)(a+J_{A,\mathfrak{A}},b+J_{B,\mathfrak{A}})\Big)\\ &= f(\alpha)D\Big((a+J_{A,\mathfrak{A}},b+J_{B,\mathfrak{A}})\Big)\\ &= \alpha.D\Big((a+J_{A,\mathfrak{A}},b+J_{B,\mathfrak{A}})\Big)\\ &= \alpha.\tilde{D}(a,b),\end{aligned}$$

and using Lemma 4.2, we have

$$\begin{aligned}\tilde{D}\Big((a,b).\alpha\Big) &= \tilde{D}\Big((a.\alpha, b.\alpha)\Big) = D\Big((a.\alpha + J_{A,\mathfrak{A}}, b.\alpha + J_{B,\mathfrak{A}})\Big)\\ &= D\Big((\alpha.a + J_{A,\mathfrak{A}}, \alpha.b + J_{B,\mathfrak{A}})\Big)\\ &= D\Big(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\Big)\\ &= f(\alpha)D\Big((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\Big)\\ &= D\Big((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\Big).\alpha\\ &= \tilde{D}(a,b).\alpha.\end{aligned}$$

Thus $\tilde{D}$ is a $\varphi\oplus\psi$-$\mathfrak{A}$-module derivation and from the $\varphi\oplus\psi$-$\mathfrak{A}$-module amenability of $A\oplus B$, it follows that there exists $x^*\in X^*$ such that

$$\tilde{D}(a,b) = \varphi\oplus\psi(a,b).x^* - x^*.\varphi\oplus\psi(a,b) \ \ (a\in A, b\in B).$$

It follows that

$$\begin{aligned}D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = \overline{\varphi}\oplus\overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^*\\ - x^*.\overline{\varphi}\oplus\overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).\end{aligned}$$

So D is $\overline{\varphi}\oplus\overline{\psi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi}\oplus\overline{\psi}$-amenable.

(iii) Suppose that $\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi}\oplus\overline{\psi}$-amenable. Since $\mathfrak{A}$ has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that $\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi}\oplus\overline{\psi}$-$\mathfrak{A}$-module amenable. So by (i), $A\oplus B$ is $\varphi\oplus\psi$-$\mathfrak{A}$-module amenable.

Similar relations can be obtained between the $\varphi\oplus\psi$-$\mathfrak{A}$-module approximate amenability of $A\oplus B$ and $\overline{\varphi}\oplus\overline{\psi}$-approximate amenability of $\frac{A}{J_{A,\mathfrak{A}}}\oplus\frac{B}{J_{B,\mathfrak{A}}}$. □

Proposition 4.4. *Let A be an $\mathfrak{A}$-module Banach algebra, where $\mathfrak{A}$ acts on A trivially from the left by $f\in \mathtt{Hom}_{\mathbb{C}}(\mathfrak{A})$. Let $\varphi\in\mathtt{Hom}_{\mathfrak{A}}(A)$ and $\frac{A}{J_{A,\mathfrak{A}}}$ be unital. If A is φ-$\mathfrak{A}$-module approximately amenable, then $\frac{A}{J_{A,\mathfrak{A}}}$ is $\overline{\varphi}$-approximately amenable.*

Proof. Let X be a Banach $\frac{A}{J_{A,\mathfrak{A}}}$-bimodule and $D:\frac{A}{J_{A,\mathfrak{A}}}\longrightarrow X^*$ be a $\overline{\varphi}$-derivation. Then X becomes a A-bimodule through the following actions

$$a.x = (a + J_{A,\mathfrak{A}}).x, \ x.a = x.(a + J_{A,\mathfrak{A}}) \ \ (a\in A, x\in X),$$

and X is an $\mathfrak{A}$-bimodule with f-trivial actions, that is $\alpha.x = x.\alpha = f(\alpha)x$ ($\alpha\in\mathfrak{A}, x\in X$). By Lemma 4.2, $f(\alpha)a - a.\alpha\in J_{A,\mathfrak{A}}$ ($\alpha\in\mathfrak{A}, a\in A$). So, $f(\alpha)a + J_{A,\mathfrak{A}} = a.\alpha + J_{A,\mathfrak{A}}$ ($\alpha\in\mathfrak{A}, a\in A$), and the actions of $\mathfrak{A}$ and A on X are compatible. Thus X is a commutative Banach A-$\mathfrak{A}$-module. Let $\tilde{D}:A\longrightarrow X^*$ be defined by $\tilde{D}(a) = D(a + J_{A,\mathfrak{A}})$ ($a\in A$). A similar argument as in the proof of Theorem 3.2 of [2], shows that $\tilde{D}$ is approximately φ-inner. So, D is approximately $\overline{\varphi}$-inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}}$ is $\overline{\varphi}$-approximately amenable. □

Theorem 4.5. *Let $\mathfrak{A}$ have a bounded approximate identity, and let A and B be $\mathfrak{A}$-module Banach algebras, where $\mathfrak{A}$ acts on A and B trivially from the left. Let $\varphi \in \mathtt{Hom}_{\mathfrak{A}}(A)$, $\psi \in \mathtt{Hom}_{\mathfrak{A}}(B)$, and let $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ be unital. Then A is φ-$\mathfrak{A}$-module approximately amenable and B is ψ-$\mathfrak{A}$-module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.*

Proof. Suppose that A is φ-$\mathfrak{A}$-module approximately amenable and B is ψ-$\mathfrak{A}$-module approximately amenable. By Proposition 4.4, $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are $\overline{\varphi}$-approximately amenable and $\overline{\psi}$-approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A} = \mathbb{C}$, we conclude that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$-approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$-$\mathfrak{A}$-module approximately amenable. Then by Proposition 3.7, A is φ-$\mathfrak{A}$-module approximately amenable and B is ψ-$\mathfrak{A}$-module approximately amenable. □

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