Iranian Journal of Mathematical Sciences and Informatics
Vol. 12, No. 2 (2017), pp 127-139
DOI: 10.7508/ijmsi.2017.2.009

Order Almost Dunford-Pettis Operators on Banach Lattices

Halimeh Ardakani*, S. M. Sadegh Modarres Mosadegh

Department of Mathematics, Yazd University, Yazd, Iran.

E-mail: halimeh_ardakani@yahoo.com
E-mail: smodarres@yazd.ac.ir

Abstract. By introducing the concepts of order almost Dunford-Pettis and almost weakly limited operators in Banach lattices, we give some properties of them related to some well known classes of operators, such as, order weakly compact, order Dunford-Pettis, weak and almost Dunford-Pettis and weakly limited operators. Then, we characterize Banach lattices E and F on which each operator from E into F that is order almost Dunford-Pettis and weak almost Dunford-Pettis is an almost weakly limited operator.

Keywords: Order Dunford-Pettis operator, Weakly limited operator, Almost Dunford-Pettis set.

2000 Mathematics subject classification: 46B40, 46B42, 47B60.

1. Introduction

A subset A of a Banach space X is called limited (resp. Dunford–Pettis), if every weak* null (resp. weak null) sequence (x_n^*) in X^* converges uniformly on A, that is

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

Also, a subset A of a Banach lattice E is called almost limited (resp. almost Dunford–Pettis) [9, 8] if every disjoint weak* null (resp. disjoint weakly null)

*Corresponding Author

Received 18 March 2015; Accepted 28 December 2015

sequence (x_n^*) in E^* converges uniformly on A.
We know that every limited (resp. almost limited) subset of a Banach lattice E is Dunford–Pettis (resp. almost Dunford–Pettis), but the converse of these assertions, in general, are false. The reader can be find some useful and additional properties of limited (resp. almost limited) and Dunford–Pettis (resp. almost Dunford–Pettis) sets in [2, 6, 8, 9].
Based on the concept of Dunford–Pettis (resp. limited) sets, the class of order Dunford–Pettis (resp. order limited) operators is defined in [3, 7, 12]. In fact, an operator T from a Banach lattice E into a Banach space X is said to be order Dunford–Pettis (resp. order limited) if it carries each order bounded subset of E into a Dunford–Pettis (resp. limited) set of X, i.e., if for each $x \in E^+$, the subset $T([-x, x])$ is Dunford–Pettis (resp. limited) in X.
An operator $T : X \to E$ is called limited (resp. almost limited), whenever $T(B_X)$ is a limited (resp. almost limited) set. An operator $T : X \to Y$ is called weakly limited whenever $T(B_X)$ is a Dunford–Pettis set in Y [14].
An operator T from a Banach lattice E into a Banach space Y is Dunford–Pettis (resp. almost Dunford–Pettis), if it carries weakly null (resp. disjoint weakly null) sequences in E to norm null ones [1, 17]. It is clear that T is weakly limited if and only if T^* is Dunford–Pettis.
The aim of this paper is to introduce new classes of operators that we call order almost Dunford–Pettis and almost weakly limited operators and give some interesting applications. Also we will give some equivalent conditions for $T(A)$ to be an almost Dunford–Pettis set, where A is an almost Dunford–Pettis (solid) subset of a Banach lattice E and T is an operator from E to X.
It is evident that if E is a Banach lattice, then its dual E^*, endowed with the dual norm and pointwise order, is also a Banach lattice. The norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized net (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice is said to be σ–Dedekind complete if for its countable subset that is bounded above has a supremum. A subset A of E is called solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$ and the solid hull of A is the smallest solid set including A and is exactly the set $Sol(A) = \{y \in E : |y| \leq |x|, \text{ for some } x \in A\}$.
Throughout this article, X and Y denote the arbitrary Banach spaces and X^* refers to the dual of the Banach space X. Also E and F denote arbitrary Banach lattices and $E^+ = \{x \in E : x \geq 0\}$ refers to the positive cone of the Banach lattice E and B_E is the closed unit ball of E. If a, b belong to a Banach lattice E and $a \leq b$, the interval $[a, b]$ is the set of all $x \in E$ such that $a \leq x \leq b$. A subset of a Banach lattice is called order bounded if it is contained in an order interval. The lattice operations in E are weakly sequentially continuous, if for every weakly null sequence (x_n) in E, $|x_n| \to 0$ for $\sigma(E, E^*)$. The lattice

operations in E^* are weak sequentially continuous, if for every weak* null sequence (f_n) in E^*, $|f_n| \to 0$ for $\sigma(E^*, E)$.
We refer the reader to [1, 15] for unexplained terminologies on Banach lattice theory and positive operators.

2. Order Almost Dunford-Pettis and Almost Weakly Limited Operators

In this section we will define new classes of operators so called order almost Dunford–Pettis and almost weakly limited operators and establish some additional properties of them related to some operators.

Definition 2.1. An operator T from E into F is said to be order almost Dunford–Pettis if it carries each order bounded subset of E into an almost Dunford–Pettis set in F.

Note that there exist operators which are order almost Dunford–Pettis, but fail to be order Dunford–Pettis. Indeed, $Id_{\ell_\infty} : \ell_\infty \to \ell_\infty$ is order almost Dunford–Pettis, but it is not order Dunford–Pettis (because $[-e, e] = B_{\ell_\infty}$ is almost Dunford–Pettis, but it is not Dunford–Pettis). It is clear that each order Dunford–Pettis operator is order almost Dunford–Pettis.
As in [1], an operator T from X into Y is said to be weak Dunford–Pettis, if it carries relatively weakly compact sets in X to Dunford–Pettis ones.
By a simple proof we can investigate that each weakly limited operator $T : E \to X$ is order Dunford–Pettis and weak Dunford–Pettis.

Definition 2.2. An operator T from X into E is said to be almost weakly limited whenever $T(B_X)$ is an almost Dunford–Pettis set in E.

It is clear that T is almost weakly limited if and only if T^* is almost Dunford–Pettis.
An operator $T : X \to E$ is called weak and almost Dunford–Pettis if T carries each relatively weakly compact set in X to an almost Dunford–Pettis set in E, equivalently, for every weakly null sequence $(x_n) \subset X$, and every disjoint weak null sequence $(f_n) \subset E^*$ we have $f_n(Tx_n) \to 0$ [5].
By a simple proof we can investigate that each almost weakly limited operator is order almost Dunford–Pettis and weak and almost Dunford–Pettis.

Example 2.3. Every weakly limited operator is almost weakly limited, but the converse is false, in general. In fact, since the closed unit ball $L^1[0, 1]$ is almost Dunford–Pettis, but it is not Dunford–Pettis, $Id_{L^1[0,1]} : L^1[0, 1] \to L^1[0, 1]$ is almost weakly limited, but it is not weakly limited.
Also, the identity operator $Id_{\ell_\infty} : \ell_\infty \to \ell_\infty$ is almost weakly limited, but it is not weakly limited.

A Banach lattice E has the Schur (resp. positive Schur) property, if every weakly null (resp. weakly null with positive terms) sequence in E is a norm null [1, 18].

Proposition 2.4. *Every operator T from a Banach space X into a Banach lattice E such that E^* has the Schur (resp. positive Schur) property is weakly limited (resp. almost weakly limited).*

Proof. We note that dual Banach lattice E^* has the Schur (resp. positive Schur) property if and only if the closed unit ball B_E is Dunford–Pettis (resp. almost Dunford–Pettis) set. □

Example 2.5. In every Banach lattice we have the following assertions:

(a) Each limited operator is weakly limited, but the converse is false, in general. Indeed $Id_{c_0} : c_0 \to c_0$ is weakly limited and it is not limited.
(b) Each limited operator is almost limited, but the converse is false, in general. Indeed $Id_{\ell_\infty} : \ell_\infty \to \ell_\infty$ is almost limited, but it is not limited.
(c) Each almost limited operator is almost weakly limited, but the converse is false, in general. Indeed $Id_c : c \to c$ is almost weakly limited, but it is not almost limited.

By [15, Definition 3.6.1], a subset A in a Banach lattice E is L-weakly compact, if every disjoint sequence in $Sol(A)$ is norm null. Every L-weakly compact set is relatively weakly compact and the converse holds for Banach lattices with the positive Schur property [18]. An operator T from a Banach space X into a Banach lattice E is L-weakly compact, if $T(B_X)$ is an L-weakly compact set in E.

Theorem 2.6. *Every L-weakly compact operator is almost weakly limited, but the converse is false, in general.*

Proof. By [8, Proposition 2.8], every L-weakly compact set in a Banach lattice is an almost Dunford–Pettis set. So every every L-weakly compact operator is almost weakly limited.
Also by [9, Theorem 4.2], every L-weakly compact operator on a Banach lattice is an almost limited operator and so is an almost weakly limited operator.
The converse is false. In fact, $Id_{c_0} : c_0 \to c_0$ is almost weakly limited operator and it is not L-weakly compact, because the closed unit ball c_0 is an almost Dunford–Pettis set, but it is not an L-weakly compact set. □

Remember that a Banach lattice E is an AL-space if $x \wedge y = 0$ in E implies $\|x + y\| = \|x\| + \|y\|$.

Theorem 2.7. [8] *Let E be an AL-space. Then for a norm bounded subset A of E, the following statements are equivalent.*

(a) *A is L-weakly compact.*

(b) *A is relatively weakly compact.*
(c) *A is Dunford–Pettis.*
(d) *A is almost Dunford–Pettis.*

An operator T from E into X is said to be an order weakly compact operator, if it carries order intervals in E to relatively weakly compact sets in X. By [13], an operator T from E into F is said to be order almost limited if it carries each order bounded subset of E into an almost limited set of F.

Theorem 2.8. *Every order weakly compact operator from a Banach lattice E into an AL-space E is an order almost limited and order Dunford–Pettis operator.*

Proof. For each $x \in E^+$, $T[-x, x]$ is relatively weakly compact and so it is L-weakly compact, by Theorem 2.7. Now, by [9] it is an almost limited set; that is, T is order almost limited.
Also by Theorem 2.7, every relatively weakly compact set in an AL-space is Dunford–Pettis and so each order weakly compact operator from a Banach lattice E into an AL-space E is an order Dunford–Pettis operator. □

By [11] an operator T from X into E is said to be weak and almost limited operator, if it carries relatively weakly compact sets in X to almost limited ones.

Theorem 2.9. *Every L-weakly compact operator from a Banach space X to a Banach lattice E is a weak and almost limited operator.*

Proof. If T is an L-weakly compact operator then $T(B_X)$ is an L-weakly compact set in E and by [9], it is an almost limited set. Since each relatively weakly compact set is bounded, so T carries each relatively weakly compact sets in X to almost limited ones. □

Theorem 2.10. *Every L-weakly compact operator T from a Banach lattice E into an AL-space is an order Dunford–Pettis operator.*

Proof. If T is an L-weakly compact operator from a Banach lattice E into an AL-space, then $T(B_E)$ is an L-weakly compact set and so by Theorem 2.7, it is a Dunford–Pettis set; that is, T is an order Dunford–Pettis operator. □

There is an order Dunford–Pettis operator which is not L-weakly compact. In fact, $Id_{c_0} : c_0 \to c_0$ is an order Dunford–Pettis operator and it is not L-weakly compact, because the closed unit ball c_0 is a Dunford–Pettis set, but it is not an L-weakly compact set.
Recall that an element x belonging to a Riesz space E is discrete, if $x > 0$ and $|y| \leq x$ implies $y = tx$ for some real number t. If every order interval $[0, y]$ in E contains a discrete element, then E is said to be a discrete Riesz space. An

operator T from E into X is said to be an AM-compact operator, if it carries order intervals in E to relatively compact sets in X [15].
It is clear that every AM-compact operator is an order Dunford–Pettis and so order almost Dunford–Pettis operator, but the converse is false, in general. We need the following lemma.

Lemma 2.11. [16] *Let T be an operator from a Banach space into a discrete Banach lattice with order continuous norm. Then the following statements are equivalent.*

(a) *T is almost limited.*
(b) *T is L-weakly compact.*
(c) *T is limited.*
(d) *T is compact.*

Theorem 2.12. *Every order almost Dunford–Pettis operator T from a Banach lattice E into a discrete AL-space F is AM-compact and this condition on F can not be removed.*

Proof. For each $x \in E^+$, $T[-x, x]$ is almost Dunford–Pettis set and by Theorem 2.7 it is L-weakly compact and also by Theorem 2.11 it is relatively compact. Hence T is an AM-compact operator.
There is an order almost Dunford–Pettis operator which is not AM-compact. Indeed $Id_{\ell_\infty} : \ell_\infty \to \ell_\infty$ is an order almost Dunford–Pettis operator, but it is not AM-compact. Every order Dunford–Pettis operator is not AM-compact, in general. Indeed $Id_c : c \to c$ is an order Dunford–Pettis operator, but it is not AM-compact. □

Corollary 2.13. *We have the following statements.*

(a) *Every almost limited operator from a Banach space X into a discrete Banach lattice E with order continuous norm is limited.*
(b) *Every almost weakly limited operator from a Banach space X into an AL-space E is weakly limited.*
(c) *Every almost weakly limited operator from a Banach space X into an AL-space E is almost limited.*
(d) *Every weakly limited operator from a Banach space X into a Grothendieck space is limited.*

Theorem 2.14. *Every order bounded operator on a Banach lattice is an order almost Dunford–Pettis operator.*

Proof. We note that every order bounded operator maps order intervals into order intervals. By [8, Corollary 2.2], each order interval of a Banach lattice is an almost Dunford–Pettis set. □

Remember that a Banach lattice E is an $AM-$ space if $x \wedge y = 0$ in E implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$.

Corollary 2.15. *If E is a AM-space with unit, then the identity operator Id_E is almost weakly limited.*

Proof. Since in an AM-space E with unit, the closed unit ball B_E is an order bounded set and so it is almost Dunford–Pettis, then the identity operator Id_E is almost weakly limited. □

Theorem 2.16. *Every order almost limited operator from E to F is an order almost Dunford–Pettis operator, and the converse holds, if F is an AL-space.*

Proof. Since each almost limited set in a Banach lattice is an almost Dunford–Pettis set, so each order almost limited operator is order almost Dunford–Pettis, but the converse is false, in general. Indeed, $Id_c : c \to c$ is order almost Dunford–Pettis, but it is not order almost limited (because $[-1, 1] = B_c$ is almost Dunford–Pettis, but it is not almost limited).
If T is an order almost Dunford–Pettis operator from E to an AL-space, then for each $x \in E^+$, $T[-x, x]$ is almost Dunford–Pettis and by Theorem 2.7, it is L-weakly compact and so almost limited. Hence T is order almost limited. □

Theorem 2.17. *For an operator T from a Banach lattice into an AL-space, the following are equivalent:*

(a) *T is order almost Dunford–Pettis,*
(b) *T is order Dunford–Pettis,*
(c) *T is order weakly compact,*
(d) *T is order almost limited.*

Proof. It follows from Theorem 2.7. □

We recall that a Banach lattice E has the weak Dunford–Pettis property if every weakly compact operator on E is an almost Dunford–Pettis operator. A Banach lattice E has the weak Dunford–Pettis property if and only if every relatively weakly compact set in E is an almost Dunford–Pettis set [8, 19].

Theorem 2.18. *Every order weakly compact operator T from a Banach lattice into a Banach lattice F with the weak Dunford–Pettis property is an order almost DP operator, and this condition on F is essential.*

Proof. Since every relatively weakly compact set in a Banach lattice F with the weak Dunford–Pettis is an almost Dunford–Pettis set, so every order weakly compact operator T from a Banach lattice into a Banach lattice F with the weak Dunford–Pettis property is order almost Dunford–Pettis operator, but an order weakly compact operator is not necessarily order almost Dunford–Pettis. In fact, the closed unit ball B_{ℓ_2} of the Banach lattice ℓ_2 is a relatively weakly compact set in ℓ_2, but it is not almost Dunford–Pettis; that is, there exist a relatively weakly compact set in ℓ_2 which is not almost Dunford–Pettis. Since $L^1[0, 1]$ has order continuous norm by [1], every order interval in $L^1[0, 1]$ is relatively weakly compact. So each operator $T : L^1[0,1] \to \ell_2$ is order weakly compact, but it is not order almost Dunford–Pettis (and so it is not order almost limited). □

Every order almost Dunford–Pettis operator is not an order weakly compact operator, in general. Indeed $Id_{\ell_\infty} : \ell_\infty \to \ell_\infty$ is an order almost Dunford–Pettis operator, but it is not order weakly compact.

Theorem 2.19. *Every operator T from a Banach lattice E with order continuous norm into a Banach lattice with the weak Dunford–Pettis property is an order almost Dunford–Pettis operator.*

Proof. Since E has order continuous norm by [1], for each $x \in E^+$ order interval $[-x, x]$ is relatively weakly compact. So $T[-x, x]$ is relatively weakly compact and so it is almost Dunford–Pettis. Hence T is an order almost Dunford–Pettis operator. □

A Banach lattice E is said to be a KB-space, whenever every increasing norm bounded sequence of E^+ is norm convergent and it is called a dual Banach lattice if $E = G^*$ for some Banach lattice G. A Banach lattice E is called a dual KB-space if E is a dual Banach lattice and E is a KB-space. It is clear that each KB-space has an order continuous norm.

Theorem 2.20. *Every almost weakly limited operator from a Banach space X into a Banach lattice E is almost limited, if one of the following assertions is valid.*

(a) *The norm of the topological bidual E^{**} is order continuous.*
(b) *E is a dual KB-space.*

Proof. If E^{**} has order continuous norm or E is a dual KB-space, then by [8, Theorem 2.9], every almost Dunford–Pettis set in E is L-weakly compact and so it is almost limited. So every almost weakly limited operator from X into E is almost limited. □

Note that there exist operators which are order almost Dunford–Pettis and weak and almost Dunford–Pettis, but fail to be (almost) weakly limited. Indeed, $Id_{\ell_1} : \ell_1 \to \ell_1$ is order almost Dunford–Pettis and weak almost Dunford–Pettis, but it is not (almost) weakly limited, because ℓ_∞ does not have the (positive) Schur property and so the closed unit ball B_{ℓ_1} is not an (almost) Dunford–Pettis set.

In the following result we characterize Banach lattices E and F on which each operator from E into F which is order almost Dunford–Pettis and weak almost Dunford–Pettis, is almost weakly limited. To establish this result, we will need the following lemma.

Lemma 2.21. *For a Banach lattice E, E^* does not have the positive Schur property if and only if there exist a disjoint weakly null sequence (f_n) in $(E^*)^+$ and a sequence (x_n) in B_{E^+} and $\epsilon > 0$ such that $|f_n(x_n)| > \epsilon$, for all n.*

Theorem 2.22. *The following assertions are equivalent:*

(a) *each order almost Dunford–Pettis and weak and almost Dunford–Pettis operator $T : E \to F$ is almost weakly limited;*
(b) *E^* has order continuous norm or F^* has the positive Schur property.*

Proof. $(a) \Rightarrow (b)$. We use the technique of [13, Theorem 4.2]. Assume (b) is false, i.e, the norm of E^* is not order continuous and F^* does not have the positive Schur property. By [15] we may assume that ℓ_1 is a closed sublattice of E and there is a positive projection P from E into ℓ_1.
On the other hand, from precedding lamma it follows that there exist a disjoint weakly null sequence (f_n) in $(F^*)^+$ and a sequence (x_n) in B_{F^+} and $\epsilon > 0$ such that $|f_n(x_n)| > \epsilon$, for all n. Now, we consider the operator $T = SoP : E \to \ell_1 \to F$ where S is the operator defined by

$$S : \ell_1 \to F, \qquad (\lambda_n) \to \sum_{n=1}^{\infty} \lambda_n x_n.$$

Since ℓ_1 has the Schur property, then by [13, Theorem 4.2], T is weak almost limited and order almost limited. So T is weak and almost Dunford–Pettis and order almost Dunford–Pettis.
Now, we show that the operator T is not almost weakly limited. Indeed, let (f_n) be a disjoint weakly null sequence in F^*. As the operator $P : E \to \ell_1$ is surjective, there exists $\delta > 0$ such that $\delta.(B_{\ell_1}) \subset P(B_E)$. Hence

$$\begin{aligned}\|T^* f_n\| = \sup_{x \in B_E} |T^*(f_n)(x)| = \sup_{x \in B_E} |f_n(T(x))| &= \sup_{x \in B_E} |f_n \ o \ S(P(x))| \\ \geq \delta |f_n \ o \ S(e_n)| \geq \delta |f_n(x_n)| > \delta.\epsilon\end{aligned}$$

(where (e_n) is the canonical basis of ℓ_1). Then $\|T^* f_n\| > \delta.\epsilon$ for all n. So T is not almost weakly limited.
$(b) \Rightarrow (a)$. Let (f_n) be a disjoint weakly null sequence of F^*. Since T is order almost Dunford–Pettis, $|T^* f_n|(x) \to 0$ for each $x \in E^+$. Since E^* has order continuous norm, by [10, Corollary 2.7] every norm bounded disjoint sequence $(x_n) \subset E^+$ is weakly null. Hence, as T is a weak almost Dunford–Pettis operator, $T^* f_n(x_n) = f_n(T(x_n)) \to 0$. Therefore T is an almost Dunford–Pettis operator.
If F^* has the positive Schur property, the closed unit ball B_F is an almost Dunford–Pettis set and clearly every operator $T : E \to F$ is almost weakly limited. □

Note that continuous linear images of Dunford–Pettis (resp. limited) sets or sequences are Dunford–Pettis (resp. limited), but the same conclusion is false for almost Dunford–Pettis sets (resp. almost limited) or sequences, in general. In the following theorem, we stablish some conditions which guarantees the continuous linear images of almost limited (resp. almost Dunford–Pettis) sets are also almost limited (resp. almost Dunford–Pettis).
Recall from [15] that a positive linear operator $T : E \to F$ between two Banach lattices is almost interval preserving, if $T[0, x]$ is dense in $[0, Tx]$, for every $x \in E^+$.

Theorem 2.23. *Let $T : E \to F$ be an almost interval preserving operator and let A be an almost Dunford–Pettis (resp. almost limited) subset of E. Then $T(A)$ is almost Dunford–Pettis (resp. almost limited) in F.*

Proof. Let (y_n^*) be a disjoint weakly null (resp. disjoint weak* null) sequence in F^*. By [15, Theorem 1.4.19], T^* is lattice homomorphism and so $(T^* y_n^*)$ is a disjoint weakly null (resp. disjoint weak* null) sequence in E^*. Since A is almost Dunford–Pettis (resp. almost limited),

$$\limsup_{x \in A} |\langle Tx, y_n^* \rangle| = \limsup_{x \in A} |\langle x, T^* y_n^* \rangle| \to 0.$$

This completes the proof. $\square$

Corollary 2.24. *Every almost interval preserving operator $T : E \to F$ between two Banach lattices is an order almost Dunford–Pettis operator.*

Proof. By [8, Corollary 2.2], for each $x \in E^+$, order interval $[-x, x]$ is an almost Dunford–Pettis set, and by Theorem 2.23, $T[-x, x]$ is an almost Dunford–Pettis set. $\square$

Theorem 2.25. *Every almost interval preserving operator T on an AM-space E with unit is almost weakly limited.*

Proof. If E is AM-space E with unit, then every norm bounded set in E is order bounded and by [8, Corollary 2.2], it is almost Dunford–Pettis. So by Theorem 2.23, the operator T on E is almost weakly limited. $\square$

By the tecnique in the proof of [16, Theorem 2.7], we have the following theorem that is already established in [4, Theorem 3.14].

Theorem 2.26. *Let E and F be two Banach lattices such that E^* has the weakly sequentially continuous lattice operations. If $T : E \to F$ is an operator, then $T(A)$ is an almost Dunford–Pettis set in F, whenever A is an almost Dunford–Pettis solid set in E.*

According to [16, Lemma 2.2], for almost limited sets, we have the following lemma for almost Dunford–Pettis sets.

Lemma 2.27. *Let A be a norm bounded subset of a Banach lattice E. If for every $\epsilon > 0$ there exists some almost Dunford–Pettis subset A_ϵ of E such that $A \subset A_\epsilon + \epsilon B_E$, then A is almost Dunford–Pettis.*

Theorem 2.28. *Let E, F and G be three Banach lattices. Then*

(a) *the class of order almost Dunford–Pettis operators is a norm closed vector subspace of the space $L(E, F)$ of all operators from E into F.*
(b) *if $T : E \to F$ is an order almost Dunford–Pettis operator, then for each almost interval preserving operator $S : F \to G$, the composed operator SoT is order almost Dunford–Pettis.*
(c) *if $T : E \to F$ is an order bounded operator, then for each order almost Dunford–Pettis operator $S : F \to G$, the composed operator SoT is order almost Dunford–Pettis.*

Proof. (a). Clearly the class of order almost Dunford–Pettis operators is a vector subspace of $L(E, F)$ and by Lemma 2.27, this class is also norm closed.
(b). Let $T : E \to F$ be an order almost Dunford–Pettis operator. Then for each $x \in E^+$, $T([-x, x])$ is almost Dunford–Pettis. Since S is almost interval preserving, $S(T([-x, x]))$ is an almost Dunford–Pettis set in F, and so SoT is order almost Dunford–Pettis.
(c). Let $T : E \to F$ be an order bounded operator. Then for each $x \in E^+$, $T([-x, x])$ is order bounded. Since S is order almost Dunford–Pettis, then $S(T([-x, x]))$ is an almost DP set in F, and so SoT is order almost Dunford–Pettis. □

By similar techniques of [7], we have the following theorems.

Theorem 2.29. *Let T be an operator from a Banach lattice E into a Banach lattice F. If T^* is almost Dunford–Pettis, then T is order almost Dunford–Pettis.*

Proof. Let (f_n) be a disjoint weakly null sequence of F^*. As the adjoint T^* is almost Dunford–Pettis, we deduce that $\|T^* f_n\| \to 0$. So for each $x \in E^+$ order interval $T[-x, x]$ is an almost Dunford–Pettis; that is,

$$\sup_{z \in T[-x,x]} |f_n(z)| = \sup_{y \in [-x,x]} |T^* f_n(y)| = |T^* f_n|(x) \to 0.$$

We deduce that T is order Dunford-Pettis. □

Theorem 2.30. *Let T be an operator from a Banach lattice E into a Banach lattice F. If T^* is almost Dunford–Pettis, then T is weak and almost Dunford–Pettis.*

Proof. Let (x_n) be a weakly null sequence of E and (f_n) be a disjoint weakly null sequence in F^*. We have to prove that $f_n(T(x_n)) \to 0$. As T^* is almost Dunford–Pettis, then $\|T^* f_n\| \to 0$. On the other hand, since (x_n) is a weakly null sequence of E, then (x_n) is norm bounded and by the inequality $|T^*(f_n)(x_n)| = |f_n(T(x_n))| \leq \|T^* f_n\|$, we conclude that T is weak and almost Dunford–Pettis. $\square$

Theorem 2.31. *The following assertions are equivalent:*

(a) *Each order almost Dunford–Pettis and weak and almost Dunford–Pettis operator $T : E \to F$ has an adjoint almost Dunford–Pettis operator;*
(b) *E^* has order continuous norm or F^* has the positive Schur property.*

Proof. The proof is similar to [7, Theorem 3.1]. $\square$

Acknowledgments

The authors are grateful to the referees for their careful reading and suggestions for improving this paper.

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