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Integral Inequalities for Riemann-Liouville Fractional Integrals of a Function With Respect to Another Function

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Abstract. In this article, we obtain generalizations for Gruss type integral inequality by using h(x)-Riemann-Liouville fractional integrals.

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1. Introduction

If f and g are two continuous functions on $[a,b]$ satisfying $m \leq f(t) \leq M$ and $p \leq g(t) \leq P$ for all $t \in [a,b]$, $m, M, p, P \in \mathbb{R}$,

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt \right| \leq \frac{1}{4}(M-m)(P-p). \tag{1.1}$$

(1.1) inequality is well-known in literature as Grüss Inequality. It is defined as the integral inequality that establishes as a connection between the product of two functions and the product of the integrals [1].

Grüss type inequalities are now vast and many extensions of the classical inequalities were intensively studied by many authors. Integral inequalities

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and applications have been addressed extensively by several researchers. For example, we refer the reader to [4, 5, 6, 7, 8, 9] and the references cited therein. Also, there are many extensions of Grüss type inequalities by using Riemann-Liouville fractional integrals [2, 13, 14, 16].

2. Preliminaries

In this section, we will give some definitions, lemmas and theorems which we use later in this article.

Definition 2.1. [12] Let $f \in L_1[0,\infty)$.The Riemann- Liouville fractional integral of order $\alpha \geq 0$ is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\, dx, \tag{2.1}$$
$$I^0 f(t) = f(t),$$

where Γ is the gamma function.

Definition 2.2. [10, 11] A function $f(t)$ is said to be in the $L_{p,k}[0,\infty)$ space if

$$L_{p,k}[0,\infty) = \left\{ f : \|f\|_{L_{p,k[0,\infty)}} = \left(\int_a^b |f(t)|^p\, t^k dt \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty,\ k \geq 0 \right\}. \tag{2.2}$$

For $k = 0$,

$$L_p[0,\infty) = \left\{ f : \|f\|_{L_{p[0,\infty)}} = \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty \right\}.$$

Definition 2.3. [10, 11] Let $f \in L_{1,k}[0,\infty)$. The Generalized Riemann-Liouville fractional integral $I^{\alpha,k} f(x)$ of order $\alpha \geq 0$ and $k \geq 0$ is defined by

$$I^{\alpha,k} f(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (x^{k+1} - t^{k+1})^{\alpha-1} t^k f(t) dt, \tag{2.3}$$
$$I^{0,k} f(x) = f(x),$$

where Γ is the gamma function.

Definition 2.4. Let $f \in L_1[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$ and also derivative $h'(x)$ is continuous on $[0,\infty)$ and $h(0) = 0$. The space $X_h^p(0,\infty)$ $(1 \leq p < \infty)$ of those real-valued Lebesque measurable functions f on $[0,\infty)$ for which

$$\|f\|_{X_h^p} = \left(\int_0^\infty |f(t)|^p\, h'(t) dt \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty \tag{2.4}$$

and for the case $p = \infty$

$$\|f\|_{X_h^\infty} = ess \sup_{0\le t<\infty} [h'(t)f(t)].$$

In particular, when $h(x) = x$ $(1 \le p < \infty)$ the space $X_h^p(0,\infty)$ coincides with the $L_p[0,\infty)$-space and also if we take $h(x) = \frac{x^{k+1}}{k+1}$ $(1 \le p < \infty,\ k \ge 0)$ the space $X_h^p(0,\infty)$ coincides with the $L_{p,k}[0,\infty)$-space.

Definition 2.5. [10, 11] Let $f \in X_h^p(0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$ and also derivative $h'(x)$ is continuous on $[0,\infty)$ and $h(0) = 0$. The Riemann-Liouville fractional integral of a function $f(x)$ with respect to another function $h(x)$ is defined by

$$I_h^\alpha f(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (h(t) - h(x))^{\alpha-1} f(x)\, h'(x) dx. \tag{2.5}$$

For the convenience of establishing our results, we give the semi-group property:

$$I_h^\alpha I_h^\beta f(t) = I_h^{\alpha+\beta} f(t)\,, \quad \alpha \ge 0,\ \beta \ge 0, \tag{2.6}$$

which implies the commutative property

$$I_h^\alpha I_h^\beta f(t) = I_h^\beta I_h^\alpha f(t)\,. \tag{2.7}$$

From Definition 5, if $f(t) = h^\gamma(t)$, then we have

$$I_h^\alpha \left[h^\gamma(t)\right] = \frac{h^{\gamma+\alpha}(t)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)},\ \alpha \ge 0,\ \beta \ge 0. \tag{2.8}$$

$(2.5) - (2.8)$ results for (2.5) Riemann-Liouville fractional integral are reduced (2.1) fractional integral and its properties when $h(x) = x$.

In [15]some special applications and convergency cases of $(2,5)$ are studied.

Dahmani et al. [2] gave the following fractional integral inequalities for the integral (2.1) in Definition 1.

Theorem 2.6. *Let f,$g \in L[0,\infty)$ and satisfy the following conditions:*

$$m \le f(t) \le M, \quad p \le g(t) \le P, \quad t \in [0,\infty), \quad m, M, p, P \in \mathbb{R},$$

then for all $t > 0$, $\alpha > 0$, $\beta > 0$,

$$i)\ \left|\tfrac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha (fg)(t) - I^\alpha f(t) I^\alpha g(t)\right| \le (\tfrac{t^\alpha}{\Gamma(\alpha+1)})^2 (M-m)(P-p), \tag{2.9}$$

$$\begin{aligned} ii)\ \ &\left(\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\beta}(fg)(t)+\tfrac{t^{\beta}}{\Gamma(\beta+1)}I^{\alpha}(fg)(t)-I^{\alpha}f(t)I^{\beta}g(t)-I^{\beta}f(t)I^{\alpha}g(t)\right)^{2} \\ &\leq\Big\{\left(M\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}-I^{\alpha}f(t)\right)\left(I^{\beta}f(t)-m\tfrac{t^{\beta}}{\Gamma(\beta+1)}\right) \\ &\qquad+\left(I^{\alpha}f(t)-m\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\tfrac{t^{\beta}}{\Gamma(\beta+1)}-I^{\beta}f(t)\right)\Big\} \\ &\qquad\times\Big\{\left(P\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}-I^{\alpha}g(t)\right)\left(I^{\beta}g(t)-p\tfrac{t^{\beta}}{\Gamma(\beta+1)}\right) \\ &\qquad+\left(I^{\alpha}g(t)-p\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(P\tfrac{t^{\beta}}{\Gamma(\beta+1)}-I^{\beta}g(t)\right)\Big\}. \end{aligned} \tag{2.10}$$

Jessada Tariboon et al. [3] gave the following fractional integral inequalities for functions in Theorem 2, Theorem 3, Theorem 4, and Lemma 1 which are bounded with integrable functions.

Theorem 2.7. *Let $f \in L[0,\infty)$. Suppose that*
(H_1) there exist two integrable functions φ_1, φ_2 on $[0,\infty)$ such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t),\ \forall t \in [0,\infty).$$

Then for $t>0$, $\alpha,\beta>0$, one has

$$I^{\beta}\varphi_1(t)I^{\alpha}f(t)+I^{\alpha}\varphi_2(t)I^{\beta}f(t) \geq I^{\alpha}\varphi_2(t)I^{\beta}\varphi_1(t)+I^{\alpha}f(t)I^{\beta}f(t). \tag{2.11}$$

Theorem 2.8. *Let f,$g \in L[0,\infty)$. Assume that*
(H_1) holds and moreover one assumes that
(H_2)there exist two integrable functions ψ_1, ψ_2 on $[0,\infty)$ such that

$$\psi_1(t) \leq g(t) \leq \psi_2(t),\ \forall t \in [0,\infty).$$

Then for $t>0$, $\alpha,\beta>0$, the following inequalities hold:

$$\begin{array}{ll} (a) & I^{\beta}\psi_1(t)I^{\alpha}f(t)+I^{\alpha}\varphi_2(t)I^{\beta}g(t) \geq I^{\beta}\psi_1(t)I^{\alpha}\varphi_2(t)+I^{\alpha}f(t)I^{\beta}g(t), \\ (b) & I^{\beta}\varphi_1(t)I^{\alpha}g(t)+I^{\alpha}\psi_2(t)I^{\beta}f(t) \geq I^{\beta}\varphi_1(t)I^{\alpha}\psi_2(t)+I^{\beta}f(t)I^{\alpha}g(t), \\ (c) & I^{\alpha}\varphi_2(t)I^{\beta}\psi_2(t)+I^{\alpha}f(t)I^{\beta}g(t) \geq I^{\alpha}\varphi_2(t)I^{\beta}g(t)+I^{\beta}\psi_2(t)I^{\alpha}f(t), \\ (d) & I^{\alpha}\varphi_1(t)I^{\beta}\psi_1(t)+I^{\alpha}f(t)I^{\beta}g(t) \geq I^{\alpha}\varphi_1(t)I^{\beta}g(t)+I^{\beta}\psi_1(t)I^{\alpha}f(t). \end{array} \tag{2.12}$$

Lemma 2.9. *Let $f,\varphi_1,\varphi_2 \in L[0,\infty)$. Assume that the condition (H_1)holds. Then, for $t>0$, $\alpha>0$, we have*

$$\begin{aligned} \tfrac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\alpha}f^2(t)-(I^{\alpha}f(t))^2 = \ & \{I^{\alpha}\varphi_2(t)-I^{\alpha}f(t))(I^{\alpha}f(t)-I^{\alpha}\varphi_1(t)) \\ & -\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\alpha}((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\Big\} \\ & +\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\alpha}\varphi_1(t)f(t)-I^{\alpha}\varphi_1(t)I^{\alpha}f(t) \\ & +\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\alpha}\varphi_2(t)f(t)-I^{\alpha}\varphi_2(t)I^{\alpha}f(t) \\ & +I^{\alpha}\varphi_1(t)I^{\alpha}\varphi_2(t)-\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}I^{\alpha}\varphi_1(t)\varphi_2(t). \end{aligned} \tag{2.13}$$

Theorem 2.10. *Let* $f, g, \varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2 \in L[0,\infty)$ *and satisfy the conditions* (H_1) *and* (H_2) *on* $[0,\infty)$. *Then for all* $t>0,\ \alpha>0,$ *one has*

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)} I^{\alpha}(fg)(t) - I^{\alpha}f(t)I^{\alpha}g(t)\right| \leq \sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\ \psi_2)}, \tag{2.14}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned} T(u,v,w) = \quad & (I^{\alpha}w(t) - I^{\alpha}u(t))(I^{\alpha}u(t) - I^{\alpha}v(t)) \\ & + \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} I^{\alpha}(v(t)u(t)) - I^{\alpha}v(t)I^{\alpha}u(t) \\ & + \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} I^{\alpha}(w(t)u(t)) - I^{\alpha}w(t)I^{\alpha}u(t) \\ & + I^{\alpha}v(t)I^{\alpha}w(t) - \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} I^{\alpha}(v(t)w(t)). \end{aligned} \tag{2.15}$$

Main Results: In this section, we will obtain Grüss type inequalities by using (2.5) $h(x)$-Riemann-Liouville fractional integral. Our first result is the following theorem.

Theorem 2.11. *Let* $f \in X_h^p(0,\infty)$ *and* $h(x)$ *be an increasing and positive monotone function on* $[0,\infty)$, *and also derivative* $h'(x)$ *is continuous on* $[0,\infty)$ *and also* $h(0)=0$. *Suppose that there exist two integrable functions* $\varphi_1,\ \varphi_2$ *on* $[0,\infty)$ $t>0,\ \alpha,\beta>0$ *such that*

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t) \quad \forall t \in [0,\infty). \tag{2.16}$$

Then we obtain the following inequality

$$I_h^{\beta}\varphi_1(t)I_h^{\alpha}f(t) + I_h^{\alpha}\varphi_2(t)I_h^{\beta}f(t) \geq I_h^{\alpha}\varphi_2(t)I_h^{\beta}\varphi_1(t) + I_h^{\alpha}f(t)I_h^{\beta}f(t). \tag{2.17}$$

Proof. From (3.1), for all $x \geq 0,\ y \geq 0$, we have

$$(\varphi_2(x) - f(x))(f(y) - \varphi_1(y)) \geq 0,$$

$$\varphi_2(x)f(y) + \varphi_1(y)f(x) \geq \varphi_1(y)\varphi_2(x) + f(x)f(y). \tag{2.18}$$

If we multiply both sides of (3.3) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)}{\Gamma(\alpha)}$, and integrate with respect to x on $(0,t)$, we obtain

$$f(y)I_h^{\alpha}\varphi_2(t) + \varphi_1(y)I_h^{\alpha}f(t) \geq \varphi_1(y)I_h^{\alpha}\varphi_2(t) + f(y)I_h^{\alpha}f(t). \tag{2.19}$$

If we multiply both sides of (3.4) by $\dfrac{(h(t)-h(y))^{\beta-1}h'(y)}{\Gamma(\beta)}$, and integrate with respect to y on $(0,t)$, we get

$$I_h^{\beta}\varphi_1(t)I_h^{\alpha}f(t) + I_h^{\alpha}\varphi_2(t)I_h^{\beta}f(t) \geq I_h^{\alpha}\varphi_2(t)I_h^{\beta}\varphi_1(t) + I_h^{\alpha}f(t)I_h^{\beta}f(t). \tag{2.20}$$

This proves the theorem. $\square$

Corollary 2.12. *Let $h(x) = x$ in Theorem 5, then we have the inequality (2.11) in Theorem 2.*

Corollary 2.13. *Let $h(x) = \frac{x^{k+1}}{k+1}$, $k \geq 0$ in Theorem 5, then we have the results in [13].*

Corollary 2.14. *Let $f \in X_h^p(0,\infty)$. Suppose that $m \leq f(t) \leq M$, $\forall t \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 0$, $\alpha > 0$, $\beta > 0$, we have*

$$\begin{aligned} &m\frac{h^{\beta}(t)}{\Gamma(\beta+1)}I_h^{\alpha}f(t) + M\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\beta}f(t) \\ &\geq Mm\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}\frac{h^{\beta}(t)}{\Gamma(\beta+1)} + I_h^{\alpha}f(t)I_h^{\beta}f(t). \end{aligned} \tag{2.21}$$

Theorem 2.15. *Let f and g be two integrable functions on $[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative $h'(x)$ is continuous on $[0,\infty)$ and also $h(0) = 0$, $t > 0$, $\alpha, \beta > 0$. Suppose that (3.1) holds and moreover assume that there exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t) \leq g(t) \leq \psi_2(t), \quad \forall t \in [0,\infty). \tag{2.22}$$

Then the following inequalities hold:

$$\begin{array}{ll} (a) & I_h^{\beta}\psi_1(t)I_h^{\alpha}f(t) + I_h^{\alpha}\varphi_2(t)I_h^{\beta}g(t) \geq I_h^{\beta}\psi_1(t)I_h^{\alpha}\varphi_2(t) + I_h^{\alpha}f(t)I_h^{\beta}g(t), \\ (b) & I_h^{\beta}\varphi_1(t)I_h^{\alpha}g(t) + I_h^{\alpha}\psi_2(t)I_h^{\beta}f(t) \geq I_h^{\beta}\varphi_1(t)I_h^{\alpha}\psi_2(t) + I_h^{\beta}f(t)I_h^{\alpha}g(t), \\ (c) & I_h^{\alpha}\varphi_2(t)I_h^{\beta}\psi_2(t) + I_h^{\alpha}f(t)I_h^{\beta}g(t) \geq I_h^{\alpha}\varphi_2(t)I_h^{\beta}g(t) + I_h^{\beta}\psi_2(t)I_h^{\alpha}f(t), \\ (d) & I_h^{\alpha}\varphi_1(t)I_h^{\beta}\psi_1(t) + I_h^{\alpha}f(t)I_h^{\beta}g(t) \geq I_h^{\alpha}\varphi_1(t)I_h^{\beta}g(t) + I_h^{\beta}\psi_1(t)I_h^{\alpha}f(t). \end{array} \tag{2.23}$$

Proof. From (3.1) and (3.7) for $\forall t \in [0,\infty)$, we have

$$(\varphi_2(x) - f(x))(g(y) - \psi_1(y)) \geq 0,$$

then

$$\varphi_2(x)g(y) + \psi_1(y)f(x) \geq \psi_1(y)\varphi_2(x) + f(x)g(y). \tag{2.24}$$

If we multiply both sides of (3.9) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)}{\Gamma(\alpha)}$ and integrate with respect to x on $(0,t)$, we get

$$g(y)I_h^{\alpha}\varphi_2(t) + \psi_1(y)I_h^{\alpha}f(t) \geq \psi_1(y)I_h^{\alpha}\varphi_2(t) + g(y)I_h^{\alpha}f(t). \tag{2.25}$$

If we multiply both sides of (3.10) by $\dfrac{(h(t)-h(y))^{\beta-1}h'(y)}{\Gamma(\beta)}$ and integrate with respect to y on $(0,t)$, we get

$$I_h^\beta \psi_1(t) I_h^\alpha f(t) + I_h^\alpha \varphi_2(t) I_h^\beta g(t) \geq I_h^\alpha \varphi_2(t) I_h^\beta \psi_1(t) + I_h^\alpha f(t) I_h^\beta g(t). \tag{2.26}$$

This proves (a).

To prove $(b)-(d)$, we use the following inequalities:

$$(b)\ (\psi_2(x) - g(x))(f(y) - \varphi_1(y)) \geq 0,$$

$$(c)\ (\varphi_2(x) - f(x))(g(y) - \psi_2(y)) \leq 0,$$

$$(d)\ (\varphi_1(x) - f(x))(g(y) - \psi_1(y)) \leq 0.$$

□

The following inequalities are the special case of Theorem 6.

Corollary 2.16. *Let $f, g \in X_h^p(0,\infty)$, $t > 0$, $\alpha > 0$, $\beta > 0$. Suppose that there exist real constants m, M, n, N, such that*

$$m \leq f(t) \leq M, \quad n \leq g(t) \leq N, \quad \forall t \in [0,\infty).$$

Then we have

$$\begin{aligned}
(a^*)\ & n\frac{h^\beta(t)}{\Gamma(\beta+1)} I_h^\alpha f(t) + M\frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\beta g(t) \\
& \geq nM\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\frac{h^\beta(t)}{\Gamma(\beta+1)} + I_h^\alpha f(t) I_h^\beta g(t), \\
(b^*)\ & m\frac{h^\beta(t)}{\Gamma(\beta+1)} I_h^\alpha g(t) + N\frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\beta f(t) \\
& \geq mN\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\frac{h^\beta(t)}{\Gamma(\beta+1)} + I_h^\beta f(t) I_h^\alpha g(t), \\
(c^*)\ & NM\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\frac{h^\beta(t)}{\Gamma(\beta+1)} + I_h^\alpha f(t) I_h^\beta g(t) \\
& \geq M\frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\beta g(t) + N\frac{h^\beta(t)}{\Gamma(\beta+1)} I_h^\alpha f(t), \\
(d^*)\ & mn\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\frac{h^\beta(t)}{\Gamma(\beta+1)} + I_h^\alpha f(t) I_h^\beta g(t) \\
& \geq m\frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\beta g(t) + n\frac{h^\beta(t)}{\Gamma(\beta+1)} I_h^\alpha f(t).
\end{aligned} \tag{2.27}$$

Corollary 2.17. *Let $f, g \in L_1[0,\infty)$ and $h(x) = x$, $t > 0$, $\alpha > 0$, $\beta > 0$. Suppose that there exist real constants m, M, n, N, such that*

$$m \leq f(t) \leq M, \quad n \leq g(t) \leq N, \quad \forall t \in [0,\infty).$$

Then we have the results in [3].

Corollary 2.18. *Let $f, g \in L_{1,k}[0,\infty)$ and $h(x) = \frac{x^{k+1}}{k+1}$, $k \geq 0$, $t > 0$, $\alpha > 0$, $\beta > 0$. Suppose that there exist real constants m, M, n, N such that*

$$m \leq f(t) \leq M, \quad n \leq g(t) \leq N, \quad \forall t \in [0,\infty).$$

Then we have the results in [13].

Lemma 2.19. *Let $f \in X_h^p(0,\infty)$ and assume φ_1, φ_2 be two integrable functions on $[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative $h'(x)$ is continuous on $[0,\infty)$ and also $h(0) = 0$, $t > 0$, $\alpha > 0$. Suppose that the condition* (3.1) *holds. Then*

$$\begin{aligned}
&\frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha f^2(t) - (I_h^\alpha f(t))^2 \\
&\qquad = (I_h^\alpha \varphi_2(t) - I_h^\alpha f(t))(I_h^\alpha f(t) - I_h^\alpha \varphi_1(t)) \\
&\qquad\quad - \frac{h^\alpha(t)}{\Gamma(\alpha+1)} (I_h^\alpha \varphi_2(t) - I_h^\alpha f(t))(I_h^\alpha f(t) - I_h^\alpha \varphi_1(t)) \\
&\qquad\quad + \frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha(\varphi_1(t) f(t)) - I_h^\alpha \varphi_1(t) I_h^\alpha f(t) \\
&\qquad\quad + \frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha(\varphi_2(t) f(t)) - I_h^\alpha \varphi_2(t) I_h^\alpha f(t) \\
&\qquad\quad + I_h^\alpha(\varphi_1(t)\varphi_2(t)) - \frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha(\varphi_1(t)\varphi_2(t))
\end{aligned} \tag{2.28}$$

Proof. For any $x, y > 0$, we have

$$\begin{aligned}
&(\varphi_2(y) - f(y))(f(x) - \varphi_1(x)) + (\varphi_2(x) - f(x))(f(y) - \varphi_1(y)) \\
&-(\varphi_2(x) - f(x))(f(x) - \varphi_1(x)) - (\varphi_2(y) - f(y))(f(y) - \varphi_1(y)) \\
&\qquad = f^2(x) + f^2(y) - 2f(x)f(y) + \varphi_2(y)f(x) + \varphi_1(x)f(y) \\
&\qquad\quad -\varphi_1(x)\varphi_2(y) + \varphi_2(x)f(y) + \varphi_1(y)f(x) - \varphi_1(y)\varphi_2(x) \\
&\qquad\quad -\varphi_2(x)f(x) + \varphi_1(x)\varphi_2(x) - \varphi_1(x)f(x) - \varphi_2(y)f(y) \\
&\qquad\quad +\varphi_1(y)\varphi_2(y) - \varphi_1(y)f(y).
\end{aligned} \tag{2.29}$$

If we multiply both sides of (3.14) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)}{\Gamma(\alpha)}$ and integrate with respect to x on $(0,t)$, we get

$$\begin{aligned}
&(\varphi_2(y) - f(y))(I_h^\alpha f(t) - I_h^\alpha \varphi_1(t)) + (I_h^\alpha \varphi_2(t) - I_h^\alpha f(t))(f(y) - \varphi_1(y)) \\
&-I_h^\alpha(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) - (\varphi_2(y) - f(y))(f(y) - \varphi_1(y)) \frac{h^\alpha(t)}{\Gamma(\alpha+1)} \\
&\quad = I_h^\alpha f^2(t) + f^2(y)\frac{h^\alpha(t)}{\Gamma(\alpha+1)} - 2f(y) I_h^\alpha f(t) + \varphi_2(y) I_h^\alpha f(t) + f(y) I_h^\alpha \varphi_1(t) \\
&\qquad -\varphi_2(y) I_h^\alpha \varphi_1(t) + f(y) I_h^\alpha \varphi_2(t) + \varphi_1(y) I_h^\alpha f(t) - \varphi_1(y) I_h^\alpha \varphi_2(t) \\
&\qquad -I_h^\alpha(\varphi_2(t) f(t)) + I_h^\alpha(\varphi_1(t)\varphi_2(t)) - I_h^\alpha(\varphi_1(t) f(t)) - \varphi_2(y) f(y) \frac{h^\alpha(t)}{\Gamma(\alpha+1)} \\
&\qquad +\varphi_1(y)\varphi_2(y)\frac{h^\alpha(t)}{\Gamma(\alpha+1)} - \varphi_1(y) f(y)\frac{h^\alpha(t)}{\Gamma(\alpha+1)}.
\end{aligned} \tag{2.30}$$

If we multiply both sides of (3.15) by $\dfrac{(h(t)-h(y))^{\alpha-1}h'(y)}{\Gamma(\alpha)}$ and integrate with respect to y on $(0,t)$, we get

$$
\begin{aligned}
&(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\\
&+(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\\
&-(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\\
&-(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\\
&\qquad=\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha f^2(t)+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha f^2(t)\\
&\qquad-2I_h^\alpha f(t)I_h^\alpha f(t)+I_h^\alpha\varphi_2(t)I_h^\alpha f(t)+I_h^\alpha\varphi_1(t)I_h^\alpha f(t)\\
&\qquad-I_h^\alpha\varphi_1(t)I_h^\alpha\varphi_2(t)+I_h^\alpha\varphi_2(t)I_h^\alpha f(t)+I_h^\alpha\varphi_1(t)I_h^\alpha f(t)\\
&\qquad-I_h^\alpha\varphi_1(t)I_h^\alpha\varphi_2(t)-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_2(t)f(t))\\
&\qquad+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)\varphi_2(t))-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)f(t))\\
&\qquad-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_2(t)f(t))+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)\varphi_2(t))\\
&\qquad-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)f(t)).
\end{aligned}
\tag{2.31}
$$

This proves lemma. □

Corollary 2.20. *Let $h(x)=x$ in Lemma 2, then we have inequality (2.13) in Lemma 1.*

Corollary 2.21. *Let $f\in X_h^p(0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative $h'(x)$ is continuous on $[0,\infty)$ and also $h(0)=0$. Suppose that $m\le f(t)\le M, \forall t\in[0,\infty)$ and $m,M\in\mathbb{R}$. Then for, $t>0,\ \alpha>0,\ \beta>0,$ we have*

$$
\begin{aligned}
&\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha f^2(t)-(I_h^\alpha f(t))^2\\
&=\left(M\frac{h^\alpha(t)}{\Gamma(\alpha+1)}-I_h^\alpha f(t)\right)\left(I_h^\alpha f(t)-m\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\right)\\
&-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha((M-f(t))(f(t)-m)).
\end{aligned}
\tag{2.32}
$$

Corollary 2.22. *Let $f\in L_{1,k}[0,\infty)$ and $h(x)=\frac{x^{k+1}}{k+1}$. Suppose that $m\le f(t)\le M$, $\forall t\in[0,\infty)$ and $m,M\in\mathbb{R}$.Then for $k\ge0$, $t>0$, $\alpha>0$, $\beta>0$, we have the results in* [13].

Theorem 2.23. *Let $f,g,\varphi_1,\varphi_2,\psi_1$ and ψ_2 be six integrable functions on $[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative*

$h'(x)$ is continuous on $[0,\infty)$ and also $h(0)=0$. Satisfying the conditions (3.1) *and* (3.7) *on $[0,\infty)$. Then for all $t>0$, $\alpha>0$, one has*

$$\left|\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(f(t)g(t))-I_h^{\alpha}f(t)I_h^{\alpha}g(t)\right| \leq \sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)}\ , \tag{2.33}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned}T(u,v,w) &= (I_h^{\alpha}w(t)-I_h^{\alpha}u(t))(I_h^{\alpha}u(t)-I_h^{\alpha}v(t))\\ &+\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(v(t)u(t))-I_h^{\alpha}v(t)I_h^{\alpha}u(t)+\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(w(t)u(t))\\ &-I_h^{\alpha}w(t)I_h^{\alpha}u(t)+I_h^{\alpha}v(t)I_h^{\alpha}w(t)-\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(v(t)w(t)).\end{aligned} \tag{2.34}$$

Proof. Let f and g be two integrable functions defined $[0,\infty)$ satisfying (3.1) and (3.7). Define

$$H(x,y)=(f(x)-f(y))(g(x)-g(y)),\quad x,y\in(0,t),\ t>0. \tag{2.35}$$

Multiplying both sides of (3.20) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)(h(t)-h(y))^{\alpha-1}h'(y)}{2\Gamma^2(\alpha)}$, $x,y\in(0,t)$ and integrating the resulting identity with respect to x and y, from 0 to t, we can state that

$$\begin{aligned}&\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}H(x,y)h'(y)h'(x)dxdy\\ &\qquad=\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(f(t)g(t))-I_h^{\alpha}f(t)I_h^{\alpha}g(t).\end{aligned} \tag{2.36}$$

Applying the Cauchy-Schwarz inequality to (3.21), we have

$$\begin{aligned}&\left(\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}(f(x)-f(y))(g(x)-g(y))h'(y)h'(x)dxdy\right)^2\\ &\leq\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}(f(x)-f(y))^2h'(y)h'(x)dxdy\\ &\times\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}(g(x)-g(y))^2h'(y)h'(x)dxdy.\end{aligned} \tag{2.37}$$

From (3.21) and (3.22) we obtain

$$\begin{aligned}&\left(\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(f(t)g(t)) - I_h^{\alpha}f(t)I_h^{\alpha}g(t)\right)^2\\ &\leq \left(\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}f^2(t) - (I_h^{\alpha}f(t))^2\right) \times \left(\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}g^2(t) - (I_h^{\alpha}g(t))^2\right).\end{aligned} \tag{2.38}$$

Since $(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \geq 0$ and $(\psi_2(t) - g(t))(g(t) - \psi_1(y)) \geq 0$, for $t \in [0, \infty)$, we have

$$\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}((\varphi_2(t) - f(t))(f(t) - \varphi_1(t))) \geq 0, \tag{2.39}$$

$$\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}((\psi_2(t) - g(t))(g(t) - \psi_1(y))) \geq 0. \tag{2.40}$$

Thus, from Lemma 2, we obtain

$$\begin{aligned}&\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}f^2(t) - (I_h^{\alpha}f(t))^2\\ &\qquad \leq (I_h^{\alpha}\varphi_2(t) - I_h^{\alpha}f(t))(I_h^{\alpha}f(t) - I_h^{\alpha}\varphi_1(t))\\ &\qquad +\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(\varphi_1(t)f(t)) - I_h^{\alpha}\varphi_1(t)I_h^{\alpha}f(t)\\ &\qquad +\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}\varphi_2(t) - I_h^{\alpha}\varphi_2(t)I_h^{\alpha}f(t)\\ &\qquad +I_h^{\alpha}\varphi_1(t)I_h^{\alpha}\varphi_2(t) - \frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(\varphi_1(t)\varphi_2(t))\\ &\qquad = T(f, \varphi_1, \varphi_2),\end{aligned} \tag{2.41}$$

$$\begin{aligned}\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}g^2(t) - (I_h^{\alpha}g(t))^2 &\leq (I_h^{\alpha}\psi_2(t) - I_h^{\alpha}g(t))(I_h^{\alpha}g(t) - I_h^{\alpha}\psi_1(t))\\ &+\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(\psi_1(t)g(t)) - I_h^{\alpha}\psi_1(t)I_h^{\alpha}g(t)\\ &+\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}\psi_2(t) - I_h^{\alpha}\psi_2(t)I_h^{\alpha}g(t)\\ &+I_h^{\alpha}\psi_1(t)I_h^{\alpha}\psi_2(t) - \frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(\psi_1(t)\psi_2(t))\\ &= T(g, \psi_1, \psi_2).\end{aligned} \tag{2.42}$$

From (3.22), (3.26), and (3.27), we get (3.18). $\square$

Corollary 2.24. *If* $T(f, \varphi_1, \varphi_2) = T(f, m, M)$, $T(g, \psi_1, \psi_2) = T(g, p, P)$ *and* $m, M, p, P \in \mathbb{R}$, *then inequality* (3.18) *reduces to*

$$\begin{aligned}&\left|\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(f(t)g(t)) - I_h^{\alpha}f(t)I_h^{\alpha}g(t)\right|\\ &\quad \leq \left(\frac{h^{\alpha}(t)}{2\Gamma(\alpha+1)}\right)^2 (M-m)(P-p).\end{aligned} \tag{2.43}$$

Corollary 2.25. *Let* $h(x)=\frac{x^{k+1}}{k+1},\ k\geq 0$ *if* $T(f,\varphi_1,\varphi_2)=T(f,m,M),\ T(g,\psi_1,\psi_2)=T(g,p,P)$ *and* $m,M,p,P\in\mathbb{R}$*, then we have the results in* [13].

Corollary 2.26. *Let* $h(x)=x$*, if* $T(f,\varphi_1,\varphi_2)=T(f,m,M),\ T(g,\psi_1,\psi_2)=T(g,p,P)$ *and* $m,M,p,P\in\mathbb{R}$*, then we have the results in* [3].

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References

1. G. Grüss, Uber das maximum des absoluten Betrages von

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$$

, *Math. Z*, **39**, (1935), 215-226 .
2. Z. Dahmani, L. Tabharit, S. Taf, New Generalisations of Grüss inequality using Riemann-Liouville fractional integrals, *Bulletin of Mathematical Anslysis and Applications*, **2**(3), (2010), 93-99.
3. J. Tariboon, S. K. Ntouyas, W. Sudsutad, Some new Riemann-Liouville fractional integral inequalities, *International Journal of Mathematics and Mathematical Sciences*, **2014**, Article ID 869434, (2014).
4. H. Yue, Ostrowski Inequality for Fractional Integrals and Related Fractional Inequalities, *TJMM5*, **5**(1), (2013), 85-89.
5. S. Belarbi, Z. Dahmani, On some new fractional integral inequalities, *Journal of Inequalities in Pure and Applied Mathematics,* **10** (3), (2009), article 86.
6. Z. Dahmani, New inequalities in fractional integrals, *International Journal of Nonlinear Science,* **9** (4), (2010), 493-497.
7. Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, *Nonlinear Science Letters A*, **1** (2), (2010), 155-160.
8. P. -L Butzer, A. -A. Kilbas, J. -J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *Journal of Mathematical Analysis and Applications,* **269**(2), (2002), 387-400.
9. M. -Z Sarıkaya, H. Ogunmez, On new inequalities via Riemann-Liouville fractional integration, *Abstract and Applied Analysis,* **2012**, Article ID 428983, (2012), 10 pages.
10. H. Yıldırım, Z. Kırtay, "Ostrowski Inequality for Generalized Fractional Integral and Related Inequalities," *Malaya Journal of Matematik,* **2** (3) (2014), 322-329.
11. U. -N. Katugampola, Approach to a generalized fractional integral, *Applied Mathematics and Computation,* **218**(3), (2011), 860-865.
12. S. -G Samko, A. -A Kilbas, O. -I Marichev, Fractional Integral and Derivatives, *Theory and Applications, Gordon and Breach, Yverdon et alibi.*, (1993).
13. E. Kacar, H. Yıldırım, Grüss Type Integral Inequalities for Generalized Riemann-Liouville Fractional Integrals, *IJPAM.*, **101**(1), (2015), 55-70.
14. S. Kılınc, H. Yıldırım, Generalized Fractional Integral Inequalities Involving Hypergeometric Operators, *IJPAM.*, **101**(1), (2015), 71-82.
15. H.H.G. Hashem, On The Solution of a Generalized Fractional Order Integral Equation and Some Applications, *Journal of Calculus and Applications,* **6**(1) Jan. (2015), 120-130.

16. G. Anastassiou, M.R. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, Montgomery identities for fractional integrals and related fractional inequalities, *J. Ineq. Pure and Appl. Math.*, **10**(4), (2009).