

## Distance-Balanced Closure of Some Graphs

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ABSTRACT. In this paper we prove that any distance-balanced graph  $G$  with  $\Delta(G) \geq |V(G)| - 3$  is regular. Also we define notion of distance-balanced closure of a graph and we find distance-balanced closures of trees  $T$  with  $\Delta(T) \geq |V(T)| - 3$ .

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### 1. INTRODUCTION

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . We denote  $|V(G)|$  by  $n$ . The set of neighbors of a vertex  $v \in V(G)$  is denoted by  $N_G(v)$ , and  $N_G[v] = N_G(v) \cup \{v\}$ . The degree of a vertex  $v$  is denoted by  $\deg_G(v)$  and minimum degree and maximum degree of  $G$  denoted by  $\delta(G)$  and  $\Delta(G)$ ,

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respectively. The distance  $d_G(u, v)$  between vertices  $u$  and  $v$  is the length of a shortest path between  $u$  and  $v$  in  $G$ . The diameter  $\text{diam}(G)$  of graph  $G$  is defined as  $\max\{d_G(u, v) : u, v \in V(G)\}$ . The notion of distance is studied in several works in graph theory (See [2] and the references therein) and many research works are based on the concepts related to this notion (See for instance [8] and [10]).

For an edge  $xy$  of a graph  $G$ ,  $W_{xy}^G$  is the set of vertices which are closer to  $x$  than  $y$ , more formally

$$W_{xy}^G = \{u \in V(G) \mid d_G(u, x) < d_G(u, y)\}.$$

Moreover,  ${}_xW_y^G$  is the set of vertices of  $G$  that have equal distances to  $x$  and  $y$ , that is

$${}_xW_y^G = \{u \in V(G) \mid d_G(u, x) = d_G(u, y)\}.$$

These sets play important roles in metric graph theory, see for instance [1, 3, 4, 5]. Since  $x$  always belongs to  $W_{xy}^G$ , for convenience we let  $U_{xy}^G = W_{xy}^G \setminus \{x\}$ . *Distance-balanced* graphs are introduced in [9] as graphs for which  $|W_{xy}^G| = |W_{yx}^G|$  (or equivalently  $|U_{xy}^G| = |U_{yx}^G|$ ) for every pair of adjacent vertices  $x, y \in V(G)$ .

In [9], the parameter  $b(G)$  of a graph  $G$  is introduced as the smallest number of the edges which can be added to  $G$  such that the obtained graph is distance-balanced. Since the complete graph is distance-balanced, this parameter is well-defined. We call graph  $G$  a *distance-balanced closure* of  $H$  if  $G$  is distance-balanced and  $H$  is a spanning subgraph of  $G$  with  $|E(G)| = b(H) + |E(H)|$ ; in other words, a distance-balanced closure of  $H$  is a distance-balanced graph  $G$  which contains  $H$  as a spanning subgraph and has minimum number of edges. As mentioned in [9], the computation of  $b(G)$  is quite hard in general but it might be interesting in some special cases. In this paper we compute  $b(G)$  for all trees  $T$  with  $\Delta(T) \geq |V(T)| - 3$ . In Section 2, we compute that distance-balanced closure of graphs  $G$  with  $\Delta(G) = n - 1$ . In Section 3, and Section 4, we concern graphs  $G$  with  $\Delta(G) = n - 2$  and  $\Delta(G) = n - 3$ , respectively. Then we compute  $b(T)$  for all trees  $T$  with  $\Delta(T) = n - 2$  and  $\Delta(T) = n - 3$ .

Here we mention some more definitions and notations about trees. Let  $P_n$  denoted the path with  $n$  vertices. A tree which has exactly one vertex of degree greater than two is said to be *starlike*. The vertex of maximum degree is called the *central vertex*. We denote by  $S(n_1, n_2, \dots, n_k)$  a starlike tree in which removing the central vertex leaves disjoint paths  $P_{n_1}, P_{n_2}, \dots, P_{n_k}$ . We say that  $S(n_1, n_2, \dots, n_k)$  has branches of length  $n_1, n_2, \dots, n_k$ . It is obvious that  $S(n_1, n_2, \dots, n_k)$  has  $n_1 + n_2 + \dots + n_k + 1$  vertices. For simplicity a starlike with  $\alpha_i$  branches of length  $n_i$  ( $1 \leq i \leq k$ ) is denoted by  $S(n_1^{\alpha_1}, n_2^{\alpha_2}, \dots, n_k^{\alpha_k})$ .

2. DISTANCE-BALANCED GRAPHS WITH MAXIMUM DEGREE  $n - 1$ 

In this section we prove that for any graph  $G$  with  $\Delta(G) = n - 1$ , the only distance-balanced closure of  $G$  is the complete graph  $K_n$ . The following result is very useful in this paper. It is in fact a slight modification of Corollary 2.3 of [9].

**Theorem 2.1.** *Let  $G$  be a graph with diameter at most 2 and  $H$  be a distance-balanced graph such that  $G$  is a spanning subgraph of  $H$ . Then  $H$  is a regular graph. Moreover, every regular graph with diameter at most 2 is distance-balanced.*

**Corollary 2.2.** *For every integer  $m \geq 1$ , the graph  $K_{1,m}$  has a unique distance-balanced closure which is isomorphic to  $K_{m+1}$ , hence,  $b(K_{1,m}) = \binom{m+1}{2} - m$ .*

*Proof.* Let  $G$  be a distance-balanced closure of  $K_{1,m}$ . By Theorem 2.1,  $G$  is a regular graph and since  $K_{1,m}$  has a vertex of degree  $m$ ,  $G$  should be  $m$ -regular, hence  $G \cong K_{m+1}$ .  $\square$

The following is an immediate conclusion of Theorem 2.1.

**Corollary 2.3.** *Let  $G$  be a graph with  $n$  vertices and  $\Delta(G) = n - 1$ . Then the graph  $G$  has a unique distance balanced closure. Moreover, this closure is isomorphic to  $K_n$ .*

3. DISTANCE-BALANCED GRAPHS WITH MAXIMUM DEGREE  $n - 2$ 

In this section, we prove that any distance-balanced graph  $G$  with  $\Delta(G) = n - 2$  is a regular graph using this, we construct a distance-balanced closure of  $T$  where  $T$  is a tree with this property (that is  $\Delta(T) = n - 2$ ) and then compute  $b(T)$ .

The following Lemma will be used occasionally in this paper and the proof is easily deduced from the definition of  $U_{xy}^G$ .

**Lemma 3.1.** *Let  $x$  and  $y$  be two adjacent vertices of a graph  $G$ , then  $U_{xy}^G \cap N_G(y) = \emptyset$  (or  $U_{xy}^G \subseteq V(G) \setminus N_G[y]$ ). Furthermore,  $N_G[y] \setminus U_{yx}^G \subseteq N_G[x]$ .*

**Theorem 3.2.** *Let  $T = S(2, 1^{m-1})$  be a starlike tree and  $H$  be a distance-balanced graph containing  $T$  as a spanning subgraph. Then  $\text{diam}(H) \leq 2$ , hence,  $H$  is an  $r$ -regular graph for some  $m \leq r \leq m + 1$ .*

*Proof.* Suppose that the vertices of  $T$  are labeled as shown in Figure 1. If  $oy \in E(H)$ , then  $H$  contains  $K_{1,m+1}$  as a spanning subgraph, so by Corollary 2.2,  $H \cong K_{m+2}$ .

So, we may assume that  $oy \notin E(H)$  (and consequently  $\text{diam}(H) \neq 1$ ). We prove  $\text{diam}(H) = 2$ . For this, it is enough to show that  $d(y, x_i) \leq 2$  for  $i = 2, \dots, m$ . Let  $i$  be an integer with  $2 \leq i \leq m$ ; using the fact that  $y$  is the only vertex which is not adjacent to  $o$  in  $H$  and using Lemma 3.1, we

conclude that  $U_{x_i o}^H \subseteq \{y\}$ ; If  $U_{x_i o}^H = \{y\}$ , then  $x_i y \in E(H)$  and  $d_H(x_i, y) = 1$ , hence in this case  $d_H(x_i, y) \leq 2$ . Otherwise,  $U_{x_i o}^H = \emptyset$ , in this case, for every  $z \in V(H) \setminus \{o, x_i\}$  we have  $d_H(z, x_i) = d_H(z, o)$ , particularly,  $d_H(z, x_i) = d_H(y, o) = 2$ . Hence,  $\text{diam}(H) = 2$ , as required. The result is now concluded using Theorem 2.1.

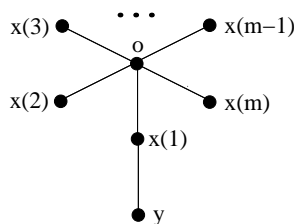


Figure 1

□

**Theorem 3.3.** Let  $T = S(2, 1^{m-1})$  be a starlike of order  $m + 2$ . Then

$$b(T) = \begin{cases} \frac{m^2}{2} - 1 & \text{if } m \text{ is even;} \\ \binom{m+1}{2} & \text{otherwise.} \end{cases}$$

*Proof.* Suppose that the vertices of  $T$  be labeled as in Figure 1, and let  $\bar{T}$  be a distance-balanced closure of  $T$ . First, suppose that  $m$  is an odd integer; Since there is no  $m$ -regular graph of order  $m + 2$ , by Theorem 3.2,  $\bar{T} \cong K_{m+2}$ .

Now, suppose that  $m$  is an even integer. Let  $H = K_{m+2}$  be a complete graph with vertex set  $V(T)$  and  $M$  be a complete matching of  $H$  which contains the edge  $oy$ . Then  $H \setminus M$ , is an  $m$ -regular graph with diameter 2 and contains  $T$  as a spanning subgraph. Hence, by Theorem 2.1 and Theorem 3.2,  $\bar{T} = H \setminus M$ , is a distance-balanced closure of  $T$  and  $b(T) = \frac{m^2}{2} - 1$ . □

**Corollary 3.4.** Let  $G$  be a connected graph of order  $n$ , with  $\Delta(G) = n - 2$  and  $H$  be a distance-balanced graph which contains  $G$  as a spanning subgraph. Then  $H$  is either an  $(n - 2)$ -regular graph or the complete graph  $K_n$ .

*Proof.* In this case  $S(2, 1^{n-3})$  is an spanning subgraph of  $G$ . So, by Theorem 3.2,  $H$  is either an  $(n - 2)$ -regular graph or the complete graph  $K_n$ . □

#### 4. DISTANCE-BALANCED GRAPHS WITH MAXIMUM DEGREE $n - 3$

In this section we will prove that every distance-balanced graph with  $\Delta(G) = n - 3$  is regular. Moreover, by constructing distance-balanced closure of trees with  $\Delta(T) = n - 3$  we compute  $b(T)$  for these trees.

**Theorem 4.1.** Let  $T = S(2^2, 1^{m-2})$  be a starlike of order  $m + 3$  and  $H$  be a distance-balanced graph which contains  $T$  as a spanning subgraph. Then  $\text{diam}(H) \leq 2$ . Moreover,  $H$  is an  $r$ -regular graph with  $r \geq m$ .

*Proof.* Suppose the vertices of  $T$  are labeled as in Figure 2. If either  $oy$  or  $oz$  be an edge of  $H$ , then by Theorem 3.2,  $H$  is an  $r$ -regular graph with  $r \geq m + 1$ , which proves this theorem. So, suppose that  $oy, oz \notin E(H)$ .

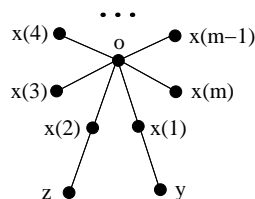


Figure 2

By Lemma 3.1, for each  $1 \leq i \leq m$ ,  $U_{x_i o}^H \subseteq \{y, z\}$ . Now, we prove that  $\deg_H(x_i) = m$ , for each  $i = 1, \dots, m$ . For this, we consider three possible cases:

- Case 1.**  $|U_{x_i o}^H| = 0$ : Then  $U_{x_i o}^H = \emptyset$  and  $U_{o x_i}^H = \emptyset$ . Hence, by Lemma 3.1,  $N_H(x_i) = \{o, x_1, x_2, \dots, x_m\} \setminus \{x_i\}$  and  $\deg_H(x_i) = m$ .
- Case 2.**  $|U_{x_i o}^H| = 1$ : Without loss of generality we can assume that  $U_{x_i o}^H = \{y\}$ . Then there is an integer  $1 \leq j \leq m$  such that  $U_{o x_i}^H = \{x_j\}$ . Since  $d_H(o, y) = 2$ ,  $yx_i \in E(H)$ . Hence, using Lemma 3.1,  $N_H(x_i) = \{o, y, x_1, x_2, \dots, x_m\} \setminus \{x_j\}$  and  $\deg_H(x_i) = m$ .
- Case 3.**  $|U_{x_i o}^H| = 2$ : We have  $U_{x_i o}^H = \{y, z\}$ . Since  $d_H(o, y) = d_H(o, z) = 2$ , we conclude that  $yx_i, zx_i \in E(H)$ . Since  $|U_{o x_i}^H| = 2$ , there are integers  $j$  and  $k$  such that  $U_{o x_i}^H = \{x_j, x_k\}$ . Hence, by Lemma 3.1, we have  $N_H(x_i) = \{o, y, z, x_1, x_2, x_3, \dots, x_m\} \setminus \{x_i, x_j, x_k\}$  and  $\deg_H(x_i) = m$ .

Next, we prove that  $\deg_H(y) \geq m - 3$  and  $\deg_H(z) \geq m - 3$ . From  $\deg_H(x_1) = m$  and Lemma 3.1, it concludes that  $|U_{y x_1}^H| \leq 2$ , hence,  $|U_{x_1 y}^H| \leq 2$ , which means that there are at most two elements in  $N_H(x_1) \setminus N_H[y]$ . Using this and Lemma 3.1, we provide  $\deg_H(y) \geq m - 3$ . With a similar argument, the inequality  $\deg_H(z) \geq m - 3$  is concluded.

Now, by using  $\deg_H(y), \deg_H(z) \geq m - 3$ ,  $\deg_H(x_i) \geq m$ , ( $i = 1, \dots, m$ ), and  $oy, oz \notin E(H)$ , hence every two nonadjacent vertices have a common neighbor, provided that  $m \geq 7$ . This means that  $\text{diam}(H) = 2$ , which proves the result in case  $m \geq 7$ , using Theorem 2.1. For the cases,  $3 \leq m \leq 6$ , through a case by case inspection (by using  $\deg_H(x_i) \geq m$ ,  $i = 1, \dots, m$ ), the same result is obtained.  $\square$

**Theorem 4.2.** For the starlike tree  $T = S(2^2, 1^{m-2})$  of order  $m + 3$ ,  $b(G) = \frac{m^2+m-4}{2}$ .

*Proof.* Let the vertices of  $T$  be labeled as in Figure 2 and  $\bar{T}$  be a distance-balanced closure of  $T$ . Now, we are going to construct  $\bar{T}$ . Let  $H = K_{m+3}$  be a complete graph with the same vertex set as  $H$ . Omit the edges of cycles  $C_1 = x_1x_2x_3 \dots x_mx_1$  and  $C_2 = oyzo$  from  $H$  to obtain  $\bar{T} = H \setminus (C_1 \cup C_2)$ . Now,  $\bar{G}$  is an  $m$ -regular graph with diameter 2, which contains  $T$  as a spanning subgraph, so by Theorem 4.1 and Theorem 2.1,  $\bar{T}$  is a distance-balanced closure of  $T$  and  $b(T) = \frac{m^2+m-4}{2}$ .  $\square$

**Theorem 4.3.** Let  $T$  be the tree of Figure 3 and  $H$  be a distance-balanced graph which contains  $T$  as a spanning subgraph. Then  $\text{diam}(H) \leq 2$ , hence,  $H$  is a regular graph. Moreover,  $b(T) = \frac{m^2+m-4}{2}$ .

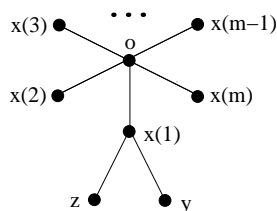
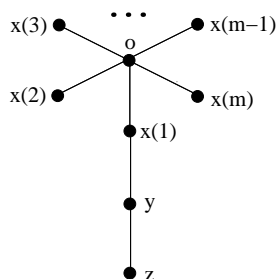


Figure 3

*Proof.* If either  $oy \in E(H)$  or  $oz \in E(H)$ , then by Theorem 3.2,  $\text{diam}(H) \leq 2$  and  $H$  is a regular graph. So, suppose that neither  $oy$  nor  $oz$  is in  $E(H)$ . Since  $|W_{x_1y}^H| = |W_{yx_1}^H|$  and  $o \in W_{x_1y}^H$ , there exists a vertex  $x_i$ ,  $i \neq 1$ , such that  $yx_i \in E(H)$ . Therefore, graph  $H$  contains graph  $S(2^2, 1^{m-2})$  as a spanning subgraph and using Theorem 4.1,  $\text{diam}(H) \leq 2$  and  $H$  is a regular graph. Furthermore, the graph introduced in the proof of Theorem 4.2, is also distance-balanced closure of  $T$ . Hence  $b(T) = \frac{m^2+m-4}{2}$ .  $\square$

**Theorem 4.4.** Consider the starlike tree  $T = S(3, 1^{m-1})$  of order  $m + 3$  and let  $H$  be a distance-balanced graph which contains  $T$  as a spanning subgraph. Then  $H$  is an  $r$ -regular graph for some  $m \leq r \leq m + 2$ .

*Proof.* Let the vertices of  $T$  be labeled as in Figure 4.



## Figure 4

If either  $oy \in E(H)$  or  $oz \in E(H)$ , then by Theorem 3.2,  $\text{diam}(H) \leq 2$  and  $H$  is a regular graph. If  $zx_1 \in E(H)$ , then  $H$  contains the graph shown in Figure 3 as a spanning subgraph, so by Theorem 4.3,  $H$  is a regular graph. So we may assume that  $\{oy, oz, x_1z\} \cap E(H) = \emptyset$ . Since  $|W_{yz}^H| = |W_{zy}^H|$  and  $x_1 \in W_{yz}^H$ , the vertex  $z$  is adjacent to at least one vertex in  $\{x_2, x_3, \dots, x_m\}$  (because otherwise according to the structure of  $T$  we have  $V \setminus \{y, z\} \subseteq U_{yz}$ ). Hence,  $H$  contains the graph  $S(2^2, 1^{m-2})$ , as a spanning subgraph. So, by Theorem 4.1,  $\text{diam}H \leq 2$  and  $H$  is a regular graph, as desired.  $\square$

**Corollary 4.5.** *Let  $G$  be a connected graph of order  $n$  with  $\Delta(G) = n-3$ . Then every distance-balanced graph  $H$  which contains  $G$  as a spanning subgraph, is regular.*

*Proof.* Since  $\Delta(G) = n-3$ ,  $G$  contains at least one of the graphs  $S(2^2, 1^{n-2})$ ,  $S(3, 1^{n-1})$  or the graph shown in Figure 3, as a spanning subgraph. Hence, the result follows from Theorem 4.2, Theorem 4.6 and Theorem 4.3.  $\square$

**Theorem 4.6.** *For the starlike tree  $G = S(3, 1^{m-1})$  of order  $m+3$ ,  $b(G) = \frac{m^2+m-4}{2}$ .*

*Proof.* Let the vertices of  $G$  be labeled as in Figure 4 and let  $\overline{G}$  be a distance-balanced closure of  $G$ . Now, we are going to construct  $\overline{G}$ . Let  $H = K_{m+3}$  be a complete graph with the same vertex set as  $G$ . Omit the edges of cycles  $C_1 = x_3x_4 \dots x_mx_3$  and  $C_2 = oxy_2x_1zo$  from  $H$  to obtain  $\overline{G} = H \setminus (C_1 \cup C_2)$ . Then the graph  $\overline{G}$  is an  $n$ -regular graph with diameter 2, which contains  $G$  as a spanning subgraph. So by Theorem 4.4,  $\overline{G}$  is a distance-balanced closure of  $G$  and  $b(G) = \frac{m^2+m-4}{2}$ .  $\square$

**Conclusion.** In previous sections, we have proved that any connected distance-balanced graph  $G$  with  $\Delta(G) \geq |V(G)| - 3$ , is a regular graph, moreover, distanced-closure of such a graph  $G$  is a smallest regular graph which contains  $G$ . This helped us to find a distance-balanced closure of trees  $T$  with  $\Delta(T) \geq |V(T)| - 3$  and to compute  $b(T)$  for such trees.

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