

Generalized Approximate Amenability of Direct Sum of Banach Algebras

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ABSTRACT. In the present paper for two \mathfrak{A} -module Banach algebras A and B , we investigate relations between φ - \mathfrak{A} -module approximate amenability of A , ψ - \mathfrak{A} -module approximate amenability of B , and $\varphi \oplus \psi$ - \mathfrak{A} -module approximate amenability of $A \oplus B$ (l^1 -direct sum of A and B), where $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$.

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1. INTRODUCTION

The notion of approximate amenable Banach algebras was introduced and extensively studied by Ghahramani and Loy in [5]. They showed in [6] that if A and B are approximately amenable Banach algebras and one of A or B has a bounded approximate identity, then $A \oplus B$ is approximately amenable, but in general the direct sum of two approximately amenable Banach algebras need not be approximately amenable (see [7]).

The concept of module amenable Banach algebras was introduced by Amini in [1], and the notion of module approximate amenable Banach algebras was studied by Pourmahmood and Bodaghi in [15]. Recently, some authors have

studied φ -derivations, and φ -amenability of Banach algebra A , whenever φ is a continuous homomorphism on A (see [8, 9, 10, 11, 12]).

The aim of the present paper is to investigate generalized approximate amenability of $A \oplus B$.

The organization of this paper is as follows:

Section 2 is devoted to the notations and definitions which are needed throughout the paper.

In section 3 for \mathfrak{A} -module Banach algebras A and B where each has a bounded approximate identity we show that A is φ - $\mathfrak{A}^\#$ -module approximately amenable and B is ψ - $\mathfrak{A}^\#$ -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - $\mathfrak{A}^\#$ -module approximately amenable.

In section 4 we show that if \mathfrak{A} has a bounded approximate identity and $\frac{A}{J_{A, \mathfrak{A}}}$ and $\frac{B}{J_{B, \mathfrak{A}}}$ are unital, then A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.

2. PRELIMINARIES

Let \mathfrak{A} and A be Banach algebras such that A is a Banach \mathfrak{A} -bimodule with compatible actions given by

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \quad (a, b \in A, \alpha \in \mathfrak{A}).$$

Let X be a Banach A -bimodule and a Banach \mathfrak{A} -bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x, \quad (\alpha.x).a = \alpha.(x.a)$$

$$(a \in A, \alpha \in \mathfrak{A}, x \in X), \quad (2.1)$$

and similar for the right or two-sided actions. Then we say that X is a Banach A - \mathfrak{A} -module. A Banach A - \mathfrak{A} -module X is called commutative A - \mathfrak{A} -module, if $\alpha.x = x.\alpha$ ($\alpha \in \mathfrak{A}, x \in X$). Note that in general, A does not satisfy the compatibility condition $a.(\alpha.b) = (a.\alpha).b$ ($a, b \in A, \alpha \in \mathfrak{A}$).

If X is a commutative Banach A - \mathfrak{A} -module, then so is X^* , where the actions of A and \mathfrak{A} on X^* are defined as follows

$$\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \quad \langle a.f, x \rangle = \langle f, x.a \rangle \quad (a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*),$$

and similar for the right actions.

Let A and B be Banach \mathfrak{A} -bimodules. Then a \mathfrak{A} -module morphism from A to B is a norm continuous map $h : A \rightarrow B$ with $h(a \pm b) = h(a) \pm h(b)$ which is multiplicative, that is

$$h(\alpha.a) = \alpha.h(a), \quad h(a.\alpha) = h(a).\alpha, \quad h(ab) = h(a)h(b) \quad (a \in A, b \in B, \alpha \in \mathfrak{A}).$$

We denote by $\text{Hom}_{\mathfrak{A}}(A, B)$, the space of all such morphism and denote $\text{Hom}_{\mathfrak{A}}(A, A)$ by $\text{Hom}_{\mathfrak{A}}(A)$. In the case that $\mathfrak{A} = \mathbb{C}$, we denote $\text{Hom}_{\mathbb{C}}(A, B)$ by $\text{Hom}(A, B)$ and denote $\text{Hom}_{\mathbb{C}}(A, A)$ by $\text{Hom}(A)$.

Let X be a Banach A -bimodule and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$. A bounded map $D : A \rightarrow X$ is called a φ - \mathfrak{A} -module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a).\varphi(b) + \varphi(a).D(b) \quad (a, b \in A), \quad (2.2)$$

and

$$D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}). \quad (2.3)$$

Although D in general is not linear, but still its boundedness implies its norm continuity.

Let X be a commutative Banach A - \mathfrak{A} -module. For every $x \in X$ define ad_x^φ by $ad_x^\varphi(a) = \varphi(a).x - x.\varphi(a)$ ($a \in A$). It is easily seen that ad_x^φ is a φ - \mathfrak{A} -module derivation. A φ - \mathfrak{A} -module derivation D is called φ -inner if there is $x \in X$ such that $D(a) = ad_x^\varphi(a)$ ($a \in A$) and is called approximately φ -inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha ad_{x_\alpha}^\varphi(a)$ ($a \in A$). A Banach algebra A is called φ - \mathfrak{A} -module amenable if for any commutative Banach A - \mathfrak{A} -module X , each φ - \mathfrak{A} -module derivation $D : A \rightarrow X^*$ is φ -inner, and A is called φ - \mathfrak{A} -module approximately amenable if each φ - \mathfrak{A} -module derivation $D : A \rightarrow X^*$ is approximately φ -inner (see [1, 15]).

In the case that $\mathfrak{A} = \mathbb{C}$, φ - \mathfrak{A} -module derivations (resp. φ - \mathfrak{A} -module amenable Banach algebras, φ - \mathfrak{A} -module approximately amenable Banach algebras) are called φ -derivation (resp. φ -amenable, φ -approximately amenable) (see [9, 10]).

3. $\varphi \oplus \psi$ -MODULE APPROXIMATE AMENABILITY OF THE DIRECT SUM OF BANACH ALGEBRAS

We commence this section with the following remark from [1]:

Remark 3.1. Assume that A has a bounded approximate identity $(e_\alpha)_\alpha$, and let $M_{\mathfrak{A}}(A)$ denotes the algebra of \mathfrak{A} -multipliers of A , that is $M_{\mathfrak{A}}(A) = \{(T_1, T_2) : T_1, T_2 \in L_{\mathfrak{A}}(A) : T_1(ab) = T_1(a)b, T_2(ab) = aT_2(b) \ (a, b \in A)\}$, where $L_{\mathfrak{A}}(A)$ is the space of all \mathfrak{A} -module morphisms on A . Then $M_{\mathfrak{A}}(A)$ is an A - \mathfrak{A} -module and A embeds in $M_{\mathfrak{A}}(A)$ via $a \mapsto (L_a, R_a)$, where $L_a(b) = ab, R_a(b) = ba$ ($a, b \in A$). For any element $T = (T_1, T_2)$ of $M_{\mathfrak{A}}(A)$ it is easy to see that $\|T_1\| = \|T_2\|$ and if we put $\|T\|$ equal to this common value, then $M_{\mathfrak{A}}(A)$ becomes a Banach A - \mathfrak{A} -module, and A is dense in $M_{\mathfrak{A}}(A)$ in the strict topology.

Before proving our next proposition we note that if $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, then by continuity of φ in the strict topology, it can be extended to an \mathfrak{A} -homomorphism $\tilde{\varphi} : M_{\mathfrak{A}}(A) \rightarrow M_{\mathfrak{A}}(A)$ defined by $\tilde{\varphi}(L_a, R_a) = (L_{\varphi(a)}, R_{\varphi(a)})$.

Proposition 3.2. *Let A be an \mathfrak{A} -module Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$. Then A is φ - \mathfrak{A} -module approximately amenable if and only if $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$ - \mathfrak{A} -module approximately amenable.*

Proof. Let $M_{\mathfrak{A}}(A)$ be $\tilde{\varphi}$ - \mathfrak{A} -module approximately amenable and let $D : A \rightarrow X^*$ be a φ - \mathfrak{A} -module derivation for some commutative Banach A - \mathfrak{A} -module X . Then by the following actions

$$T.x = \lim_{\alpha} T_1(e_\alpha).x, \quad x.T = \lim_{\alpha} x.T_2(e_\alpha) \quad (x \in X, T = (T_1, T_2) \in M_{\mathfrak{A}}(A)),$$

X is a commutative Banach $M_{\mathfrak{A}}(A)$ - \mathfrak{A} -module and by continuity of D in the strict topology, it can be extended to a bounded $\tilde{\varphi}$ - \mathfrak{A} -derivation $\tilde{D} : M_{\mathfrak{A}}(A) \rightarrow X^*$, defined by $\tilde{D}(L_a, R_a) = D(a)$. From the $\tilde{\varphi}$ - \mathfrak{A} -module approximate amenability of $M_{\mathfrak{A}}(A)$, it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$\tilde{D}(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)).$$

Hence for every $a \in A$ we have

$$\begin{aligned} D(a) &= \tilde{D}(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a)) \\ &= \lim_{\beta} ((L_{\varphi(a)}, R_{\varphi(a)}).x_\beta^* - x_\beta^*.(L_{\varphi(a)}, R_{\varphi(a)})) \\ &= \lim_{\beta} (\lim_{\alpha} L_{\varphi(a)}(e_\alpha).x_\beta^* - \lim_{\alpha} x_\beta^*.R_{\varphi(a)}(e_\alpha)) \\ &= \lim_{\beta} (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)). \end{aligned}$$

This means that D is approximately φ -inner and so A is φ - \mathfrak{A} -module approximately amenable.

Conversely, Suppose that A is φ - \mathfrak{A} -module approximately amenable. Let X be a commutative Banach $M_{\mathfrak{A}}(A)$ - \mathfrak{A} -module and let $D : M_{\mathfrak{A}}(A) \rightarrow X^*$ be a $\tilde{\varphi}$ - \mathfrak{A} -module derivation. We consider the module actions of A on X by

$$a.x = (L_a, R_a).x, \quad x.a = x.(L_a, R_a) \quad (a \in A, x \in X). \quad (3.1)$$

Thus X is a commutative Banach A - \mathfrak{A} -module. Define $\tilde{D} : A \rightarrow X^*$ by $\tilde{D}(a) = D(L_a, R_a)$ ($a \in A$). It is easy to see that \tilde{D} is a φ - \mathfrak{A} -module derivation and from the φ - \mathfrak{A} -module approximate amenability of A , it follows that there exists a net $(x_\beta^*)_\beta \subset X^*$ such that

$$\tilde{D}(a) = \lim_{\beta} (\varphi(a).x_\beta^* - x_\beta^*.\varphi(a)) \quad (a \in A).$$

Then $D(L_a, R_a) = \lim_{\beta} (\tilde{\varphi}(L_a, R_a).x_\beta^* - x_\beta^*.\tilde{\varphi}(L_a, R_a))$. Now by the continuity of D and $\tilde{\varphi}$, and density of A in $M_{\mathfrak{A}}(A)$ in the strict topology, we conclude that

$$D(T) = \lim_{\beta} (\tilde{\varphi}(T).x_\beta^* - x_\beta^*.\tilde{\varphi}(T)) \quad (T \in M_{\mathfrak{A}}(A)).$$

So D is a approximately $\tilde{\varphi}$ -inner. Therefore $M_{\mathfrak{A}}(A)$ is $\tilde{\varphi}$ - \mathfrak{A} -module approximately amenable. \square

Let I be a closed ideal of a Banach algebra A with a bounded approximate identity $(e_\alpha)_\alpha$, and let X be a commutative Banach I - \mathfrak{A} -module. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi|_I \subset I$, then X is a commutative Banach A - \mathfrak{A} -module with the following actions

$$a.x = \lim_{\alpha} \varphi(e_\alpha)a.x, \quad x.a = \lim_{\alpha} x.\varphi(e_\alpha)a \quad (a \in A, x \in X). \quad (3.2)$$

Proposition 3.3. *Let I be a closed ideal of an \mathfrak{A} -module Banach algebra A which has a bounded approximate identity $\{e_\alpha\}$, and let I be \mathfrak{A} -invariant, i.e. $\mathfrak{A}.I \subseteq I$. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ be such that $\varphi|_I \subset I$. If A is φ - \mathfrak{A} -module approximately amenable, then I is $\varphi|_I$ - \mathfrak{A} -module approximately amenable.*

Proof. Let X be a commutative Banach $M_{\mathfrak{A}}(I)$ - \mathfrak{A} -module, and $D : M_{\mathfrak{A}}(I) \rightarrow X^*$ be a $\tilde{\varphi}$ - \mathfrak{A} -module derivation. By the same actions as (3.1), we can consider X as a commutative Banach I - \mathfrak{A} -module. So, by (3.2), X is a commutative Banach A - \mathfrak{A} -module. By definition of $M_{\mathfrak{A}}(I)$, there is an \mathfrak{A} -module morphism $h : A \rightarrow M_{\mathfrak{A}}(I)$ and $D \circ h$ is a module derivation on A , so it is approximately φ -inner. Hence D is approximately $\tilde{\varphi}$ -inner. Since I has a bounded approximate identity, by Proposition 3.2, I is $\varphi|_I$ - \mathfrak{A} -module approximately amenable. \square

Let A and B be \mathfrak{A} -module Banach algebras. It is well known that $A \oplus B$, the l^1 -direct sum of A and B , is a Banach algebra with respect to the canonical multiplication defined by $(a, b)(c, d) := (ac, bd)$, and is a Banach \mathfrak{A} -bimodule by the following actions

$$\alpha.(a, b) := (\alpha.a, \alpha.b), \quad (a, b).\alpha := (a.\alpha, b.\alpha) \quad (\alpha \in \mathfrak{A}, a \in A, b \in B).$$

We note that if $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, then $\varphi \oplus \psi : A \oplus B \rightarrow A \oplus B$ defined by $\varphi \oplus \psi(a, b) = (\varphi(a), \psi(b))$ is an \mathfrak{A} -morphism on $A \oplus B$.

Lemma 3.4. *Let A be a unital \mathfrak{A} -module Banach algebra, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, and let $D : A \rightarrow X^*$ be a φ - \mathfrak{A} -module derivation for some commutative Banach A - \mathfrak{A} -module X . If the left (resp. right, two-sided) action of $\varphi(A)$ on X^* is zero, then D is φ -inner.*

Proof. Let e_A be the identity of A and let the left (resp. right, two-sided) action of $\varphi(A)$ on X^* is zero. We can easily show that $D = ad_{-D(e)}^{\varphi}$ (resp. $D = ad_{D(e)}^{\varphi}, D = 0$). So D is φ -inner. \square

The proof of the following proposition is adopted from that of Proposition 2.7 of [5].

Proposition 3.5. *Let A and B be unital \mathfrak{A} -module Banach algebras with identities e_A and e_B , respectively, and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$ such that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$, and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$). If A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable, then $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.*

Proof. Let X be a commutative Banach $A \oplus B$ - \mathfrak{A} -module and let $D : A \oplus B \rightarrow X^*$ be a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation. Write $Y_1 = \varphi(e_A).X^*.\varphi(e_A)$, $Y_2 = \psi(e_B).X^*.\psi(e_B)$, $Y_3 = \varphi(e_A).X^*.\psi(e_B)$, $Y_4 = \psi(e_B).X^*.\varphi(e_A)$, $Y_5 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\varphi(e_A)$, $Y_6 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.\psi(e_B)$, $Y_7 = \varphi(e_A).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$, $Y_8 = \psi(e_B).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$, $Y_9 = (1 - \varphi(e_A))(1 - \psi(e_B)).X^*.(1 - \varphi(e_A))(1 - \psi(e_B))$ and let $\pi_j : X^* \rightarrow Y_j$ be the associated projections. Thus $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 \oplus Y_5 \oplus Y_6 \oplus Y_7 \oplus Y_8 \oplus Y_9$. Consider the derivations $D_j = \pi_j \circ D$, so $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$. From the fact that $\varphi(e_A).\alpha = \alpha.\varphi(e_A)$ ($\alpha \in \mathfrak{A}$), and $\psi(e_B).\alpha = \alpha.\psi(e_B)$ ($\alpha \in \mathfrak{A}$), one can easily check that Y_j for $j = 1, \dots, 9$ is a commutative Banach $A \oplus B$ - \mathfrak{A} -module. Since the action of $\varphi(A) \oplus \psi(B)$ on (at least) one side on Y_5 (resp. Y_6, Y_7, Y_8, Y_9) is zero, by Lemma 3.4, we conclude that D_5 (resp. D_6, D_7, D_8, D_9) is approximately $\varphi \oplus \psi$ -inner.

From the φ - \mathfrak{A} -module approximate amenability of A , it follows that the $\varphi \oplus \psi$ - \mathfrak{A} -module derivation $A \oplus 0 \rightarrow \varphi(e_A).X^*.\varphi(e_A)$ is approximately $\varphi \oplus \psi$ -inner and since the action of $0 \oplus \psi(B)$ on $\varphi(e_A).X^*.\varphi(e_A)$ is zero, we conclude that D_1 is approximately $\varphi \oplus \psi$ -inner. Similarly, the $\varphi \oplus \psi$ - \mathfrak{A} -module derivation $D_2 : A \oplus B \rightarrow \psi(e_B).X^*.\psi(e_B)$ is approximately $\varphi \oplus \psi$ -inner.

The right action of $\varphi(A) \oplus 0$ on $\varphi(e_A).X^*.\psi(e_B)$ is zero. Hence, by Lemma 3.4, $D_3|_{A \oplus 0}$ is $\varphi \oplus \psi$ -inner. So there exists $\xi \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3|_{A \oplus 0}(a, 0) = \varphi(a).\xi - \xi.\varphi(a) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B),$$

for every $a \in A$ and $b \in B$. Similarly, there exists $\eta \in \varphi(e_A).X^*.\psi(e_B)$ such that

$$D_3|_{0 \oplus B}(0, b) = \psi(b).\eta - \eta.\psi(b) = -\varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)),$$

for every $a \in A$ and $b \in B$. Hence

$$D_3(a, b) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B)(\varphi(a), \psi(b)).$$

Since $D_3(e_A, e_B) = 0$, it follows that

$$0 = D_3(e_A, e_B) = \varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\eta.\psi(e_B).$$

Then for every $a \in A$ and $b \in B$, we have

$$D_3(a, b) = (\varphi(a), \psi(b))\varphi(e_A).\xi.\psi(e_B) - \varphi(e_A).\xi.\psi(e_B)(\varphi(a), \psi(b)).$$

Thus D_3 is $\varphi \oplus \psi$ -inner. The same argument holds for the $\varphi \oplus \psi$ - \mathfrak{A} -module derivation $D_4 : A \oplus B \rightarrow \psi(e_B).X^*.\varphi(e_A)$. Therefore D is approximately $\varphi \oplus \psi$ -inner, and so $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable. \square

Lemma 3.6. *Let A and B be \mathfrak{A} -module Banach algebras, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. If there is a h in $\text{Hom}_{\mathfrak{A}}(A, B)$ such that $h \circ \varphi = \psi \circ h$ and the range of h is a dense subset of B , then φ - \mathfrak{A} -module approximate amenability of A implies ψ - \mathfrak{A} -module approximate amenability of B .*

Proof. Let $D : B \rightarrow X^*$ be a ψ - \mathfrak{A} -module derivation for some commutative Banach B - \mathfrak{A} -module X . Then by the following actions

$$a \bullet x = h(a).x, \quad x \bullet a = x.h(a) \quad (a \in A, x \in X),$$

X is a commutative Banach A - \mathfrak{A} -module. Let $\tilde{D} = D \circ h : A \rightarrow X^*$. One can easily prove that D is a φ - \mathfrak{A} -module derivation. From the φ - \mathfrak{A} -module approximate amenability of A , it follows that there exists a net $(x_\alpha^*)_\alpha$ in X^* such that $\tilde{D}(a) = \lim_\alpha (\varphi(a) \bullet x_\alpha^* - x_\alpha^* \bullet \varphi(a))$ ($a \in A$). Now continuity and density of $h(A)$ in B , imply that D is approximately ψ -inner. Therefore B is ψ - \mathfrak{A} -module approximately amenable. \square

Proposition 3.7. *Let A and B be \mathfrak{A} -module Banach algebras, $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. If A is not φ - \mathfrak{A} -module approximately amenable or B is not ψ - \mathfrak{A} -module approximately amenable, then $A \oplus B$ is not $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.*

Proof. Suppose that A is not φ - \mathfrak{A} -module approximately amenable. The projection map $\pi : A \oplus B \rightarrow A$ determines an \mathfrak{A} -module epimorphism of $A \oplus B$ onto A such that $\pi \circ (\varphi \oplus \psi) = \varphi \circ \pi$. So, if $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable, then by Lemma 3.6, A is φ - \mathfrak{A} -module approximately amenable. This contradicts the fact that A is not φ - \mathfrak{A} -module approximately amenable. Therefore $A \oplus B$ is not $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable. Similarly, we can prove the result for B . \square

Let \mathfrak{A} be a non-unital Banach algebra. Then $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$, the unitization of \mathfrak{A} is a unital Banach algebra which contains \mathfrak{A} as a closed ideal. Let A be a Banach \mathfrak{A} -bimodule. Then A is a Banach $\mathfrak{A}^\#$ -module with the following module actions:

$$(\alpha, \lambda).a = \alpha.a + \lambda a, \quad a.(\alpha, \lambda) = a.\alpha + \lambda a \quad (\lambda \in \mathbb{C}, \alpha \in \mathfrak{A}, a \in A).$$

Let $A^\# = (A \oplus \mathfrak{A}^\#, \bullet)$, where the multiplication \bullet is defined through

$$(a, u) \bullet (b, v) = (ab + a.v + u.b, uv) \quad (a, b \in A, u, v \in \mathfrak{A}^\#).$$

Then with the actions defined by

$$u.(a, v) = (u.a, uv), \quad (a, v).u = (a.u, vu) \quad (a \in A, u, v \in \mathfrak{A}^\#),$$

$A^\#$ is a unital $\mathfrak{A}^\#$ -module Banach algebra with the identity $1_{A^\#} = (0, 1_{\mathfrak{A}^\#})$ (see [4]).

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$, one defines $\varphi^\# : A^\# \rightarrow A^\#$ by $\varphi^\#(a, u) = (\varphi(a), u)$ ($(a, u) \in A^\#$), then $\varphi^\# \in \text{Hom}_{\mathfrak{A}^\#}(A^\#)$.

The following proposition generalizes Proposition 2.7 of [5].

Theorem 3.8. *Let A and B be \mathfrak{A} -module Banach algebras and each has a bounded approximate identity. Let $\varphi \in \text{Hom}_{\mathfrak{A}^\#}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}^\#}(B)$. Then A*

is $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and B is $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}^\#$ -module approximately amenable.

Proof. Suppose that A is $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and B is $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. By Proposition 12 of [13], $A^\#$ is $\varphi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable and $B^\#$ is $\psi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable, so by Proposition 3.5, $A^\# \oplus B^\#$ is $\varphi^\# \oplus \psi^\#\text{-}\mathfrak{A}^\#$ -module approximately amenable. Since $A \oplus B$ is a closed $\mathfrak{A}^\#$ -invariant ideal in $A^\# \oplus B^\#$, the result follows from Proposition 3.3.

For the converse, suppose that $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. Then by Proposition 3.7, A is $\varphi\text{-}\mathfrak{A}^\#$ -module approximately amenable and B is $\psi\text{-}\mathfrak{A}^\#$ -module approximately amenable. \square

4. $\varphi \oplus \psi$ -MODULE APPROXIMATE AMENABILITY AND $\varphi \oplus \psi$ -AMENABILITY OF DIRECT SUM OF BANACH ALGEBRAS

We start this section with the following definition:

Definition 4.1. We say the Banach algebra \mathfrak{A} acts trivially on A from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where f is a multiplicative linear functional on \mathfrak{A} .

We assume that $J_{A,\mathfrak{A}}$ is the closed linear span of

$$\{(a.\alpha)b - a(\alpha.b) \mid \alpha \in \mathfrak{A}, a, b \in A\},$$

in A . It follows immediately that $J_{A,\mathfrak{A}}$ is both A -submodule and \mathfrak{A} -submodule of A . So $\frac{A}{J_{A,\mathfrak{A}}}$ is both Banach A -module and \mathfrak{A} -module (see page 346 of [14]).

To prove our next result we need to quote the following lemma from [2].

Lemma 4.2. Let A be a Banach algebra and Banach \mathfrak{A} -module with compatible actions, and J_0 be a closed ideal of A such that $J_{A,\mathfrak{A}} \subseteq J_0$. If $\frac{A}{J_0}$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $a.\alpha - \alpha.a \in J_0$, i.e., $\frac{A}{J_0}$ is commutative Banach \mathfrak{A} -module.

Before we turn to our next result we note that if for every $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, one defines $\bar{\varphi} : \frac{A}{J_{A,\mathfrak{A}}} \rightarrow \frac{A}{J_{A,\mathfrak{A}}}$ by $\bar{\varphi}(a + J_{A,\mathfrak{A}}) = \varphi(a) + J_{A,\mathfrak{A}}$, then $\bar{\varphi} \in \text{Hom}_{\mathfrak{A}}(\frac{A}{J_{A,\mathfrak{A}}})$.

Theorem 4.3. Let A and B be \mathfrak{A} -module Banach algebras and let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\psi \in \text{Hom}_{\mathfrak{A}}(B)$. Then the following statements are valid:

- (i) $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module amenable (resp. $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module approximately amenable) if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}\text{-}\mathfrak{A}$ -module amenable (resp. $\bar{\varphi} \oplus \bar{\psi}\text{-}\mathfrak{A}$ -module approximately amenable).
- (ii) Let \mathfrak{A} acts on A and B trivially from the left by $f \in \text{Hom}_{\mathbb{C}}(\mathfrak{A})$. Suppose that $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are unital, and $A \oplus B$ is $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module amenable (resp. $\varphi \oplus \psi\text{-}\mathfrak{A}$ -module approximately amenable), then $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -amenable (resp. $\bar{\varphi} \oplus \bar{\psi}$ -approximately amenable).

(iii) Let \mathfrak{A} have a bounded approximately identity and $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$ -amenable (resp. $\overline{\varphi} \oplus \overline{\psi}$ -approximately amenable). Then $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module amenable (resp. $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable).

Proof. (i) Let $A \oplus B$ be $\varphi \oplus \psi$ - \mathfrak{A} -module amenable, and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ be $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module derivation for some commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ - \mathfrak{A} -module X . Then X becomes a $A \oplus B$ -bimodule through the following actions

$$(a, b).x := (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x \quad (a \in A, b \in B, x \in X), \tag{4.1}$$

and

$$x.(a, b) := x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B, x \in X). \tag{4.2}$$

Hence X is a commutative Banach $A \oplus B$ - \mathfrak{A} -module. Define $\tilde{D} : A \oplus B \rightarrow X^*$ by

$$\tilde{D}(a, b) = D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B).$$

It is easy to check that, \tilde{D} is a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation. From the $\varphi \oplus \psi$ - \mathfrak{A} -module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus

$$\begin{aligned} D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\overline{\varphi} \oplus \overline{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}). \end{aligned}$$

This means that D is $\overline{\varphi} \oplus \overline{\psi}$ -inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module amenable.

Conversely, suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module amenable. Let $D : A \oplus B \rightarrow X^*$ be a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation for some commutative Banach $A \oplus B$ - \mathfrak{A} -module X . We consider the following module actions of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ on X ,

$$(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x := (a, b).x, \quad x.(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) := x.(a, b),$$

for all $a \in A, b \in B$ and $x \in X$. Using (2.1) and the commutativity of X , we have $J_{A,\mathfrak{A}}X = J_{B,\mathfrak{A}}X = XJ_{A,\mathfrak{A}} = XJ_{B,\mathfrak{A}} = 0$. Thus $(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}})X = X(J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}) = 0$. So X is a commutative Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ - \mathfrak{A} -module. Define $\tilde{D} : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ by

$$\tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) = D(a, b) \quad (a \in A, b \in B).$$

Also using (2.2) and (2.3) we see that D vanishes on $J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. Hence \tilde{D} is well defined. One can easily check that \tilde{D} is a $\overline{\varphi} \oplus \overline{\psi}$ - \mathfrak{A} -module derivation.

Now from the $\bar{\varphi} \oplus \bar{\psi}$ - \mathfrak{A} -module amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$, it follows that there exists $x^* \in X^*$ such that

$$\begin{aligned} \tilde{D}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) \quad (a \in A, b \in B). \end{aligned}$$

It follows that

$$D(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

Thus D is $\varphi \oplus \psi$ -inner. So $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module amenable.

Similarly, we can show that $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable if and only if $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - \mathfrak{A} -module approximately amenable.

(ii) Let $A \oplus B$ be $\varphi \oplus \psi$ - \mathfrak{A} -module amenable and let $D : \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}} \rightarrow X^*$ be a derivation for some Banach $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ -bimodule X . Then X becomes a $A \oplus B$ -bimodule through the actions as (4.1) and (4.2). Also X is an \mathfrak{A} -bimodule with f -trivial actions, that is

$$\alpha.x = x.\alpha = f(\alpha)x \quad (\alpha \in \mathfrak{A}, x \in X).$$

Then X is a commutative Banach $A \oplus B$ - \mathfrak{A} -module. Define

$$\Gamma : \frac{A \oplus B}{I} \rightarrow \frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}, \quad (a, b) + I \mapsto (a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}),$$

where $I = J_{A,\mathfrak{A}} \oplus J_{B,\mathfrak{A}}$. It is routinely checked that Γ defines an \mathfrak{A} -bimodule morphism. Let $\Pi : A \oplus B \rightarrow \frac{A \oplus B}{I}$ be the quotient map, and let $\tilde{D} := D \circ \Gamma \circ \Pi : A \oplus B \rightarrow X^*$. For every $(a, b), (a', b') \in A \oplus B$, we may easily prove that

$$\tilde{D}((a, b)(a', b')) = \tilde{D}(a, b).\varphi \oplus \psi(a', b') + \varphi \oplus \psi(a, b).\tilde{D}(a', b'),$$

and for every $(a, b) \in A \oplus B$, and $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned} \tilde{D}(\alpha.(a, b)) &= \tilde{D}((\alpha.a, \alpha.b)) = \tilde{D}((f(\alpha)a, f(\alpha)b)) \\ &= D((f(\alpha)a + J_{A,\mathfrak{A}}, f(\alpha)b + J_{B,\mathfrak{A}})) \\ &= D(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= f(\alpha)D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= \alpha.D((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})) \\ &= \alpha.\tilde{D}(a, b), \end{aligned}$$

and using Lemma 4.2, we have

$$\begin{aligned} \tilde{D}\left((a, b).\alpha\right) &= \tilde{D}\left((a.\alpha, b.\alpha)\right) = D\left((a.\alpha + J_{A,\mathfrak{A}}, b.\alpha + J_{B,\mathfrak{A}})\right) \\ &= D\left((\alpha.a + J_{A,\mathfrak{A}}, \alpha.b + J_{B,\mathfrak{A}})\right) \\ &= D\left(f(\alpha)(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) \\ &= f(\alpha)D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right) \\ &= D\left((a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}})\right).\alpha \\ &= \tilde{D}(a, b).\alpha. \end{aligned}$$

Thus \tilde{D} is a $\varphi \oplus \psi$ - \mathfrak{A} -module derivation and from the $\varphi \oplus \psi$ - \mathfrak{A} -module amenability of $A \oplus B$, it follows that there exists $x^* \in X^*$ such that

$$\tilde{D}(a, b) = \varphi \oplus \psi(a, b).x^* - x^*.\varphi \oplus \psi(a, b) \quad (a \in A, b \in B).$$

It follows that

$$\begin{aligned} D(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}) &= \bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}).x^* \\ &\quad - x^*.\bar{\varphi} \oplus \bar{\psi}(a + J_{A,\mathfrak{A}}, b + J_{B,\mathfrak{A}}). \end{aligned}$$

So D is $\bar{\varphi} \oplus \bar{\psi}$ -inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -amenable.

(iii) Suppose that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -amenable. Since \mathfrak{A} has a bounded approximate identity, by Proposition 2.1 of [1], we conclude that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ - \mathfrak{A} -module amenable. So by (i), $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module amenable.

Similar relations can be obtained between the $\varphi \oplus \psi$ - \mathfrak{A} -module approximate amenability of $A \oplus B$ and $\bar{\varphi} \oplus \bar{\psi}$ -approximate amenability of $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$. \square

Proposition 4.4. *Let A be an \mathfrak{A} -module Banach algebra, where \mathfrak{A} acts on A trivially from the left by $f \in \text{Hom}_{\mathbb{C}}(\mathfrak{A})$. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$ and $\frac{A}{J_{A,\mathfrak{A}}}$ be unital. If A is φ - \mathfrak{A} -module approximately amenable, then $\frac{A}{J_{A,\mathfrak{A}}}$ is $\bar{\varphi}$ -approximately amenable.*

Proof. Let X be a Banach $\frac{A}{J_{A,\mathfrak{A}}}$ -bimodule and $D : \frac{A}{J_{A,\mathfrak{A}}} \rightarrow X^*$ be a $\bar{\varphi}$ -derivation. Then X becomes a A -bimodule through the following actions

$$a.x = (a + J_{A,\mathfrak{A}}).x, \quad x.a = x.(a + J_{A,\mathfrak{A}}) \quad (a \in A, x \in X),$$

and X is an \mathfrak{A} -bimodule with f -trivial actions, that is $\alpha.x = x.\alpha = f(\alpha)x$ ($\alpha \in \mathfrak{A}, x \in X$). By Lemma 4.2, $f(\alpha)a - a.\alpha \in J_{A,\mathfrak{A}}$ ($\alpha \in \mathfrak{A}, a \in A$). So, $f(\alpha)a + J_{A,\mathfrak{A}} = a.\alpha + J_{A,\mathfrak{A}}$ ($\alpha \in \mathfrak{A}, a \in A$), and the actions of \mathfrak{A} and A on X are compatible. Thus X is a commutative Banach A - \mathfrak{A} -module. Let $\tilde{D} : A \rightarrow X^*$ be defined by $\tilde{D}(a) = D(a + J_{A,\mathfrak{A}})$ ($a \in A$). A similar argument as in the proof of Theorem 3.2 of [2], shows that \tilde{D} is approximately φ -inner. So, D is approximately $\bar{\varphi}$ -inner. Therefore $\frac{A}{J_{A,\mathfrak{A}}}$ is $\bar{\varphi}$ -approximately amenable. \square

Theorem 4.5. *Let \mathfrak{A} have a bounded approximate identity, and let A and B be \mathfrak{A} -module Banach algebras, where \mathfrak{A} acts on A and B trivially from the left. Let $\varphi \in \text{Hom}_{\mathfrak{A}}(A)$, $\psi \in \text{Hom}_{\mathfrak{A}}(B)$, and let $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ be unital. Then A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable if and only if $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.*

Proof. Suppose that A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable. By Proposition 4.4, $\frac{A}{J_{A,\mathfrak{A}}}$ and $\frac{B}{J_{B,\mathfrak{A}}}$ are $\bar{\varphi}$ -approximately amenable and $\bar{\psi}$ -approximately amenable, respectively. Now by using Proposition 3.5 for $\mathfrak{A} = \mathbb{C}$, we conclude that $\frac{A}{J_{A,\mathfrak{A}}} \oplus \frac{B}{J_{B,\mathfrak{A}}}$ is $\bar{\varphi} \oplus \bar{\psi}$ -approximately amenable. So, Theorem 4.3, implies that $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable.

Conversely, suppose that $A \oplus B$ is $\varphi \oplus \psi$ - \mathfrak{A} -module approximately amenable. Then by Proposition 3.7, A is φ - \mathfrak{A} -module approximately amenable and B is ψ - \mathfrak{A} -module approximately amenable. \square

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