Distance-Balanced Closure of Some Graphs

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Abstract. In this paper we prove that any distance-balanced graph \( G \) with \( \Delta(G) \geq |V(G)| - 3 \) is regular. Also we define notion of distance-balanced closure of a graph and we find distance-balanced closures of trees \( T \) with \( \Delta(T) \geq |V(T)| - 3 \).

Keywords: Distances in graphs, Distance-balanced graphs, Distance-balanced closure.


1. Introduction

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). We denote \( |V(G)| \) by \( n \). The set of neighbors of a vertex \( v \in V(G) \) is denoted by \( N_G(v) \), and \( N^G_G[v] = N_G(v) \cup \{v\} \). The degree of a vertex \( v \) is denoted by \( \deg_G(v) \) and minimum degree and maximum degree of \( G \) denoted by \( \delta(G) \) and \( \Delta(G) \).

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respectively. The distance $d_G(u, v)$ between vertices $u$ and $v$ is the length of a shortest path between $u$ and $v$ in $G$. The diameter $\text{diam}(G)$ of graph $G$ is defined as $\max\{d_G(u, v) : u, v \in V(G)\}$. The notion of distance is studied in several works in graph theory (See [2] and the references therein) and many research works are based on the concepts related to this notion (See for instance [8] and [10]).

For an edge $xy$ of a graph $G$, $W^G_{xy}$ is the set of vertices which are closer to $x$ than $y$, more formally

$$W^G_{xy} = \{u \in V(G) | d_G(u, x) < d_G(u, y)\}.$$ 

Moreover, $xW^G_y$ is the set of vertices of $G$ that have equal distances to $x$ and $y$, that is

$$xW^G_y = \{u \in V(G) | d_G(u, x) = d_G(u, y)\}.$$ 

These sets play important roles in metric graph theory, see for instance [1, 3, 4, 5]. Since $x$ always belongs to $W^G_{xy}$, for convenience we let $U^G_{xy} = W^G_{xy} \setminus \{x\}$. Distance-balanced graphs are introduced in [9] as graphs for which $|W^G_{xy}| = |W^G_{yx}|$ (or equivalently $|U^G_{xy}| = |U^G_{yx}|$) for every pair of adjacent vertices $x, y \in V(G)$.

In [9], the parameter $b(G)$ of a graph $G$ is introduced as the smallest number of the edges which can be added to $G$ such that the obtained graph is distance-balanced. Since the complete graph is distance-balanced, this parameter is well-defined. We call graph $G$ a distance-balanced closure of $H$ if $G$ is distance-balanced and $H$ is a spanning subgraph of $G$ with $|E(G)| = b(H) + |E(H)|$; in other words, a distance-balanced closure of $H$ is a distance-balanced graph $G$ which contains $H$ as a spanning subgraph and has minimum number of edges. As mentioned in [9], the computation of $b(G)$ is quite hard in general but it might be interesting in some special cases. In this paper we compute $b(G)$ for all trees $T$ with $\Delta(T) \geq |V(T)| - 3$. In Section 2, we compute that distance-balanced closure of graphs $G$ with $\Delta(G) = n - 1$. In Section 3, and Section 4, we concern graphs $G$ with $\Delta(G) = n - 2$ and $\Delta(G) = n - 3$, respectively. Then we compute $b(T)$ for all trees $T$ with $\Delta(T) = n - 2$ and $\Delta(T) = n - 3$.

Here we mention some more definitions and notations about trees. Let $P_n$ denote the path with $n$ vertices. A tree which has exactly one vertex of degree greater than two is said to be starlike. The vertex of maximum degree is called the central vertex. We denote by $S(n_1, n_2, \ldots, n_k)$ a starlike tree in which removing the central vertex leaves disjoint paths $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$. We say that $S(n_1, n_2, \ldots, n_k)$ has branches of length $n_1, n_2, \ldots, n_k$. It is obvious that $S(n_1, n_2, \ldots, n_k)$ has $n_1 + n_2 + \ldots + n_k + 1$ vertices. For simplicity a starlike with $\alpha_i$ branches of length $n_i$ ($1 \leq i \leq k$) is denoted by $S(n_1^{\alpha_1}, n_2^{\alpha_2}, \ldots, n_k^{\alpha_k})$. 


2. Distance-Balanced Graphs with Maximum Degree $n - 1$

In this section we prove that for any graph $G$ with $\Delta(G) = n - 1$, the only distance-balanced closure of $G$ is the complete graph $K_n$. The following result is very useful in this paper. It is in fact a slight modification of Corollary 2.3 of [9].

**Theorem 2.1.** Let $G$ be a graph with diameter at most 2 and $H$ be a distance-balanced graph such that $G$ is a spanning subgraph of $H$. Then $H$ is a regular graph. Moreover, every regular graph with diameter at most 2 is distance-balanced.

**Corollary 2.2.** For every integer $m \geq 1$, the graph $K_{1,m}$ has a unique distance-balanced closure which is isomorphic to $K_{m+1}$, hence, $b(K_{1,m}) = \left(\frac{m+1}{2}\right) - m$.

**Proof.** Let $G$ be a distance-balanced closure of $K_{1,m}$. By Theorem 2.1, $G$ is a regular graph and since $K_{1,m}$ has a vertex of degree $m$, $G$ should be $m$-regular, hence $G \cong K_{m+1}$. □

The following is an immediate conclusion of Theorem 2.1.

**Corollary 2.3.** Let $G$ be a graph with $n$ vertices and $\Delta(G) = n - 1$. Then the graph $G$ has a unique distance balanced closure. Moreover, this closure is isomorphic to $K_n$.

3. Distance-Balanced Graphs with Maximum Degree $n - 2$

In this section, we prove that any distance-balanced graph $G$ with $\Delta(G) = n - 2$ is a regular graph using this, we construct a distance-balanced closure of $T$ where $T$ is a tree with this property (that is $\Delta(T) = n - 2$) and then compute $b(T)$.

The following Lemma will be used occasionally in this paper and the proof is easily deduced from the definition of $U_{xy}^G$.

**Lemma 3.1.** Let $x$ and $y$ be two adjacent vertices of a graph $G$, then $U_{xy}^G \cap N_G(y) = \emptyset$ (or $U_{xy}^G \subseteq V(G) \setminus N_G[y]$). Furthermore, $N_G[y] \setminus U_{yx}^G \subseteq N_G[x]$.

**Theorem 3.2.** Let $T = S(2, 1^{m-1})$ be a starlike tree and $H$ be a distance-balanced graph containing $T$ as a spanning subgraph. Then $\text{diam}(H) \leq 2$, hence, $H$ is an $r$-regular graph for some $m \leq r \leq m + 1$.

**Proof.** Suppose that the vertices of $T$ are labeled as shown in Figure 1. If $oy \in E(H)$, then $H$ contains $K_{1,m+1}$ as a spanning subgraph, so by Corollary 2.2, $H \cong K_{m+2}$.

So, we may assume that $oy \notin E(H)$ (and consequently $\text{diam}(H) \neq 1$). We prove $\text{diam}(H) = 2$. For this, it is enough to show that $d(y, xi) \leq 2$ for $i = 2, \cdots, m$. Let $i$ be an integer with $2 \leq i \leq m$; using the fact that $y$ is the only vertex which is not adjacent to $o$ in $H$ and using Lemma 3.1, we
conclude that $U^H_{x_i,o} \subseteq \{y\}$; If $U^H_{x_i,o} = \{y\}$, then $x_iy \in E(H)$ and $d_H(x_i, y) = 1$, hence in this case $d_H(x_i, y) \leq 2$. Otherwise, $U^H_{x_i,o} = \emptyset$, in this case, for every $z \in V(H) \setminus \{o, x_i\}$ we have $d_H(z, x_i) = d_H(z, o)$, particularly, $d_H(z, x_i) = d_H(y, o) = 2$. Hence, $\text{diam}(H) = 2$, as required. The result is now concluded using Theorem 2.1.

\[ \text{Figure 1} \]

**Theorem 3.3.** Let $T = S(2, 1^{m-1})$ be a starlike of order $m + 2$. Then

\[ b(T) = \begin{cases} \frac{m^2}{2} - 1 & \text{if } m \text{ is even;} \\ \frac{m+1}{2} & \text{otherwise.} \end{cases} \]

*Proof.* Suppose that the vertices of $T$ be labeled as in Figure 1, and let $\overline{T}$ be a distance-balanced closure of $T$. First, suppose that $m$ is an odd integer; Since there is no $m$-regular graph of order $m + 2$, by Theorem 3.2, $\overline{T} \cong K_{m+2}$.

Now, suppose that $m$ is an even integer. Let $H = K_{m+2}$ be a complete graph with vertex set $V(T)$ and $M$ be a complete matching of $H$ which contains the edge $oy$. Then $H \setminus M$, is an $m$-regular graph with diameter 2 and contains $T$ as a spanning subgraph. Hence, by Theorem 2.1 and Theorem 3.2, $\overline{T} = H \setminus M$, is a distance-balanced closure of $T$ and $b(T) = \frac{m^2}{2} - 1$. \[ \square \]

**Corollary 3.4.** Let $G$ be a connected graph of order $n$, with $\Delta(G) = n - 2$ and $H$ be a distance-balanced graph which contains $G$ as a spanning subgraph. Then $H$ is either an $(n - 2)$-regular graph or the complete graph $K_n$.

*Proof.* In this case $S(2, 1^{n-3})$ is an spanning subgraph of $G$. So, by Theorem 3.2, $H$ is either an $(n - 2)$-regular graph or the complete graph $K_n$. \[ \square \]

4. Distance-Balanced Graphs with Maximum Degree $n - 3$

In this section we will prove that every distance-balanced graph with $\Delta(G) = n - 3$ is regular. Moreover, by constructing distance-balanced closure of trees with $\Delta(T) = n - 3$ we compute $b(T)$ for these trees.
Theorem 4.1. Let $T = S(2^2, 1^{m-2})$ be a starlike of order $m + 3$ and $H$ be a distance-balanced graph which contains $T$ as a spanning subgraph. Then $\text{diam}(H) \leq 2$. Moreover, $H$ is an $r$-regular graph with $r \geq m$.

Proof. Suppose the vertices of $T$ are labeled as in Figure 2. If either $oy$ or $oz$ be an edge of $H$, then by Theorem 3.2, $H$ is an $r$-regular graph with $r \geq m + 1$, which proves this theorem. So, suppose that $oy, oz \notin E(H)$.

By Lemma 3.1, for each $1 \leq i \leq m$, $U^H_{x(i)} \subseteq \{y, z\}$. Now, we prove that $\deg_H(x_i) = m$, for each $i = 1, \cdots, m$. For this, we consider three possible cases:

Case 1. $|U^H_{x(i)}| = 0$: Then $U^H_{x(i)} = \emptyset$ and $U^H_{ox(i)} = \emptyset$. Hence, by Lemma 3.1, $N_H(x_i) = \{o, x_1, x_2, \ldots, x_m\} \setminus \{x_i\}$ and $\deg_H(x_i) = m$.

Case 2. $|U^H_{x(i)}| = 1$: Without loss of generality we can assume that $U^H_{x(i)} = \{y\}$. Then there is an integer $1 \leq j \leq m$ such that $U^H_{ox(i)} \subseteq \{x_j\}$. Since $d_H(o, y) = 2$, $yx_i \in E(H)$. Hence, using Lemma 3.1, $N_H(x_i) = \{o, y, x_1, x_2, \ldots, x_m\} \setminus \{x_j\}$ and $\deg_H(x_i) = m$.

Case 3. $|U^H_{x(i)}| = 2$: We have $U^H_{x(i)} = \{y, z\}$. Since $d_H(o, y) = d_H(o, z) = 2$, we conclude that $yx_i, xz_i \in E(H)$. Since $|U^H_{ox(i)}| = 2$, there are integers $j$ and $k$ such that $U^H_{ox(i)} = \{x_j, x_k\}$. Hence, by Lemma 3.1, we have $N_H(x_i) = \{o, y, z, x_1, x_2, \ldots, x_m\} \setminus \{x_j, x_k\}$ and $\deg_H(x_i) = m$.

Next, we prove that $\deg_H(y) \geq m - 3$ and $\deg_H(z) \geq m - 3$. From $\deg_H(x_1) = m$ and Lemma 3.1, it concludes that $|U^H_{yx_2}| \leq 2$, hence, $|U^H_{yx_1}| \leq 2$, which means that there are at most two elements in $N_H(x_1) \setminus N_H[y]$. Using this and Lemma 3.1, we provide $\deg_H(y) \geq m - 3$. With a similar argument, the inequality $\deg_H(z) \geq m - 3$ is concluded.

Now, by using $\deg_H(y), \deg_H(z) \geq m - 3$, $\deg_H(x_i) \geq m$, $(i = 1, \cdots, m)$, and $oy, oz \notin E(H)$, hence every two nonadjacent vertices have a common neighbor, provided that $m \geq 7$. This means that $\text{diam}(H) = 2$, which proves the result in case $m \geq 7$, using Theorem 2.1. For the cases, $3 \leq m \leq 6$, through a case by case inspection (by using $\deg_H(x_i) \geq m$, $i = 1, \cdots, m$) the same result is obtained. \hfill $\Box$
Theorem 4.2. For the starlike tree $T = S(2^2, 1^{m-2})$ of order $m + 3$, $b(G) = \frac{m^2 + m - 4}{2}$.

Proof. Let the vertices of $T$ be labeled as in Figure 2 and $\overline{T}$ be a distance-balanced closure of $T$. Now, we are going to construct $\overline{T}$. Let $H = K_{m+3}$ be a complete graph with the same vertex set as $H$. Omit the edges of cycles $C_1 = x_1x_2x_3\ldots x_mx_1$ and $C_2 = oyso$ from $H$ to obtain $\overline{T} = H \setminus (C_1 \cup C_2)$. Now, $\overline{T}$ is an $m$-regular graph with diameter 2, which contains $T$ as a spanning subgraph, so by Theorem 4.1 and Theorem 2.1, $\overline{T}$ is a distance-balanced closure of $T$ and $b(T) = \frac{m^2 + m - 4}{2}$. □

Theorem 4.3. Let $T$ be the tree of Figure 3 and $H$ be a distance-balanced graph which contains $T$ as a spanning subgraph. Then $\text{diam}(H) \leq 2$, hence, $H$ is a regular graph. Moreover, $b(T) = \frac{m^2 + m - 4}{2}$.

Proof. If either $oy \in E(H)$ or $oz \in E(H)$, then by Theorem 3.2, $\text{diam}(H) \leq 2$ and $H$ is a regular graph. So, suppose that neither $oy$ nor $oz$ is in $E(H)$. Since $|W_{x_{1},y}| = |W_{y_{x_{1}}}|$ and $o \in W_{x_{1},y}$, there exists a vertex $x_{i}$, $i \neq 1$, such that $yx_{i} \in E(H)$. Therefore, graph $\tilde{H}$ contains graph $S(2^2, 1^{m-2})$ as a spanning subgraph and using Theorem 4.1, $\text{diam}(H) \leq 2$ and $H$ is a regular graph. Furthermore, the graph introduced in the proof of Theorem 4.2, is also distance-balanced closure of $T$. Hence $b(T) = \frac{m^2 + m - 4}{2}$. □

Theorem 4.4. Consider the starlike tree $T = S(3, 1^{m-1})$ of order $m + 3$ and let $H$ be a distance-balanced graph which contains $T$ as a spanning subgraph. Then $H$ is an $r$-regular graph for some $m \leq r \leq m + 2$.

Proof. Let the vertices of $T$ be labeled as in Figure 4.
If either $oy \in E(H)$ or $oz \in E(H)$, then by Theorem 3.2, $\text{diam}(H) \leq 2$ and $H$ is a regular graph. If $xz_1 \in E(H)$, then $H$ contains the graph shown in Figure 3 as a spanning subgraph, so by Theorem 4.3, $H$ is a regular graph. Hence, $H$ contains the graph $S(2^2, 1^{m-2})$, as a spanning subgraph. So, by Theorem 4.1, $\text{diam} H \leq 2$ and $H$ is a regular graph, as desired. □

**Corollary 4.5.** Let $G$ be a connected graph of order $n$ with $\Delta(G) = n-3$. Then every distance-balanced graph $H$ which contains $G$ as a spanning subgraph, is regular.

**Proof.** Since $\Delta(G) = n-3$, $G$ contains at least one of the graphs $S(2^2, 1^{n-2})$, $S(3, 1^{n-1})$ or the graph shown in Figure 3, as a spanning subgraph. Hence, the result follows from Theorem 4.2, Theorem 4.6 and Theorem 4.3. □

**Theorem 4.6.** For the starlike tree $G = S(3, 1^{m-1})$ of order $m+3$, $b(G) = \frac{m^2 + m - 4}{2}$.

**Proof.** Let the vertices of $G$ be labeled as in Figure 4 and let $\overline{G}$ be a distance-balanced closure of $G$. Now, we are going to construct $\overline{G}$. Let $H = K_{m+3}$ be a complete graph with the same vertex set as $G$. Omit the edges of cycles $C_1 = x_1 x_3 \ldots x_m x_3$ and $C_2 = oyx_2x_3z_0$ from $H$ to obtain $\overline{G} = H \setminus (C_1 \cup C_2)$. Then the graph $\overline{G}$ is an $n$-regular graph with diameter 2, which contains $G$ as a spanning subgraph. So by Theorem 4.4, $\overline{G}$ is a distance-balanced closure of $G$ and $b(G) = \frac{m^2 + m - 4}{2}$. □

**Conclusion.** In previous sections, we have proved that any connected distance-balanced graph $G$ with $\Delta(G) \geq |V(G)| - 3$, is a regular graph, moreover, distance-closure of such a graph $G$ is a smallest regular graph which contains $G$. This helped us to find a distance-balanced closure of trees $T$ with $\Delta(T) \geq |V(T)| - 3$ and to compute $b(T)$ for such trees.

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**References**

2. F. Buckley, F. Harary, Distance in graphs, Addison-Wesley, 1990.