Distance-Balanced Closure of Some Graphs

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Abstract. In this paper we prove that any distance-balanced graph $G$ with $\Delta(G) \geq |V(G)| - 3$ is regular. Also we define notion of distance-balanced closure of a graph and we find distance-balanced closures of trees $T$ with $\Delta(T) \geq |V(T)| - 3$.

Keywords: Distances in graphs, Distance-balanced graphs, Distance-balanced closure.


1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote $|V(G)|$ by $n$. The set of neighbors of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ is denoted by $\deg_G(v)$ and minimum degree and maximum degree of $G$ denoted by $\delta(G)$ and $\Delta(G)$.
respectively. The distance \( d_G(u, v) \) between vertices \( u \) and \( v \) is the length of a shortest path between \( u \) and \( v \) in \( G \). The diameter \( \text{diam}(G) \) of graph \( G \) is defined as \( \max\{d_G(u, v) : u, v \in V(G)\} \). The notion of distance is studied in several works in graph theory (See [2] and the references therein) and many research works are based on the concepts related to this notion (See for instance [8] and [10]).

For an edge \( xy \) of a graph \( G \), \( W^G_{xy} \) is the set of vertices which are closer to \( x \) than \( y \), more formally

\[
W^G_{xy} = \{ u \in V(G) | d_G(u, x) < d_G(u, y) \}.
\]

Moreover, \( zW^G_y \) is the set of vertices of \( G \) that have equal distances to \( x \) and \( y \), that is

\[
zW^G_y = \{ u \in V(G) | d_G(u, x) = d_G(u, y) \}.
\]

These sets play important roles in metric graph theory, see for instance [1, 3, 4, 5]. Since \( x \) always belongs to \( W^G_{xy} \), for convenience we let \( U^G_{xy} = W^G_{xy} \setminus \{x\} \).

Distance-balanced graphs are introduced in [9] as graphs for which \( T \) all trees \( \Delta(T) \geq |V(T)| - 3 \). In Section 2, we compute that distance-balanced closure of graphs \( G \) with \( \Delta(G) = n - 1 \). In Section 3, and Section 4, we concern graphs \( G \) with \( \Delta(G) = n - 2 \) and \( \Delta(G) = n - 3 \), respectively. Then we compute \( b(G) \) for all trees \( T \) with \( \Delta(T) = n - 2 \) and \( \Delta(T) = n - 3 \).

Here we mention some more definitions and notations about trees. Let \( P_n \) denoted the path with \( n \) vertices. A tree which has exactly one vertex of degree greater than two is said to be starlike. The vertex of maximum degree is called the central vertex. We denote by \( S(n_1, n_2, \ldots, n_k) \) a starlike tree in which removing the central vertex leaves disjoint paths \( P_{n_1}, P_{n_2}, \ldots, P_{n_k} \). We say that \( S(n_1, n_2, \ldots, n_k) \) has branches of length \( n_1, n_2, \ldots, n_k \). It is obvious that \( S(n_1, n_2, \ldots, n_k) \) has \( n_1 + n_2 + \ldots + n_k + 1 \) vertices. For simplicity a starlike with \( \alpha_i \) branches of length \( n_i \) (\( 1 \leq i \leq k \)) is denoted by \( S(n_1^{\alpha_1}, n_2^{\alpha_2}, \ldots, n_k^{\alpha_k}) \).
2. DISTANCE-BALANCED GRAPHS WITH MAXIMUM DEGREE $n - 1$

In this section we prove that for any graph $G$ with $\Delta(G) = n - 1$, the only distance-balanced closure of $G$ is the complete graph $K_n$. The following result is very useful in this paper. It is in fact a slight modification of Corollary 2.3 of [9].

**Theorem 2.1.** Let $G$ be a graph with diameter at most 2 and $H$ be a distance-balanced graph such that $G$ is a spanning subgraph of $H$. Then $H$ is a regular graph. Moreover, every regular graph with diameter at most 2 is distance-balanced.

**Corollary 2.2.** For every integer $m \geq 1$, the graph $K_{1,m}$ has a unique distance-balanced closure which is isomorphic to $K_{m+1}$, hence, $b(K_{1,m}) = \frac{(m+1)!}{2} - m$.

**Proof.** Let $G$ be a distance-balanced closure of $K_{1,m}$. By Theorem 2.1, $G$ is a regular graph and since $K_{1,m}$ has a vertex of degree $m$, $G$ should be $m$-regular, hence $G \cong K_{m+1}$. \qed

The following is an immediate conclusion of Theorem 2.1.

**Corollary 2.3.** Let $G$ be a graph with $n$ vertices and $\Delta(G) = n - 1$. Then the graph $G$ has a unique distance balanced closure. Moreover, this closure is isomorphic to $K_n$.

3. DISTANCE-BALANCED GRAPHS WITH MAXIMUM DEGREE $n - 2$

In this section, we prove that any distance-balanced graph $G$ with $\Delta(G) = n - 2$ is a regular graph using this, we construct a distance-balanced closure of $T$ where $T$ is a tree with this property (that is $\Delta(T) = n - 2$) and then compute $b(T)$.

The following Lemma will be used occasionally in this paper and the proof is easily deduced from the definition of $U_{xy}^G$.

**Lemma 3.1.** Let $x$ and $y$ be two adjacent vertices of a graph $G$, then $U_{xy}^G \cap N_G(y) = \emptyset$ (or $U_{xy}^G \subseteq V(G) \setminus N_G(y)$). Furthermore, $N_G[y] \setminus U_{xy}^G \subseteq N_G[x]$.

**Theorem 3.2.** Let $T = S(2,1^{m-1})$ be a starlike tree and $H$ be a distance-balanced graph containing $T$ as a spanning subgraph. Then $\text{diam}(H) \leq 2$, hence, $H$ is an $r$-regular graph for some $m \leq r \leq m+1$.

**Proof.** Suppose that the vertices of $T$ are labeled as shown in Figure 1. If $oy \in E(H)$, then $H$ contains $K_{1,m+1}$ as a spanning subgraph, so by Corollary 2.2, $H \cong K_{m+2}$.

So, we may assume that $oy \notin E(H)$ (and consequently $\text{diam}(H) \neq 1$). We prove $\text{diam}(H) = 2$. For this, it is enough to show that $d(y,xi) \leq 2$ for $i = 2, \ldots, m$. Let $i$ be an integer with $2 \leq i \leq m$; using the fact that $y$ is the only vertex which is not adjacent to $o$ in $H$ and using Lemma 3.1, we
conclude that $U_{x,o}^H \subseteq \{y\}$; If $U_{x,o}^H = \{y\}$, then $x_iy \in E(H)$ and $d_H(x_i, y) = 1$, hence in this case $d_H(x_i, y) \leq 2$. Otherwise, $U_{x,o}^H = \emptyset$, in this case, for every $z \in V(H) \setminus \{o, x_i\}$ we have $d_H(z, x_i) = d_H(z, o)$, particularly, $d_H(z, x_i) = d_H(y, o) = 2$. Hence, $diam(H) = 2$, as required. The result is now concluded using Theorem 2.1.

![Figure 1](image_url)

**Theorem 3.3.** Let $T = S(2, 1^{m-1})$ be a starlike of order $m+2$. Then

$$b(T) = \begin{cases} \frac{m^2}{m+1} - 1 & \text{if } m \text{ is even; } \\ \frac{m^2}{m+1} & \text{otherwise.} \end{cases}$$

**Proof.** Suppose that the vertices of $T$ be labeled as in Figure 1, and let $\overline{T}$ be a distance-balanced closure of $T$. First, suppose that $m$ is an odd integer; Since there is no $m$-regular graph of order $m+2$, by Theorem 3.2, $\overline{T} \cong K_{m+2}$.

Now, suppose that $m$ is an even integer. Let $H = K_{m+2}$ be a complete graph with vertex set $V(T)$ and $M$ be a complete matching of $H$ which contains the edge $oy$. Then $H \setminus M$, is an $m$-regular graph with diameter 2 and contains $T$ as a spanning subgraph. Hence, by Theorem 2.1 and Theorem 3.2, $\overline{T} = H \setminus M$, is a distance-balanced closure of $T$ and $b(T) = \frac{m^2}{m+1} - 1$. □

**Corollary 3.4.** Let $G$ be a connected graph of order $n$, with $\Delta(G) = n-2$ and $H$ be a distance-balanced graph which contains $G$ as a spanning subgraph. Then $H$ is either an $(n-2)$-regular graph or the complete graph $K_n$.

**Proof.** In this case $S(2, 1^{n-3})$ is a spanning subgraph of $G$. So, by Theorem 3.2, $H$ is either an $(n-2)$-regular graph or the complete graph $K_n$. □

4. Distance-Balanced Graphs with Maximum Degree $n-3$

In this section we will prove that every distance-balanced graph with $\Delta(G) = n-3$ is regular. Moreover, by constructing distance-balanced closure of trees with $\Delta(T) = n-3$ we compute $b(T)$ for these trees.
Theorem 4.1. Let $T = S(2^2, 1^{m-2})$ be a starlike of order $m + 3$ and $H$ be a distance-balanced graph which contains $T$ as a spanning subgraph. Then $\text{diam}(H) \leq 2$. Moreover, $H$ is an $r$-regular graph with $r \geq m$.

Proof. Suppose the vertices of $T$ are labeled as in Figure 2. If either $oy$ or $oz$ be an edge of $H$, then by Theorem 3.2, $H$ is an $r$-regular graph with $r \geq m + 1$, which proves this theorem. So, suppose that $oy, oz \notin E(H)$.

![Figure 2](attachment:image.png)

By Lemma 3.1, for each $1 \leq i \leq m$, $U_{x_i}^H \subseteq \{y, z\}$. Now, we prove that $\text{deg}_H(x_i) = m$, for each $i = 1, \ldots, m$. For this, we consider three possible cases:

**Case 1.** $|U_{x_1}^H| = 0$: Then $U_{x_1}^H = \emptyset$ and $U_{ox_2}^H = \emptyset$. Hence, by Lemma 3.1, $N_H(x_1) = \{o, x_1, x_2, \ldots, x_m\} \setminus \{x_i\}$ and $\text{deg}_H(x_i) = m$.

**Case 2.** $|U_{x_2}^H| = 1$: Without loss of generality we can assume that $U_{x_2}^H = \{y\}$. Then there is an integer $1 \leq j \leq m$ such that $U_{ox_2}^H = \{x_j\}$. Since $d_H(o, y) = 2$, $yx_i \in E(H)$. Hence, using Lemma 3.1, $N_H(x_1) = \{o, y, x_1, x_2, \ldots, x_m\} \setminus \{x_j\}$ and $\text{deg}_H(x_i) = m$.

**Case 3.** $|U_{x_3}^H| = 2$: We have $U_{x_3}^H = \{y, z\}$. Since $d_H(o, y) = d_H(o, z) = 2$, we conclude that $yx_i, xz_i \in E(H)$. Since $|U_{ox_3}^H| = 2$, there are integers $j$ and $k$ such that $U_{ox_3}^H = \{x_j, x_k\}$. Hence, by Lemma 3.1, we have $N_H(x_i) = \{o, y, z, x_1, x_2, x_3, \ldots, x_m\} \setminus \{x_j, x_k\}$ and $\text{deg}_H(x_i) = m$.

Next, we prove that $\text{deg}_H(y) \geq m - 3$ and $\text{deg}_H(z) \geq m - 3$. From $\text{deg}_H(x_1) = m$ and Lemma 3.1, it concludes that $|U_{y_1}^H| \leq 2$, hence, $|U_{yx_1}^H| \leq 2$, which means that there are at most two elements in $N_H(x_1) \setminus N_H[y]$. Using this and Lemma 3.1, we provide $\text{deg}_H(y) \geq m - 3$. With a similar argument, the inequality $\text{deg}_H(z) \geq m - 3$ is concluded.

Now, by using $\text{deg}_H(y), \text{deg}_H(z) \geq m - 3, \text{deg}_H(x_i) \geq m, (i = 1, \ldots, m)$, and $oy, oz \notin E(H)$, hence every two nonadjacent vertices have a common neighbor, provided that $m \geq 7$. This means that $\text{diam}(H) = 2$, which proves the result in case $m \geq 7$, using Theorem 2.1. For the cases, $3 \leq m \leq 6$, through a case by case inspection (by using $\text{deg}_H(x_i) \geq m, i = 1, \ldots, m$) the same result is obtained. □
Theorem 4.2. For the starlike tree \( T = S(2^2, 1^{m-2}) \) of order \( m + 3 \), \( b(G) = \frac{m^2 + m - 4}{2} \).

Proof. Let the vertices of \( T \) be labeled as in Figure 2 and \( \overline{T} \) be a distance-balanced closure of \( T \). Now, we are going to construct \( \overline{T} \). Let \( H = K_{m+3} \) be a complete graph with the same vertex set as \( H \). Omit the edges of cycles \( C_1 = x_1x_2x_3 \ldots x_mx_1 \) and \( C_2 = oyzo \) from \( H \) to obtain \( \overline{T} = H \setminus (C_1 \cup C_2) \). Now, \( \overline{G} \) is an \( m \)-regular graph with diameter 2, which contains \( T \) as a spanning subgraph, so by Theorem 4.1 and Theorem 2.1, \( \overline{T} \) is a distance-balanced closure of \( T \) and \( b(T) = \frac{m^2 + m - 4}{2} \). \( \square \)

Theorem 4.3. Let \( T \) be the tree of Figure 3 and \( H \) be a distance-balanced graph which contains \( T \) as a spanning subgraph. Then \( \text{diam}(H) \leq 2 \), hence, \( H \) is a regular graph. Moreover, \( b(T) = \frac{m^2 + m - 4}{2} \).

Proof. If either \( oy \in E(H) \) or \( oz \in E(H) \), then by Theorem 3.2, \( \text{diam}(H) \leq 2 \) and \( H \) is a regular graph. So, suppose that neither \( oy \) nor \( oz \) is in \( E(H) \).

Since \( |W^H_{x,y}| = |W^H_{y,z}| \) and \( o \in W^H_{z,y} \), there exists a vertex \( x_i, i \neq 1 \), such that \( yx_i \in E(H) \). Therefore, graph \( \overline{H} \) contains graph \( S(2^2, 1^{m-2}) \) as a spanning subgraph and using Theorem 4.1, \( \text{diam}(H) \leq 2 \) and \( H \) is a regular graph. Furthermore, the graph introduced in the proof of Theorem 4.2, is also distance-balanced closure of \( T \). Hence \( b(T) = \frac{m^2 + m - 4}{2} \). \( \square \)

Theorem 4.4. Consider the starlike tree \( T = S(3, 1^{m-1}) \) of order \( m + 3 \) and let \( H \) be a distance-balanced graph which contains \( T \) as a spanning subgraph. Then \( H \) is an \( r \)-regular graph for some \( m \leq r \leq m + 2 \).

Proof. Let the vertices of \( T \) be labeled as in Figure 4.
If either \( oy \in E(H) \) or \( oz \in E(H) \), then by Theorem 3.2, \( \text{diam}(H) \leq 2 \) and \( H \) is a regular graph. If \( zx_1 \in E(H) \), then \( H \) contains the graph shown in Figure 3 as a spanning subgraph, so by Theorem 4.3, \( H \) is a regular graph.

So we may assume that \( \{ oy, oz, x_1 z \} \cap E(H) = \emptyset \). Since \( |W_H yz| = |W_H zy| \) and \( x_1 \in W_H yz \), the vertex \( z \) is adjacent to at least one vertex in \( \{ x_2, x_3, \ldots, x_m \} \) (because otherwise according to the structure of \( T \) we have \( V \setminus \{ y, z \} \subseteq U_{yz} \)).

Hence, \( H \) contains the graph \( S(2^2, 1^{m-2}) \), as a spanning subgraph. So, by Theorem 4.1, \( \text{diam} H \leq 2 \) and \( H \) is a regular graph, as desired. □

**Corollary 4.5.** Let \( G \) be a connected graph of order \( n \) with \( \Delta(G) = n - 3 \). Then every distance-balanced graph \( H \) which contains \( G \) as a spanning subgraph, is regular.

**Proof.** Since \( \Delta(G) = n - 3 \), \( G \) contains at least one of the graphs \( S(2^2, 1^{n-2}) \), \( S(3, 1^{n-1}) \) or the graph shown in Figure 3, as a spanning subgraph. Hence, the result follows from Theorem 4.2, Theorem 4.6 and Theorem 4.3. □

**Theorem 4.6.** For the starlike tree \( G = S(3, 1^{m-1}) \) of order \( m + 3 \), \( b(G) = \frac{m^2 + m - 4}{2} \).

**Proof.** Let the vertices of \( G \) be labeled as in Figure 4 and let \( \overline{G} \) be a distance-balanced closure of \( G \). Now, we are going to construct \( \overline{G} \). Let \( H = K_{m+3} \) be a complete graph with the same vertex set as \( G \). Omit the edges of cycles \( C_1 = x_0 x_1 \ldots x_m x_0 \) and \( C_2 = oyx_2x_1zo \) from \( H \) to obtain \( \overline{G} = H \setminus (C_1 \cup C_2) \).

Then the graph \( \overline{G} \) is an \( n \)-regular graph with diameter 2, which contains \( G \) as a spanning subgraph. So by Theorem 4.4, \( \overline{G} \) is a distance-balanced closure of \( G \) and \( b(G) = \frac{m^2 + m - 4}{2} \). □

**Conclusion.** In previous sections, we have proved that any connected distance-balanced graph \( G \) with \( \Delta(G) \geq |V(G)| - 3 \), is a regular graph, moreover, distance-closure of such a graph \( G \) is a smallest regular graph which contains \( G \). This helped us to find a distance-balanced closure of trees \( T \) with \( \Delta(T) \geq |V(T)| - 3 \) and to compute \( b(T) \) for such trees.

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