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On the Wiener Index of Some Edge Deleted Graphs

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Abstract. The sum of distances between all the pairs of vertices in a connected graph is known as the *Wiener index* of the graph. In this paper, we obtain the Wiener index of edge complements of stars, complete subgraphs and cycles in K_n.

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1. Introduction

Let $G = (V, E)$ be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. Given two distinct vertices u, v of G, let $d(u, v)$ denote the *distance between u and v*, is the number of edges on a shortest path between

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u and v. The *Wiener index, W(G)* is a well known distance based topological index introduced as a structural discriptor for acyclic organic molecules [5]. In 1947 Harold Wiener defined $W(G)$ as the sum of the distances between all the pairs of vertices of G[4]. That is,

$$W(G) = \sum_{u<v} d(u, v),$$

equivalently, $W(G)$ of a graph G is defined as the half of the sum of the distances between every pair of vertices of G. That is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u, v), \tag{1.1}$$

where the summation extends over all possible pairs of distinct vertices u and v in $V(G)$.

For more details on $W(G)$, see [1, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14] and the references cited therein.

Since calculation of $W(G)$ of a graph G can be computationally expensive, in this paper we provide the formulae to find $W(G)$ of some class of graphs, which are obtained by deleting the edges of a complete graph K_n.

Two subgraphs G_1 and G_2 of a graph G are said to be *independent*, if $V_1 \bigcap V_2$ is an empty set. where $V(G)$ is the vertex set of G.

Let G be a graph and H be a subgraph of G. The *edge complement of H in G* is the subgraph of G obtained by deleting all the edges of H from G.

For terminology not given here, we follow [7].

2. Existing Results

Here we present some existing results, which are the motivation for the main reults of this paper.

Theorem 2.1. [2] *Let G be a connected graph with n vertices having a clique K_r of order r. Let $G(n, r)$ be the graph obtained from G by removing the edges of K_r, $0 \leq r \leq n-1$. Then*

$$W(G(n, r)) \geq \frac{1}{2}[n(n-1) + r(r-1)] \tag{2.1}$$

The equality holds if and only if $G \cong K_n$, a complete graph on n vertices.

Theorem 2.2. [2] *Let $(K_r)_i, i = 1, 2, \cdots, k$ be the independent complete subgraphs on r vertices of K_n. Let $G(n, r, k)$ be the graph obtained from complete graph K_n by removing the edges of $(K_r)_i, i = 1, 2, \cdots, k, 1 \leq k \leq \lfloor \frac{n}{r} \rfloor$ and $0 \leq r \leq n - 1$, then*

$$W(G(n, r, k)) = \frac{1}{2}[n(n-1) + kr(r-1)] \tag{2.2}$$

Theorem 2.3. [2] *Let $e_i, i = 1, 2, \cdots, k, 0 \leq k \leq n - 2$ be the edges of a complete graph K_n incident to a vertex v of K_n. Let $K_n(k)$ be the graph obtained from complete graph K_n by removing the edges $e_i, i = 1, 2, \cdots, k$, then*

$$W(K_n(k)) = \binom{n}{2} + k \tag{2.3}$$

In the sequel of this paper, we generalize the above results and in continuation, we extend the result for the edge complements of cycles in K_n.

3. Main Results

Theorem 3.1. *Let $(K_{1,m-1})_i, i = 1, 2, \ldots, k$ be the k independent star subgraphs of order m of a complete graph K_n. $G(n, m, k)$ be the graph obtained by deleting the edges of these k independent star subgraphs $(K_{1,m-1})_i, i = 1, 2, \ldots, k$ from K_n. Then for $mk \leq n$,*

$$W(G(n, m, k)) = \frac{1}{2}n(n-1) + k(m-1) \tag{3.1}$$

Proof. Let K_n be the complete graph on n vertices and let $K_{1,m-1}$ be the star subgraph on m vertices of K_n. Let $v_1, v_2, \ldots, v_m$ be the vertices of first copy of $K_{1,m-1}$, $v_m + 1, v_m + 2, \ldots, v_{2m}$ be the vertices of second copy of $K_{1,m-1}$, and so on. Let $v_{(mk-1)} + 1, v_{(mk-1)} + 2, \ldots, v_{mk}$ are the vertices of k^{th} copy of $K_{1,m-1}$. And the remaining vertices of K_n are $v_{mk} + 1, v_{mk} + 2, \ldots, v_n$.

If the edges of any star subgraph $K_{1,m-1}$ are deleted from K_n, then in the resulting subgraph,
there is 1 vertex, which is at a distance 2 with $m - 1$ vertices and at a distance 1 with $n - m$ vertices.
There are $m - 1$ vertices, which are at a distance 2 with 1 vertex and at a distance 1 with $n - 2$ vertices.
And each of the remaining $n - m$ vertices are at a distance 1 with $n - 1$ vertices.

Since in $G(n, m, k)$, the edges of k copies of independent star subgraphs $(K_{1,m-1})_i, i = 1, 2, \ldots, k$ are removed from K_n, hence in $G(n, m, k)$, there are, k vertices, $v_{im} + 1, i = 0, 1, 2, \ldots, k - 1$ are at a distance 2 with $m - 1$ vertices and at a distance 1 with $n - m$ vertices.

Each of the vertices $v_{im}+2$ to $v_{im}+m, i=0,1,2,\ldots,k-1$, are at a distance 2 with 1 vertex and at a distance 1 with $n-2$ vertices.
And each of the remaining $n-mk$ vertices are at a distance 1 with each of the $n-1$ vertices.

Therefore, using the formula (1.1), we have,

$$\begin{aligned} W(G(n,m,k)) &= \frac{1}{2}[k\big((m-1)(2)+(n-m)(1)+(m-1)(2)+(m-1)(n-2)\big) \\ &\quad +(n-mk)(n-1)] \\ &= \frac{1}{2}[n^2-n+2mk-2k] \\ &= \frac{1}{2}n(n-1)+k(m-1) \end{aligned}$$

□

Remark 3.2. For $k=1$ eqn.(3.1) reduces to eqn.(2.3).

Theorem 3.3. *Let $K_{1,m_1-1},K_{1,m_2-1},\ldots,K_{1,m_k-1}$ be the k independent star subgraphs of order $m_1,m_2,\ldots,m_k$ respectively, of a complete graph K_n. $G(n,m_1,m_2,\ldots,m_k)$ be the graph obtained from K_n by deleting the edges of atmost one copy of each of the independent stars $K_{1,m_1-1},K_{1,m_2-1},\ldots,K_{1,m_k-1}$. Then for $\sum_{i=1}^{k}m_i\le n$,*

$$W(G(n,m_1,m_2,\ldots,m_k))=\frac{1}{2}n(n-1)+\sum_{i=1}^{k}m_i-k \tag{3.2}$$

Proof. Let $v_1,v_2,\ldots,v_{m_1}$ be the vertices of K_{1,m_1-1}, $v_{m_1}+1,v_{m_1}+2,\ldots,v_{m_2}$ be the vertices of K_{1,m_2-1}, and so on $v_{(m_{k-1})}+1,v_{(m_{k-1})}+2,\ldots,v_{m_k}$ be the vertices of $K_{1,m_{k-1}}$. The remaining vertices of K_n are $v_{m_k}+1,v_{m_k}+2,\ldots,v_n$. Let $G(n,m_1,m_2,\ldots,m_k)$ be the graph obtained by deleting the edges of independent star subgraphs $K_{1,m_i-1}, i=1,2,\ldots,k$ from K_n.
In $G(n,m_1,m_2,\ldots,m_k)$,

Corresponding to the vertices of K_n from which the edges of the star K_{1,m_1-1} are deleted:
Vertex v_1 is at a distance 2 with m_1-1 vertices and at a distance 1 with $n-m_1$ vertices. v_2 to v_{m_1} are at a distance 2 with 1 vertex and at a distance 1 with $n-2$ vertices.

Corresponding to the vertices of K_n from which the edges of the star K_{1,m_2-1} are deleted:
Vertex v_{m_1+1} is at a distance 2 with m_2-1 vertices and at a distance 1 with $n-m_2$ vertices. $v_{m_1}+2$ to v_{m_2} are at a distance 2 with 1 vertex and at a

distance 1 with $n-2$ vertices. And so on.

Corresponding to the vertices of K_n from which the edges of the star K_{1,m_k-1} are deleted:
Vertex $v_{m_{k-1}+1}$ is at a distance 2 with m_k-1 vertices and at a distance 1 with $n-m_k$ vertices. $v_{m_{k-1}+2}$ to v_{m_k} are at a distance 2 with 1 vertex and at a distance 1 with $n-2$ vertices.

And each of the remaining vertices $n-\sum_{i=1}^{k} m_i$ to v_n are at a distance 1 with $n-1$ vertices.

Therefore, using the formula (1.1), we have,

$$\begin{aligned}
W\big[G(n, m_1, m_2, \ldots m_k)\big] &= \frac{1}{2}[(m_1-1)(2)+(n-m_1)(1) \\
&\quad + (m_1-1)[(1)(2)+(n-2)(1)] \\
&\quad + (m_2-1)(2)+(n-m_2)(1) \\
&\quad + (m_2-1)[(1)(2)+(n-2)(1)] \\
&\quad + \cdots + (m_k-1)(2)+(n-m_k)(1) \\
&\quad + (m_k-1)[(1)(2)+(n-2)(1)] \\
&\quad + (n-\sum_{i=1}^{k} m_i)(n-1)] \\
&= \frac{1}{2}[(m_1 n+m_1-2)+(m_2 n+m_2-2) \\
&\quad + \cdots + (m_k n+m_k-2)+(n-1)(n-\sum_{i=1}^{k} m_i)] \\
&= \frac{1}{2}[2\sum_{i=1}^{k} m_i - 2k + n(n-1)] \\
&= \frac{1}{2}n(n-1)+\sum_{i=1}^{k} m_i - k
\end{aligned}$$

□

Remark 3.4. For $m_1=m_2=\cdots=m_k=m$, eqn.(3.2) reduces to eqn.(3.1).

Corollary 3.5. *Let $G(n,(m_1,l_1),(m_2,l_2),\ldots,(m_k,l_k))$ be the graph obtained from K_n by deleting the edges of independent stars K_{1,m_1-1} (l_1 copies) ,K_{1,m_2-1} (l_2 copies), …, K_{1,m_k-1} (l_k copies). Then for $\sum_{i=1}^{k} m_i l_i \le n$,*

$$W\big[G(n,(m_1,l_1),(m_2,l_2),\ldots,(m_k,l_k)\big] = \frac{1}{2}n(n-1)+\sum_{i=1}^{k} m_i l_i - \sum_{i=1}^{k} l_i \quad (3.3)$$

Proof. If the edges of l_1 copies of the star K_{1,m_1-1}, l_2 copies of the star K_{1,m_2-1},..., l_k copies of the star K_{1,m_k-1}, are deleted from K_n, then in $G(n,(m_1,l_1),(m_2,l_2),\dots,(m_k,l_k))$,

for each of the l_1 copies of the edge deleted stars K_{1,m_1-1}, there is 1 vertex which is, at a distance 2 with m_1-1 vertices and at a distance 1 with $n-m_1$ vertices. And there are m_1-1 vertices which are, at a distance 2 with 1 vertex and at a distance 1 with $n-2$ vertices.

For each of the l_2 copies of the edge deleted stars K_{1,m_2-1}, there is 1 vertex (say v_{m_1+1}), is at a distance 2 with m_2-1 vertices and at a distance 1 with $n-m_2$ vertices. And there are m_2-1 vertices (say v_{m_1+2} to v_{m_2}) are at a distance 2 with 1 vertex and at a distance 1 with $n-2$ vertices.
and so on.

For each of the l_k copies of the edge deleted stars K_{1,m_k-1}, there is 1 vertex (say $v_{m_{k-1}+1}$) is at a distance 2 with m_k-1 vertices and at a distance 1 with $n-m_k$ vertices, and there are $v_{m_k}-1$ vertices are at a distance 2 with 1 vertex and at a distance 1 with $n-2$ vertices.

And each of the remaining vertices $n-\sum_{i=1}^{k} m_i$ to v_n are at a distance 1 with $n-1$ vertices.

Therefore, using the formula (1.1), we have,

$$\begin{aligned}
W\big[G(n,m_1,m_2,\dots m_k)\big] &= \frac{1}{2}\{l_1[(m_1-1)(2)+(n-m_1)(1)\\
&\quad+(m_1-1)\big((1)(2)+(n-2)(1)\big)]\\
&\quad+l_2\big[(m_2-1)(2)+(n-m_2)(1)\\
&\quad+(m_2-1)[(1)(2)+(n-2)(1)]\big]\\
&\quad+\dots+l_k\big[(m_k-1)(2)+(n-m_k)(1)\\
&\quad+(m_k-1)[(1)(2)+(n-2)(1)]\big]\\
&\quad+(n-\sum_{i=1}^{k} m_i)(n-1)\}\\
&= \frac{1}{2}\big[l_1(m_1n+m_1-2)+l_2(m_2n+m_2-2)+\dots\\
&\quad+l_k(m_kn+m_k-2)+(n-1)(n-\sum_{i=1}^{k} m_il_i)\big]\\
&= \frac{1}{2}n(n-1)+\sum_{i=1}^{k} m_il_i-\sum_{i=1}^{k} l_i
\end{aligned}$$

□

Theorem 3.6. *Let $K_{r_1}, K_{r_2}, \ldots, K_{r_k}$ be the complete subgraphs of order $r_1, r_2, \ldots, r_k$ respectively, of a complete graph K_n. $G(n, r_1, r_2, \ldots, r_k)$ be the graph obtained from K_n by deleting the edges of atmost one copy of each of the independent complete subgraphs $K_{r_1}, K_{r_2}, \ldots, K_{r_k}$. Then for $\sum_{i=1}^{k} r_i \le n$,*

$$W(G(n, r_1, r_2, \ldots, r_k)) = \frac{1}{2}\big[n(n-1) + \sum_{i=1}^{k} r_i^2 - \sum_{i=1}^{k} r_i\big] \tag{3.4}$$

Proof. Let $v_1, v_2, \ldots, v_{m_1}$ be the vertices of K_{r_1}, $v_{m_1}+1, v_{m_1}+2, \ldots, v_{m_2}$ be the vertices of K_{r_2}, and so on. $v_{(m_{k-1})}+1, v_{(m_{k-1})}+2, \ldots, v_{m_k}$ be the vertices of K_{r_k}. The remaining vertices of K_n are $v_{m_k}+1, v_{m_k}+2, \ldots, v_n$. Let $G(n, r_1, r_2, \ldots, r_k)$ be the graph obtained by deleting the edges of independent complete subgraphs $K_{r_1}, K_{r_2}, \ldots, K_{r_k}$ from K_n.

In $G(n, r_1, r_2, \ldots, r_k)$,

There are r_1 vertices each of which are at a distance 2 with each of the $r_1 - 1$ vertices and at a distance 1 with $n - r_1$ vertices.

There are r_2 vertices each of which are at a distance 2 with each of the $r_2 - 1$ vertices and at a distance 1 with $n - r_2$ vertices, and so on.

There are r_k vertices each of which are at a distance 2 with each of the $r_k - 1$ vertices and at a distance 1 with $n - r_k$ vertices.

And the remaining $n - \sum_{i=1}^{k} r_i$ vertices are at a distance 1 with each of the $n - 1$ vertices.

Therefore, using the formula (1.1), we have,

$$\begin{aligned}
W(G(n, r_1, r_2, \ldots r_k)) &= \frac{1}{2}\big[r_1[(r_1-1)(2) + (n-r_1)(1)] \\
&\quad + r_2[(r_2-1)(2) + (n-r_2)(1)] \\
&\quad + \cdots + r_k[(r_k-1)(2) + (n-r_k)(1)] \\
&\quad + (n - \sum_{i=1}^{k} r_i)(n-1)(1)\big] \\
&= \frac{1}{2}\big[n(n-1) + \sum_{i=1}^{k} r_i^2 - \sum_{i=1}^{k} r_i\big]
\end{aligned}$$

□

Remark 3.7. For $r_1 = r_2 = \ldots, r_k = r$, eqn.(3.4) reduces to eqn. (2.2).

Corollary 3.8. *Let $G(n, (r_1, l_1), (r_2, l_2), \ldots, (r_k, l_k))$ be the graph obtained from K_n by deleting the edges of independent complete subgraphs K_{r_1} (l_1 copies),K_{r_2} (l_2 copies), …, K_{r_k} (l_k copies). Then for $\sum_{i=1}^{k} m_i l_i \le n$,*

$$W\big[G(n,(r_1,l_1),(r_2,l_2),\dots,(r_k,l_k)\big] = \frac{1}{2}\big[n(n-1)+\sum_{i=1}^{k} l_i r_i^2 - \sum_{i=1}^{k} l_i r_i\big] \quad (3.5)$$

Proof. If the edges of independent complete subgraphs K_{r_1} (l_1 copies),K_{r_2} (l_2 copies), $\dots$, K_{r_k} (l_k copies) are deleted from K_n.

Then in $G(n,(r_1,l_1),(r_2,l_2),\dots,(r_k,l_k))$,

for each of the l_1 copies of the edge deleted complete subgraphs K_{r_1}, there are r_1 vertices each of which are at a distance 2 with each of the r_1-1 vertices and at a distance 1 with $n-r_1$ vertices.

For each of the l_2 copies of the edge deleted complete subgraphs K_{r_2}, there are r_2 vertices each of which are at a distance 2 with each of the r_2-1 vertices and at a distance 1 with $n-r_2$ vertices, and so on.

For each of the l_k copies of the edge deleted complete subgraphs K_{r_k}, there are r_k vertices each of which are at a distance 2 with each of the r_k-1 vertices and at a distance 1 with $n-r_k$ vertices.

And the remaining $n-\sum_{i=1}^{k} r_i$ vertices are at a distance 1 with each of the $n-1$ vertices.

Therefore, using the formula (1.1), we have,

$$\begin{aligned}
W\big[G(n,(r_1,l_1),(r_2,l_2),\dots,(r_k,l_k)\big] &= \frac{1}{2}\big\{l_1\big(r_1[(r_1-1)(2)+(n-r_1)(1)]\big)\\
&\quad + l_2\{r_2[(r_2-1)(2)+(n-r_2)(1)]\}\\
&\quad + \dots\dots\dots\dots\dots\dots\dots\dots + \\
&\quad + l_k\{r_k[(r_k-1)(2)+(n-r_k)(1)]\}\\
&\quad + (n-\sum_{i=1}^{k} r_i)(n-1)(1)\big\}\\
&= \frac{1}{2}\big[l_1 r_1(n+r_1-2)+l_2 r_2(n+r_2-2)\\
&\quad + \cdots + l_k r_k(n+r_k-2)\\
&\quad + (n-1)(n-\sum_{i=1}^{k} l_i r_i)\big]\\
&= \frac{1}{2}\{n(n-1)+\sum_{i=1}^{k} l_i r_i^2 - \sum_{i=1}^{k} l_i r_i\}
\end{aligned}$$

□

Theorem 3.9. *Let K_n be a complete graph of order n. Let C_p be a cycle subgraph on p vertices of K_n. $G(n,p)$ be the graph obtained from K_n by deleting the edges of C_p. Then for $p \leq n$,*

$$W(G(n,p)) = \frac{1}{2}n(n-1) + p \tag{3.6}$$

Proof. In $G(n,p)$, each of the vertex of C_p in $G(n,p)$ is at a distance 2 with 2 vertices and at a distance 1 with $n-3$ vertices. And each of the remaining $n-p$ vertex is at a distance 1 with $n-1$ vertices.
Therefore, using the formula 1.1, we have,

$$\begin{aligned} W(G(n,p)) &= \frac{1}{2}\big[p\big((2)(2) + (n-3)(1)\big)\big) + (n-p)(n-1)\big] \\ &= \frac{1}{2}n(n-1) + p \end{aligned}$$

□

4. Discussion

By means of above results, we have generalized some of the results obtained in [2], for the Wiener indices.The results obtained in [2] can be obtained by our results as special cases.

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References

1. I. Gutman, Y. Yeh, S. Lee, Y. Luo, Some recent results in the theory of Wiener number, *Indian J. Chem.*, **32**A, (1993), 651 – 661.
2. H. S. Ramane, D. S. Revankar, A. B. Ganagi, On the Wiener index of a graph, *J. Indones. Math. Soc.*, **18**(1), (2012), 57 – 66.
3. F. Buckley, F. Harary, *Distance in Graph*, Addison - Wesley, Redwood, 1990.
4. S. Nikolic, N. Trinajstic, Z. Mihalic, The Wiener index: development and applications, *Croat. Chem. Acta*, **68**, (1995), 105–129.
5. H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.*, **69**, (1947), 17–20.
6. A. A Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.*, **66**, (2001), 211 –249.
7. J. A. Bondy, U. S. R. Murthy, *Graph Theory with Applications*, Elsevier, North-Holland, 1976.
8. S. Klavzar, I. Gutman, Wiener number of vertex - weighted graphs and a chemical application, *Discrete Appl. Math.*, **80**, (1997), 73 –81.

9. B. Mohr, T. Pisansky, How to compute the Weiner index of a graph, *Journal of Mathematical Chemistry*, **2**, (1988), 267 – 277.
10. A. Graovac, T. Pisansky, On the Wiener index of a graph, *J. Math. chem.*, **8**, (1991), 53–62.
11. I. Gutman, Y. N. Yeh, S. L. Lee, Some recent results in the theory of Wiener number, *Indian J. chem.,* **32**A, (1993), 651 – 661.
12. K. Xu, M. Liu, K. C. Das, I. Gutman, B. Fortula, A survey on graphs extremal with respect to distance - based topological indices, *Match Commun. Math. Comput. Chem.*, **71**, (2014), 461 –508.
13. G. H. Fath-Tabar, A. R. Ashrafi, The hyper weiner polynomial of graphs, *Iranian Journal of Mathematical Sciences and Informaatics*, **6**, (2011), 67–74.
14. T. Doslic, A. Graovac, F. Cataldo, O. Ori, Note on some distance - based invarients for 2 - dimensional square and comb lattices, *Iranian Journal of Mathematical Sciences and Informaatics*, **5**, (2010), 61–68.