A Graphical Characterization for $SPAP$-Rings

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Abstract. Let $R$ be a commutative ring and $I$ an ideal of $R$. The zero-divisor graph of $R$ with respect to $I$, denoted by $\Gamma_I(R)$, is the simple graph whose vertex set is $\{x \in R \setminus I \mid xy \in I, \text{ for some } y \in R \setminus I\}$, with two distinct vertices $x$ and $y$ are adjacent if and only if $xy \in I$. In this paper, we state a relation between zero-divisor graph of $R$ with respect to an ideal and almost prime ideals of $R$. We then use this result to give a graphical characterization for $SPAP$-rings.

Keywords: $SPAP$-ring, Almost prime ideal, Zero-divisor graph with respect to an ideal.


1. INTRODUCTION

Throughout this paper, all rings are assumed to be commutative with identity. A graph (simple graph) $G$ is an ordered pair of disjoint sets $(V,E)$ such that $V = V(G)$ is the vertex set of $G$ and $E = E(G)$ is its edge set. A graph $F$ is called a subgraph of a graph $G$ if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A subgraph $F$ of $G$ is said to be an induced subgraph of $G$ if each edge of $G$ having its ends in $V(F)$ is also an edge of $F$. A graph in which each pair of distinct vertices is joined by an edge is called complete. There have been several studies concerning the assignment a graph to a ring, a group, a semigroup or a module, for more information see [1], [8] and [12]. The
concept of the zero-divisor graph of a commutative ring $R$ was first introduced by Beck [6]. The zero-divisor graph of a commutative ring $R$ is defined to be the graph $\Gamma(R)$, whose vertices are the non-zero zero-divisors of $R$, and where $x$ is adjacent to $y$ if $xy = 0$. In [10] Redmond has generalized the notion of the zero-divisor graph. For a given ideal $I$ of a commutative ring $R$, he defined the zero-divisor graph of $R$ with respect to $I$, denoted by $\Gamma_I(R)$, is the simple graph whose vertex set is $\{x \in R \setminus I \mid xy \in I\text{, for some } y \in R \setminus I\}$, with two distinct vertices $x$ and $y$ joined by an edge when $xy \in I$. Clearly $\Gamma_0(R) = \Gamma(R)$. Bhatwadekar and Sharma [7] defined a proper ideal $I$ of an integral domain $R$ to be almost prime if for all $a,b \in R$, $ab \in I \setminus I^2$, then either $a \in I$ or $b \in I$. Anderson and Bataineh [3], use this definition for an arbitrary commutative ring and stated a necessary and sufficient condition for a commutative Noetherian ring under which every proper ideal of $R$ is a product of almost prime ideals. Then Rostami and Nekooei [11], considered $\text{SPAP}$-rings and characterized the structure of $\text{SPAP}$-rings, in special cases. Also, they showed that $\text{SPAP}$-rings are quasi-Frobenius (a Noetherian self-injective ring), and $\text{SPAP}$-rings are an applicative class of rings in Coding Theory, for more information see [11].

In the next section, we state a relation between zero-divisor graph with respect to an ideal of $R$ and almost prime ideals of $R$. Then we state the concept of the intersection graph of ideals of $R$, and we give a graphical characterization for $\text{SPAP}$-rings.

2. MAIN RESULTS

A proper ideal $I$ in a ring $R$ is called almost prime if for all $a,b \in R$, $ab \in I \setminus I^2$ either $a \in I$ or $b \in I$. Also, a proper ideal $I$ of a ring $R$ is called weakly prime if for all $a,b \in R$ with $0 \neq ab \in I$, either $a \in I$ or $b \in I$. Clearly, every weakly prime ideal is almost prime. The following lemma which plays an important role in this paper gives a graphical characterization for almost prime ideals.

**Lemma 2.1.** Let $I$ be a proper ideal of $R$. Then $I$ is an almost prime ideal of $R$ if and only if $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$.

**Proof.** Let $I$ be an almost prime ideal of $R$ and $x \in V(\Gamma_I(R))$, then $x \in R \setminus I$, and there exists $y \in R \setminus I$ such that $xy \in I$. Thus $x,y \notin I^2$. Now if $xy \notin I^2$, then we have $xy \in I \setminus I^2$, this gives $x \in I$ or $y \in I$, a contradiction. Thus $xy \in I^2$ and so $x \in V(\Gamma_{I^2}(R))$. Now let $x,y \in V(\Gamma_I(R))$ be adjacent in $\Gamma_I(R)$, so $xy \in I$, if $xy \notin I^2$, then we have $xy \in I \setminus I^2$, this gives $x \in I$ or $y \in I$, a contradiction. Therefore, $x$ and $y$ are adjacent in $\Gamma_{I^2}(R)$. Thus $E(\Gamma_I(R)) \subseteq E(\Gamma_{I^2}(R))$. Clearly, each edge of $\Gamma_{I^2}(R)$ having its ends in $\Gamma_I(R)$ is also an edge of $\Gamma_I(R)$. Therefore, $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$. Conversely, let $\Gamma_I(R)$ be an induced subgraph of $\Gamma_{I^2}(R)$ and $ab \in I \setminus I^2$, if
If $a, b \notin I$ then, $a$ and $b$ are adjacent in $\Gamma_I(R)$ and so $a$ and $b$ are adjacent in $\Gamma_{I^2}(R)$, thus $ab \in I^2$, a contradiction. Therefore, either $a \in I$ or $b \in I$. \hfill $\Box$

The following lemma is a similar result for weakly prime ideals.

**Lemma 2.2.** Let $I$ be a proper ideal of $R$. Then $I$ is a weakly prime ideal of $R$ if and only if $\Gamma_I(R)$ is an induced subgraph of $\Gamma(R)$.

**Proof.** Let $I$ be a weakly prime ideal of $R$ and $x \in V(\Gamma_I(R))$. Then $x \in R \setminus I$, and there exists $y \in R \setminus I$ such that $xy \notin I$. If $xy \neq 0$, we have $0 \neq xy \in I$, this gives $x \in I$ or $y \in I$, a contradiction. Thus $xy = 0$, and so $x \in V(\Gamma(R))$. Now, let $x, y \in V(\Gamma_I(R))$ be adjacent in $\Gamma_I(R)$, thus $xy \in I$. Repeating the previous argument leads to $xy = 0$. Hence $x, y$ are adjacent in $\Gamma(R)$. Clearly, each edge of $\Gamma(R)$ having its ends in $\Gamma_I(R)$ is also an edge of $\Gamma_I(R)$. Therefore $\Gamma_I(R)$ is an induced subgraph of $\Gamma(R)$. Conversely, let $\Gamma_I(R)$ be an induced subgraph of $\Gamma(R)$ and $0 \neq ab \in I$ for $a, b \in R$, if $a \notin I$ and $b \notin I$ then, $a$ and $b$ are adjacent in $\Gamma_I(R)$, thus $a$ and $b$ are adjacent in $\Gamma(R)$. This gives $ab = 0$, a contradiction. Thus either $a \in I$ or $b \in I$. \hfill $\Box$

**Lemma 2.3.** Let $I$ be a proper ideal of $R$. Then $I$ is a prime ideal of $R$ if and only if $\Gamma_I(R) = \emptyset$.

**Proof.** The proof is straightforward. \hfill $\Box$

Now let $I$ be a prime ideal of $R$. Thus $\Gamma_I(R) = \emptyset$ and so $\Gamma_I(R) = \emptyset$ is an induced subgraph of $\Gamma_{I^2}(R)$ and $\Gamma(R)$, this is a graphical verification for the fact that “prime ideals are almost prime and weakly prime”.

**Definition 2.4.** A local ring $(R, m)$ is called special product of almost prime ideals ring ($SPAP$-ring), if for each $x \in m \setminus m^2$, $<x^2> = m^2$ and $m^3 = 0$.

$SPAP$-rings were first introduced in [3] by D. D. Anderson and M. Bataineh. In [3], D. D. Anderson and M. Bataineh used $SPAP$-rings to characteriz Noetherian rings whose proper ideals are a product of almost prime ideals. In general, an $SPAP$-ring is not Noetherian, see [3, Example 20]. For an $SPAP$-ring $(R, m)$, $m$ is the unique prime ideal of $R$, thus $R$ is a Noetherian ring if and only if $R$ is an Artinian ring if and only if $m$ is a finitely generated ideal of $R$.

Before proceeding, we mention the definition of the intersection graph of ideals of a ring which helps us to give a characterization for $SPAP$-rings.

**Definition 2.5.** Let $R$ be a ring, the intersection graph of ideals of $R$, denoted by $G(R)$, is the graph whose vertices are proper non-trivial ideals of $R$ and two distinct vertices are adjacent if and only if the corresponding ideals of $R$ have a non-trivial (non-zero)intersection.
Lemma 2.6. [5, Theorem 2.11.] Let \((R, m)\) be an Artinian local ring. Then the intersection graph of ideals of \(R\) is complete if and only if \(R\) has a unique minimal ideal.

For more information about intersection graph of ideals of \(R\), see [2, 5]. In the remainder of this section, we characterize Artinian local rings which \(\Gamma_I(R)\) is an induced subgraph of \(\Gamma_J(R)\) for all non-minimal ideals \(I\) of \(R\), and the intersection graph of ideal of \(R\) is complete.

Lemma 2.7. Let \((R, m)\) be an Artinian local ring and \(\Gamma_I(R)\) is an induced subgraph of \(\Gamma_J(R)\), for every non-minimal ideal \(I\) of \(R\). Then \(m^2\) is a minimal ideal of \(R\) or \(m^2 = 0\).

Proof. Let \(m^2\) be a non-minimal ideal of \(R\). Then by Lemma 2.1, \(m^2\) is an almost prime ideal of \(R\). We show that \(m^2\) must be zero in this case. For this purpose, we show that \(m^2 = m^3 = m^4\) and the Nakayama’s Lemma gives \(m^2 = 0\). If for all \(x, y \in m\), \(xy \in m^4\), we have \(m^2 \subseteq m^3\), thus \(m^2 = m^3 = m^4\).

Now let there exist \(x, y \in m\) such that \(xy \notin m^4\), so \(xy \in m^2 \setminus m^4 = m^2 \setminus (m^2)^2\), since \(m^2\) is almost prime, only one of the following cases happens:

\(x \in m^2\) and \(y \notin m^2\) or \(x \notin m^2\) and \(y \in m^2\). Suppose \(x \in m^2\) and \(y \notin m^2\).

Since \(y^2 \in m^2\), \(y \notin m^2\) and \(m^2\) is almost prime, we must have \(y^2 \notin m^4\).

Repeating the previous argument and \(y, x + y \notin m^2\) and \(y(x + y) \in m^2\) leads to \(y(x + y) \in m^4\). Thus \(xy + y^2 = y(x + y), y^2 \in m^4\), so \(xy \in m^4\), a contradiction. Thus \(m^2\) is zero or a minimal ideal. □

Now we mention the definition of a class of rings which are important in the rest of this paper.

Definition 2.8. A commutative ring \(R\) is called special principal ideal ring (SPIR), if it is a principal ideal ring with unique prime ideal and that prime ideal is nilpotent.

Mori [9] has shown that a ring has the property that every ideal is a product of prime ideals if and only if it is a finite direct product of Dedekind domains and special principal ideal rings (SPIRs)(For more information about special principal ideal ring see [9]). In the next lemma, we state a relation between SPAP-rings and SPIR rings.

Lemma 2.9. Let \((R, m)\) be an SPIR ring such that \(\Gamma_I(R)\) is an induced subgraph of \(\Gamma_J(R)\), for every non-minimal ideal \(I\) and \(m^2\) is the unique minimal ideal of \(R\). Then \((R, m)\) is an SPAP-ring.

Proof. Since \(R\) is an SPIR ring, \(m = \langle x \rangle\) for some \(x \in m\). Now let \(0 \neq J \neq m^2\) be an ideal of \(R\). If \(J = J^2\), Nakayama’s Lemma gives \(J = 0\), a contradiction. So \(J \neq J^2\), thus we can select \(y \in J \setminus J^2 \subseteq m\) such that \(J = \langle y \rangle\). Thus \(y = rx \in J \setminus J^2\), for some \(r \in R\). Since \(J \neq m^2\), Lemma
2.1 gives $J$ is an almost prime ideal of $R$ and since $y = rx \in J \setminus J^2$, we have $x \in J$ or $r \in J$. If $x \in J$, then $J = m$ and if $r \in J \subseteq m$, then we have $J = <y> = <rx> \subseteq m^2$ and since $m^2$ is the unique minimal ideal of $R$, $J = 0$ or $J = m^2$, a contradiction. This means, the set of all ideals of $R$ is $\{0, m^2, m, R\}$. Now if $m = m^2$, we have $m = 0$, a contradiction. Thus $m \neq m^2$. If $a \in m \setminus m^2$, since the set of all ideals of $R$ is $\{0, m^2, m, R\}$, we have $m = <a>$, so $m^2 = <a^2>$. Now if $m^3 \neq 0$, we have $m^2 = m^4$, and Nakayama’s Lemma gives $m = 0$, a contradiction. Thus $m^3 = 0$. This completes the proof. □

D. D. Anderson and M. Bataineh in [3], by using SPAP-rings, characterized Noetherian rings whose proper ideals are a product of almost prime ideals. Actually, they stated the following theorem.

**Theorem 2.10.** [3, Theorem 22]. Let $R$ be a Noetherian ring. Then every proper ideal of $R$ is a product of almost prime ideals if and only if $R$ is a finite direct product of Dedekind domains, SPIRs, and (Noetherian) SPAP-rings.

**Proposition 2.11.** Let $(R, m)$ be an Artinian local ring such that $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, for every non-minimal ideal $I$ of $R$ and the intersection graph of ideal of $R$ is complete. If $m^2 \neq 0$ then $R$ is an SPAP-ring.

**Proof.** Since the intersection graph of ideals of $R$ is complete, by Lemma 2.6, $R$ has a unique minimal ideal. Since $m^2 \neq 0$, Lemma 2.7 gives, $m^2$ is the unique minimal ideal of $R$. Now let $I$ be an arbitrary proper ideal of $R$, if $I$ is a non-minimal ideal of $R$, then $\Gamma_I(R)$ is an induced subgraph of $\Gamma_{I^2}(R)$, so $I$ is an almost prime ideal of $R$, by Lemma 2.1, and if $I$ is a minimal ideal of $R$, then $I = m^2$. Therefore, in all cases $I$ is finite product of almost prime ideals (note that $m$ is prime and so is almost prime), thus by Theorem 2.10, $R$ is a finite direct product of Dedekind domains, SPIR rings, and SPAP-rings. Since $R$ is a local ring, this direct product must have a single ring. Let $R$ be a Dedekind domain. Since $m^2$ is a minimal ideal of $R$, we have $m^3 = 0$ or $m^2 = m^3$, in both cases, we have $m^2 = 0$. Thus $R$ is not a Dedekind domain and Lemma 2.9, completes the proof. □

An $R$-module $M$ is said to be a multiplication $R$-module if for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. Clearly, every cyclic module is multiplication module, see [4] for more information. After stating the main result, we require the following three lemmas.

**Lemma 2.12.** Let $(R, m)$ be an SPAP-ring. If $m^2 \neq 0$, then $m^2$ is a minimal ideal of $R$.  


If \( m = m^2 \), then \( m^2 = m^3 = 0 \), a contradiction. Therefore \( m \neq m^2 \), thus there exists \( y \in m \setminus m^2 \). So \( m^2 = \langle y^2 \rangle \). Therefore, \( m^2 \) is a cyclic \( R \)-module and so it is a multiplication \( R \)-module. Now if \( J \) is a submodule (ideal of \( R \)) of \( m^2 \), there exists an ideal \( K \) of \( R \), such that \( J = Kn^2 \). If \( K = R \), then \( J = m^2 \) and if \( K \neq R \) then \( J = Kn^2 \subseteq m^3 = 0 \), hence \( J = 0 \). Therefore \( m^2 \) is a minimal ideal of \( R \).

**Lemma 2.13.** Let \( (R, m) \) be an SPAP-ring. If \( m^2 \neq 0 \) and \( I \) is a proper ideal of \( R \), then \( I = 0 \) or \( I = m^2 \) or \( I^2 = m^2 \).

**Proof.** Since \( m^2 \neq 0 \), by Lemma 2.12, \( m^2 \) is a minimal ideal of \( R \). Now let \( I \) be a proper ideal of \( (R, m) \). If \( I \subseteq m^2 \), then \( I = 0 \) or \( I = m^2 \). If \( I \nsubseteq m^2 \), then there exists \( y \in I \setminus m^2 \). So \( m^2 = \langle y^2 \rangle \), hence \( m^2 = \langle y^2 \rangle \subseteq I^2 \subseteq m^2 \). Thus \( I^2 = m^2 \).

By combining the above two lemmas, we have the following lemma.

**Lemma 2.14.** Let \( (R, m) \) be an SPAP-ring. If \( m^2 \neq 0 \), then \( m^2 \) is the unique minimal ideal of \( R \).

Now we can state the main result of this paper.

**Theorem 2.15.** Let \( (R, m) \) be an Artinian local ring with \( m^2 \neq 0 \). Then \( \Gamma_I(R) \) is an induced subgraph of \( \Gamma_{I^2}(R) \), for every non-minimal ideal \( I \) of \( R \) and the intersection graph of ideals of \( R \) is complete if and only if \( R \) is an SPAP-ring.

**Proof.** Let \( R \) be an SPAP-ring by Lemma 2.14, \( m^2 \) is the unique minimal ideal of \( R \), so by Lemma 2.6, the intersection graph of ideals of \( R \) is complete. Now let \( I \) be a proper ideal of \( R \), Lemma 2.13 gives \( I = 0 \) or \( I = m^2 \) or \( I^2 = m^2 \). If \( I \) is a non-zero non-minimal ideal of \( R \) and \( ab \in I \setminus I^2 \), for \( a, b \in R \), then \( ab \not\in I^2 = m^2 \), so \( a \) or \( b \) is not in \( m \), thus \( a \) or \( b \) is unit. Thus \( a \) or \( b \) must be in \( I \). This shows that \( I \) is an almost prime ideal of \( R \). Hence, by Lemma 2.1, \( \Gamma_I(R) \) is an induced subgraph of \( \Gamma_{I^2}(R) \). In general, the zero ideal is an almost prime of \( R \). Thus every non-minimal ideal of \( R \) is almost prime and so \( \Gamma_I(R) \) is an induced subgraph of \( \Gamma_{I^2}(R) \), for every non-minimal ideal \( I \) of \( R \).

The converse of theorem is valid by Proposition 2.11.

**Example 2.16.** Let \( k \) be an ordered field. Then for a non-empty set \( \{x_\alpha\}_{\alpha \in \Delta} \) of indeterminates. Define \( R = k[[\{x_\alpha\}_{\alpha \in \Delta}]] \), \( m = \langle \{x_\alpha\}_{\alpha \in \Delta} \rangle \), and \( J = \langle \{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2 \}_{\alpha \neq \beta}, \{x_\alpha^3\}_{\alpha} \rangle \). Let \( \overline{R} = \frac{k}{J} \). Then \( \overline{R} \) is an SPAP-ring with \( \overline{m}^2 \neq 0 \) and \( \overline{m} \) is not principal for \( |\Delta| > 1 \), see [3, Example 20]. If \( \Delta \) is a finite set, then \( \overline{R} \) is a Noethrian SPAP-ring with \( \overline{m}^2 \neq 0 \), and thus \( \Gamma_{\overline{I}}(\overline{R}) \) is an induced subgraph of \( \Gamma_{\overline{I^2}}(\overline{R}) \), for every non-minimal ideal \( \overline{I} \) of \( \overline{R} \) and the intersection graph of ideals of \( \overline{R} \) is complete.
Theorem 2.17. Let \((R, m)\) be an Artinian local ring with \(m^2 \neq 0\), such that \(\Gamma_I(R)\) is an induced subgraph of \(\Gamma_{I^2}(R)\), for every non-minimal ideal \(I\) of \(R\) and the intersection graph of ideals of \(R\) is complete. If \(\text{char}(R) \neq p^2\), for any prime number \(p\) and \(\text{char}(\frac{R}{m}) \neq 2\), then there exists a regular local ring \((S, n)\), a positive integer number \(h\), and subset \(\{x_a\}_{a=1}^h\) of \(n\) such that \(R \cong S[K]\) in which \(K\) is minimally generated by the elements \(\{x_ix_j\}_{1 \leq i < j \leq h}, \{x_i^2\}_{2 \leq j \leq \tau}\) and \(\{x_i^2u_ix_j^2\}_{\tau + 1 \leq i \leq h}\), where the \(u_i\) are unit in \(R\) and \(\tau\) is the Cohen-Macaulay type of \(R\).

Proof. By Theorem 2.15 and [11, Proposition 6.3.].

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REFERENCES