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The e-Theta Hopes

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Abstract. The largest class of hyperstructures is the H_v-structures, introduced in 1990, which proved to have a lot of applications in mathematics and several applied sciences, as well. Hyperstructures are used in the Lie-Santilli theory focusing to the hypernumbers, called e-numbers. We present the appropriate e-hyperstuctures which are defined using any map, in the sense the derivative map, called ∂-hyperstructures.

Keywords: hyperstructures, H_v-structures, Hopes, ∂-Structures.

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1. Basic Definitions

We deal with hyperstructures called H_v-structures introduced in 1990 [16],[17] which satisfy the weak axioms where the non-empty intersection replaces the equality.
Some basic definitions are the following:
In a set H equipped with a hyperoperation (abbreviation *hyperoperation*=***hope***) $. : H \times H \longrightarrow P(H) - \{\emptyset\}$, we abbreviate by
WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by
COW the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$.

The hyperstructure $(H, .)$ is called ***H_v-semigroup*** if it is WASS and is called **H_v-group** if it is reproductive H_v-semigroup, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, .)$ is called **H_v-ring** if $(+)$ and $(.)$ are WASS, the

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reproduction axiom is valid for $(+)$ and $(.)$ is *weak distributive* with respect to $(+)$, i.e.

$$x(y+z)\cap(xy+xz)\neq\emptyset,(x+y)z\cap(xz+yz)\neq\emptyset,\forall x,y,z\in R.$$

Motivation for H_v***-structures***:
The motivation for H_v-structures is the following: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group. This is the motivation to introduce the H_v-structures [16].
Specifying this motivation we remark: Let $(G,.)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,.)$ is an H_v-group, therefore we have the quotient $(G/R,.)/\beta^*$ which is a group, the *fundamental* one. Remark that the classes of the fundamental group $(G/R,.)/\beta^*$ are a union of some of the R-classes. Otherwise, the $(G/R,.)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R.

In an H_v-semigroup the *powers* of an element $h\in H$ are defined as follows

$$h^1=\{h\},h^2=h.h,...,h^n=h^\circ h^\circ ...^\circ h,$$

where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H,.)$ is called *cyclic of period s*, if there exists an element g, called *generator*, and a natural number n, the minimum one, such that

$$H=h^1\cup h^2\cup...\cup h^s$$

Analogously the cyclicity for the infinite period is defined [15]. If there is an element h and a natural number s, the minimum one, such that $H=h^s$, then $(H,.)$ is called *single-power cyclic of period s*.

For more definitions and applications on H_v-structures, see the books [2], [3] [7], [8], [17] and papers as [1], [4], [6], [11], [12], [15], [18], [19], [20], [22], [23], [24].

The fundamental relations β^*,γ^* and ε^* are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [16], [17], [18], [19]. The way to find the fundamental classes is given by analogous theorems to the following one:

Theorem 1.1. *Let $(H,.)$ be an H_v-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $x,y \subset \mathbf{u}$ where $\mathbf{u} \in \mathbf{U}$. Then the fundamental relation β^* is the transitive closure of the relation β.*

Analogous theorems for the relations γ^* in H$_v$-rings, ε^* in H$_v$-modules and H$_v$-vector spaces, are also proved. An element is called single if its fundamental class is singleton [17].

Fundamental relations are used for general definitions. Thus, an H$_v$-ring $(R,+,.)$ is called H$_v$-field if R/γ^* is a field.

Let $(H,.),(H,\otimes)$ H$_v$-semigroups defined on the same set H. (.) is called ***smaller*** than $(\otimes)$, and $(\otimes)$ greater than (.), iff there exists automorphism

$$f \in Aut(H,\otimes) \;\; such \;\; that \;\; xy \subset f(x \otimes y), \forall x \in H.$$

Then we write $. \leq \otimes$ and we say that $(H,\otimes)$ *contains* the $(H,.)$. If $(H,.)$ is a structure then it is called *basic structure* and $(H,\otimes)$ is called H$_b$-*structure*.

The Little Theorem. Greater hopes of hopes which are WASS or COW, are also WASS and COW, respectively.

Definition 1.2. Let $(H,.)$ be hypergroupoid. We say that we *remove* $h \in H$, if we consider the restriction of the hope (.) on the $H-\{h\}$. We say that an $\underline{h} \in H$ absorbs $h \in H$ if we replace h, whenever it appears, by $\underline{h}$. We say that $\underline{h} \in H$ *merges* with $h \in H$, if we take as the product of any $h \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and we consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

Definition 1.3. [23]. Let $A=(a_{ij}) \in M_{m\times n}$ be a matrix and $s,t \in N$, with $1 \leq s \leq m, 1 \leq t \leq n$. Then *helix-projection* is a map $\underline{st} : M_m \times n \longrightarrow M_s \times t :$ $A \longrightarrow A\underline{st} = (\underline{a}_{ij})$, where $A\underline{st}$ has entries

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} | 1 \leq i \leq s, 1 \leq j \leq t \;\; and \;\; \kappa,\lambda \in N, i+\lambda s \leq m, j+\lambda t \leq n\}$$

Let $A=(a_{ij}) \in M_{m\times n}, B=(b_{ij}) \in M_{u\times v}$ be matrices and $s=min(m,u)$, $t=min(n,v)$. We define a hyper-addition, called *helix-addition*, by

$$\oplus : M_{m\times n} \times M_{u\times v} \longrightarrow P(M_{s\times t}) : (A,B) \longrightarrow A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subseteq M_{s\times t}$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij}+b_{ij}) | a_{ij} \in \underline{a}_{ij} \;\; and \;\; b_{ij} \in \underline{b}_{ij}\}$.
Let Let $A=(a_{ij}) \in M_{m\times n}, B=(b_{ij}) \in M_{u\times v}$ be matrices and $s=min(n,u)$, define the *helix-multiplication*, by

$$\otimes : M_{m\times n} \times M_{u\times v} \longrightarrow P(M_{m\times v}) : (A,B) \longrightarrow A \otimes B = A\underline{ms}.B\underline{sv} = (\underline{a}_{ij}).(\underline{b}_{ij}) \subseteq M_{m\times v}$$

where $(\underline{a}_{ij}).(\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it}.b_{tj}) | a_{ij} \in \underline{a}_{ij} \;\; and \;\; b_{ij} \in \underline{b}_{ij}\}$.

The helix-addition is commutative, WASS but not associative. The helix-multiplication is WASS, not associative and it is not distributive, not even weak, to the helix-addition. For all matrices of the same type, the inclusion distributivity, is valid.

Definition 1.4. [17], [22]. Let $(F,+,.)$ be an H_v-field, $(V,+)$ be a COW H_v-group and there exists an external hope

$$. : F \times V \longrightarrow P(V) - \emptyset : (a,x) \longrightarrow ax$$

such that, for all a,b in F and x,y in V we have

$$a(x+y) \cap (ax+ay) \neq \emptyset,\ \ (a+b)x \cap (ax+bx) \neq \emptyset,\ \ (ab)x \cap a(bx) \neq \emptyset,$$

then V is called an H_v-vector space over F. In the case of an H_v-ring instead of an H_v-field then the H_v-modulo is defined. In these cases the fundamental relation ε^* is the smallest equivalence relation such that the quotient V/ε^* is a vector space over the fundamental field F/γ^*.

The general definition of an H$_v$-Lie algebra was given in [14], [21], [22] as follows:

Definition 1.5. Let $(\mathbf{L},+)$ be H$_v$-vector space over the field $(\mathbf{F},+,)$, $\varphi : \mathbf{F} \longrightarrow \mathbf{F}/\gamma^*$, the canonical map and $\omega_F = \{x \in F : \varphi(x) = 0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\varphi' : \mathbf{L} \longrightarrow \mathbf{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\varepsilon^*$. Consider the *bracket (commutator) hope:*

$$[\ ,\ ] : L \times L \longrightarrow P(L) : (x,y) \longrightarrow [x,y]$$

then $\mathbf{L}$ is an H$_v$-Lie algebra over $\mathbf{F}$ if the following axioms are satisfied:
(L1) The bracket hope is bilinear, i.e.
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1,y] + \lambda_2[x_2,y]) \neq \emptyset$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x,y_1] + \lambda_2[x,y_2]) \neq \emptyset,\ \forall x, x_1, x_2, y, y_1, y_2 \in L$ *and* $\lambda_1, \lambda_2 \in F$
(L2) $[x,x] \cap \omega_L \neq \emptyset,\ \forall x \in L$
(L3) $([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \cap \omega_L \neq \emptyset,\ \forall x,y \in L$

The *uniting elements* method was introduced by Corsini-Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

2. Some Classes of H_v-Structures

The P-hopes

A general way to define hopes, which are not always of a constant length, from given operations [15], [17] can be generalized as follows:

Definition 2.1. Let $(G, .)$ be a groupoid, then for every set $P \subset G, P \neq \emptyset$, we define the following hopes called *P-hopes*:

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy), \forall x, y \in G$$

The $(G, P), (G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called *P-hyperstructures*. The most usual case is if $(G, .)$ is semigroup, then

$$xPy = (xP)y \cup x(Py) = xPy$$

and (G, P) is a semihypergroup but we do not know for $(G, \underline{P}_r), (G, \underline{P}_l)$. In some cases, mainly depending on the choice of P, the $(G, \underline{P}_r), (G, \underline{P}_l)$ can be associative or WASS. If in G, more operations are defined then for each operation several P-hopes can be defined.

Construction. Let $(G, .)$ be abelian group and $P \subset G$, with more than one elements. We define a hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x.P.y = \{x.h.y | h \in P\} & x \neq e \quad and \quad y \neq e \\ x.y & x = e \quad or \quad y = e \end{cases}$$

Definition 2.2. Let $M = M_{m \times n}$ be a module of $m \times n$ matrices over a ring R and $P = \{P_i : i \in I\} \subseteq M$. We define, a kind of, a P- hope $\underline{P}$ on M as follows

$$\underline{P} : M \times M \longrightarrow P(M) \quad : (A, B) \longrightarrow A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq M$$

where P^t denotes the transpose of the matrix P. The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope $\underline{P}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$A\underline{P}(B + C) \subseteq A\underline{P}B + A\underline{P}C \quad for \;\; all \;\; A, B, C \in M$$

is valid. Therefore, $(M, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

Definition 2.3. Let $M = M_{m \times n}$ be a module of $m \times n$ matrices over a ring R and let take sets $S = \{s_k : k \in K\} \subseteq R, Q = \{Q_i : j \in J\} \subseteq M, P = \{P_i : i \in I\} \subseteq M$. Define three hopes as follows

$$S : R \times M \longrightarrow P(M) : (r, A) \longrightarrow r\underline{S}A = \{(r_{sk})A : k \in K\} \subseteq M$$

$$\underline{Q}_+ : M \times M \longrightarrow P(M) : (A, B) \longrightarrow A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq M$$

$$\underline{P} : M \times M \longrightarrow P(M) : (A, B) \longrightarrow A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq M$$

Then $(M, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra over R called general matrix P-*hyperalgebra*. In a similar way a generalization of this hyperalgebra can be defined if one considers an H_v-ring or an H_v-field instead of a ring and using H_v-matrices.

We present now a large class of hopes defined in any groupoid with a map f on it, which is denoted by 'theta' ∂, since the motivation is the property which the derivative has on the product of functions, see [21],[22].

The theta ∂-hopes

Definition 2.4. Let $(G, .)$ be groupoid (respectively, hypergroupoid) and $f : G \longrightarrow G$ be a map. We define a hope ∂, called *theta-hope*, on G as follows

$$x\partial y = \{f(x).y, x.f(y)\}, \forall x, y \in G.(resp.\ x\partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G)$$

If $(.)$ is commutative then (∂) is commutative. If $(.)$ is *COW*, then (∂) is *COW*.

Let $(G, .)$ be groupoid (resp. hypergroupoid) and $f : G \longrightarrow \mathbf{P}(G) - \{\emptyset\}$, multivalued map. We define the *theta-hope* (∂), on G as follows

$$x\partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G$$

Let $(G, .)$ be a groupoid and $f_i : G \longrightarrow G, i \in I$, be set of maps on G. We consider the map $f_\cup : G \longrightarrow \mathbf{P}(G)$ such that $f_\cup(x) = \{f_i(x) | i \in I\}$, called *the union* of the $f_i(x)$. We define the *union theta-hrope* (∂), on G if we consider the $f_\cup(x)$. A special case for given f, is to take the union with the identity: We consider the map $\underline{f} \equiv f \cup (id)$, so $\underline{f}(x) = \{x, f(x)\}, \forall x \in G$, which we call *b-theta-hope*. Then we have

$$x\partial y = \{xy, f(x).y, x.f(y)\}, \forall x, y \in G.$$

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for any functions $s(x), t(x)$, we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

Proposition 2.5. [21]. *If $(G, .)$ is a semigroup then:*
(a) For every f, the (∂) is WASS. If f is homomorphism then (∂) remains WASS.
(b) If f is homomorphism and projection, i.e. $f^2 = f$, then (∂) is associative.
(c) If $(G, .)$ is a semigroup then, for every f, the b-theta-operation (∂) is WASS.
(d) Reproductivity. If $(.)$ is reproductive then $(\partial)$is also reproductive, because

$$x\partial G = \bigcup_{g \in G} \{f(x).g, x.f(g)\} = G \ \ and \ \ G\partial x = \bigcup_{g \in G} \{f(g).x, g.f(x)\} = G$$

(e) Commutativity. If (.) is commutative then (∂) is commutative. If f is into the center of G, then (∂) is a commutative. If (.) is a COW then, (∂) is a COW.
(f) Unit elements. u is a right unit element if $x \in x\partial u = \{f(x).u, x.f(u)\}$. So $f(u) = e$, where e be a unit in $(G, .)$. The elements of the kernel of f, are the units of (G, ∂).
(g) Inverse elements. Let $(G, .)$ is a monoid with unit e and u be a unit in (G, ∂), then $f(u) = e$. For given x, the element $x' = (f(x))^{-1}u$ and $x'' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

Definition 2.6. Let $(R, +, .)$ be a ring and $f : R \longrightarrow R, g : R \longrightarrow R$ be two maps. We define two hopes (∂_+) and $(\partial_.)$, called both *theta-hopes*, on R as follows

$$x\partial_+ y = \{f(x) + y, x + f(y)\} \text{ and } x\partial_. y = \{g(x)y, xg(y)\}, \forall x, y \in G$$

The hyperstructure $(R, \partial_+, \partial_.)$, called *theta*, is an H_v-near-ring, i.e. satisfy all H_v-ring axioms, except the weak distributivity.

Some results and examples:

Let $(G, .)$ be group and $f(x) = a$, a constant map on G. Then $(G, \partial)/\beta^*$ is singleton. If $f(x) = e$, then $x\partial y = \{x, y\}$ which is the smallest incidence hope.

Consider all polynomials of first degree $g_i(x) = a_i x + b_i$, and as map the derivative, we have

$$g_1 \partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\},$$

so it is a hope inside the set of first degree polynomials. Moreover all polynomials $x + c$, where c be a constant, are units.

Several results can be obtained using ∂-hopes [21]:

Theorem 2.7. *(a) Consider the group of integers $(\mathbb{Z}, +)$ and $n \neq 0$ be a natural number. Take the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbb{Z} - \{0\}$. Then*

$$(\mathbb{Z}, \partial)/\beta^* \cong (\mathbb{Z}_n, +).$$

(b) Take the ring of integers $(\mathbb{Z}, +, .)$ and fix $n \neq 0$ a natural number. Consider f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbb{Z} - \{0\}$. Then $(\mathbb{Z}, \partial_+, \partial_.)$, where ∂_+ and $\partial_.$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbb{Z}, \partial_+, \partial_.)/\gamma^* \cong (\mathbb{Z}_n).$$

(c) Take the $(\mathbb{Z}, +, .)$ *and* $n \neq 0$ *a natural. Take* f *such that* $f(n) = 0$ *and* $f(x) = x, \forall x \in \mathbb{Z} - \{n\}$. *Then* $(\mathbb{Z}, \partial_+, \partial_.)$ *is an* H_v*-ring, moreover,*

$$(\mathbb{Z}, \partial_+, \partial_.)/\gamma^* \cong (\mathbb{Z}_n).$$

Special case of the above is for $n = p$*, prime, then* $(\mathbb{Z}, \partial_+, \partial_.)$ *is an* H_v*-field.*

Special case of the above is for $n = p$, prime, then $(\mathbb{Z}, \partial_+, \partial_.)$ is an H_v-field.

3. The e-Theta Hopes

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [13] and they are called e-hyperfields. A hyperstructure $(H, .)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we normally assume that for every element x, there exists an inverse element x^{-1}, i.e. $e \in x.x^{-1} \cap x^{-1}.x$. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics [9], [13], [14]. In the following we present the ∂-hyperstructures that they can be used in this theory. First we give the general definition of ∂-hopes.

Definition 3.1. Let H be a set equipped with n operations (or hopes) $\otimes_1, \otimes_2, ..., \otimes_n$ and a map (or multivalued map) $f : H \longrightarrow H$ (or $f : H \longrightarrow P(H) - \{\emptyset\}$, respectively), then n hopes $\partial_1, \partial_2, ..., \partial_n$ on H can be defined, called *theta-operations* (we rename here theta-hopes and we write ∂*-hope*) by putting

$$x\partial_i y = \{f(x) \otimes_i y, x \otimes_i f(y)\}, \quad \forall x, y \in H \;\; and \;\; i \in \{1, 2, ..., n\}$$

or, in case where $\otimes_i$ is hope or f is multivalued map, we have

$$x\partial_i y = (f(x) \otimes_i y) \cup (x \otimes_i f(y)), \quad \forall x, y \in H \;\; andi \in \{1, 2, ..., n\}.$$

(i) If $\otimes_i$ is associative then ∂_i is WASS. Indeed for any map f we have

$$(x\partial y)\partial z = \{f(x)y, xf(y)\}\partial z = \{f(f(x)y)z, f(x)yf(z), f(xf(y))z, xf(y)f(z)\}$$

and

$$x\partial(y\partial z) = x\partial\{f(y)z, yf(z)\} = \{f(x)f(y)z, xf(f(y)z), f(x)yf(z), xf(yf(z))\}$$

so

$$(x\partial y)\partial z \cap x\partial(y\partial z) = \{f(x)yf(z)\} \neq \emptyset$$

(ii) if moreover the map f is an homomorphism then on the above relations we have

$$(x\partial y)\partial z = \{f(x)y, xf(y)\}\partial z = \{f(f(x))f(y)z, f(x)yf(z), f(x)f(f(y))z, xf(y)f(z)\}$$

and

$$x\partial(y\partial z) = x\partial\{f(y)z, yf(z)\} = \{f(x)f(y)z, xf(f(y))f(z), f(x)yf(z), xf(y)f(f(z))\}$$

so again we have

$$(x\partial y)\partial z \cap x\partial(y\partial z) = \{f(x)yf(z)\} \neq \emptyset$$

(iii) if moreover the map f is an homomorphism and a projection $f^2 = f$, then we have

$$(x\partial y)\partial z = \{f(x)y, xf(y)\}\partial z = \{f(x)f(y)z, f(x)yf(z), xf(y)f(z)\}$$

and

$$x\partial(y\partial z) = x\partial\{f(y)z, yf(z)\} = \{f(x)f(y)z, f(x)yf(z), xf(y)f(z)\}$$

so we have the associativity

$$(x\partial y)\partial z = x\partial(y\partial z) = \{f(x)f(y)z, f(x)yf(z), xf(y)f(z)\}$$

Construction. Let $(G, .)$ be a group and f any map on G. We define the *e-theta hopes* (∂_e) as follows:

$$x\partial_e y = \begin{cases} \{f(x)y, xf(y)\} & x \neq e \quad and \quad y \neq e \\ f(xy) & x = e \quad and \quad y = e \end{cases}$$

The hyperstructure (G, ∂_e) is an H_v-group if f is an onto map on G.

Proof. . Let x, y, z be non unit elements of $(G, .)$. Then, supposing $e \neq f(x)y, e \neq xf(y), e \neq f(y)z, e \neq yf(z)$, we have

$$(x\partial_e y)\partial_e z = \{f(x)y, xf(y)\}\partial_e z = \{f(f(x)y)z, f(x)yf(z), f(xf(y))z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{f(y)z, yf(z)\} = \{f(x)f(y)z, xf(f(y)z), f(x)yf(z), xf(yf(z))\}$$

so

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(x)yf(z)\} \neq \emptyset$$

If $e = f(x)y$, then we have

$$(x\partial_e y)\partial_e z = \{e, xf(y)\}\partial_e z = \{f(z), f(xf(y))z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{f(y)z, yf(z)\} = \{f(x)f(y)z, xf(f(y)z), f(z), xf(yf(z))\}$$

therefore we have

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(z)\} \neq \emptyset,$$

so the hope ∂_e is WASS. □

Suppose now that f is an homomorphism, then for $e = f(x)y$, we have

$$(x\partial_e y)\partial_e z = \{e, xf(y)\}\partial_e z = \{f(z), f(x)f(f(y))z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{f(y)z, yf(z)\} = \{f(x)f(y)z, xf(f(y))f(z), f(z), xf(y)f(f(z))\}$$

so again ∂_e is WASS:

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(z)\} \neq \emptyset$$

Suppose now that f is moreover a projection then generally we have the above case (iii), but if $e = f(x)y$, then we have

$$(x\partial_e y)\partial_e z = \{e, xf(y)\}\partial_e z = \{f(z), f(x)f(y)z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{f(y)z, yf(z)\} = f(x)f(y)z, xf(y)f(z), f(z), xf(y)f(z)$$

therefore

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(z), f(x)f(y)z, xf(y)f(z)\} \neq \emptyset$$

if $e = xf(y)$, then we have

$$(x\partial_e y)\partial_e z = \{f(x)y, e\}\partial_e z = \{f(x)f(y)z, f(x)yf(z), f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{f(y)z, yf(z)\} = \{f(x)f(y)z, f(x)yf(z), f(z)\}$$

so ∂_e is associative,

if we have both $e = f(x)y$ and $e = xf(y)$, then

$$(x\partial_e y)\partial_e z = e\partial_e z = \{f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{f(y)z, yf(z)\} = \{f(x)f(y)z, f(x)yf(z), f(z)\}$$

so

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(z)\} \neq \emptyset$$

and ∂_e again is WASS.

We have analogous cases for $e = f(y)z, e = yf(z)$:

If $e = f(y)z$, then we have

$$(x\partial_e y)\partial_e z = \{f(x)y, xf(y)\}\partial_e z = \{f(f(x)y)z, f(x)yf(z), f(xf(y))z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z_) = x\partial_e\{e, yf(z)\} = \{f(x), f(x)yf(z), xf(yf(z))\}$$

therefore we have

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(x)yf(z)\} \neq \emptyset$$

so the hope ∂_e is WASS.

Suppose now that f is an homomorphism, then for $e = f(y)z$, we have

$$(x\partial_e y)\partial_e z = \{f(x)y, xf(y)\}\partial_e z = \{f(f(x)), f(x)yf(z), f(x)f(f(y))z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z_) = x\partial_e\{e, yf(z)\} = \{f(x), f(x)yf(z), xf(y)f(f(z))\}$$

so again ∂_e is WASS:

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(x)yf(z)\} \neq \emptyset$$

Suppose now that f is moreover a projection then generally we have the above case (iii), but if $e = f(y)z$, then we have

$$(x\partial_e y)\partial_e z = \{f(x)y, xf(y)\}\partial_e z = \{f(x), f(x)yf(z), f(x)f(y)z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{e, yf(z)\} = \{f(x), f(x)yf(z), xf(y)f(z)\}$$

therefore

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(x), f(x)yf(z), xf(y)f(z)\} \neq \emptyset$$

If $e = yf(z)$, then we have

$$(x\partial_e y)\partial_e z = \{f(x)y, xf(y)\}\partial_e z = \{f(f(x)y)z, f(x), f(xf(y))z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e\{f(y)z, e\} = \{f(x)f(y)z, xf(f(y)z), f(x)\}$$

so ∂_e is WASS:

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(x)\} \neq \emptyset$$

If we have both $e = f(y)z$ and $e = yf(z)$, then

$$(x\partial_e y)\partial_e z = \{f(x)y, xf(y)\}\partial_e z = \{f(f(x)y)z, f(x), f(xf(y))z, xf(y)f(z)\}$$

$$x\partial_e(y\partial_e z) = x\partial_e e = \{f(x)\}$$

so

$$(x\partial_e y)\partial_e z \cap x\partial_e(y\partial_e z) = \{f(x)\} \neq \emptyset$$

∂_e again is WASS.

Now let $x \neq e$, then

$$x\partial_e G = \{f(x)\} \cup [x\partial_e(G - \{e\})] = \{f(x)\} \cup \{f(x)y/y$$

$$\in (G - \{e\})\} \cup \{xf(y)/y \in (G - \{e\})\} = f(x)G$$

Because we remark that

$$\{f(x)y/y \in (G - \{e\})\} = f(x)(G - \{e\})$$

in which the set $f(x)(G-\{e\})$ contains all the elements of G except the element $f(x)$ and this element is in $x\partial_e G$, therefore we have $x\partial_e G = G$. The same proof for $G\partial_e x = G$. Finally, the reproductivity for the unit e is obvious since f is onto map. Thus (∂_e) is reproductive.

Remark that e is scalar unit in (G, ∂_e). Any element x of G has one or two inverses: the element $(f(x))^{-1}$ and the element y if $xf(y) = e$.

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