New Jensen and Ostrowski Type Inequalities for General
Lebesgue Integral with Applications

S. S. Dragomir\textsuperscript{a,b}

\textsuperscript{a}Mathematics, College of Engineering & Science, Victoria University,
PO Box 14428 Melbourne City, MC 8001, Australia.
\textsuperscript{b}School of Computational & Applied Mathematics, University of the
Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.

E-mail: sever.dragomir@vu.edu.au

Abstract. Some new inequalities related to Jensen and Ostrowski in-
equalities for general Lebesgue integral are obtained. Applications for
\( f \)-divergence measure are provided as well.

Keywords: Ostrowski’s inequality, Jensen’s inequality, \( f \)-Divergence measures.


1. Introduction

Let \((\Omega, \mathcal{A}, \mu)\) be a measurable space consisting of a set \(\Omega\), a \(\sigma\) – algebra \(\mathcal{A}\) of parts of \(\Omega\) and a countably additive and positive measure \(\mu\) on \(\mathcal{A}\) with values in \(\mathbb{R} \cup \{\infty\}\). Assume, for simplicity, that \(\int_{\Omega} d\mu = 1\). Consider the Lebesgue

\[
L(\Omega, \mu) := \{ f : \Omega \to \mathbb{R}, \text{ f is } \mu\text{-measurable and } \int_{\Omega} |f(t)| \, d\mu(t) < \infty \}.
\]

For simplicity of notation we write everywhere in the sequel \(\int_{\Omega} w(t) \, d\mu(t)\) instead of \(\int_{\Omega} w(t) \, d\mu(t)\).

In order to provide a reverse of the celebrated Jensen’s integral inequality
for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:
Theorem 1.1. Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on $(m, M)$ and $f : \Omega \to [m, M]$ so that $\Phi \circ f$, $f$, $\Phi' \circ f$, $(\Phi' \circ f) : f \in L(\Omega, \mu)$. Then we have the inequality:

$$0 \leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left( \int_{\Omega} f d\mu \right) \leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \leq \frac{1}{2} [\Phi' (M) - \Phi' (m)] \int_{\Omega} |f - \int_{\Omega} f d\mu| d\mu. \quad (1.1)$$

In the case of discrete measure, we have:

Corollary 1.2. Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on $(m, M)$. If $x_i \in [m, M]$ and $w_i \geq 0$ ($i = 1, \ldots, n$) with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the counterpart of Jensen’s weighted discrete inequality:

$$0 \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i \leq \frac{1}{2} [\Phi' (M) - \Phi' (m)] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|. \quad (1.2)$$

Remark 1.3. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

If $f, g : \Omega \to \mathbb{R}$ are $\mu$-measurable functions and $f, g, fg \in L(\Omega, \mu)$, then we may consider the Čebyšev functional

$$T (f, g) := \int_{\Omega} f g d\mu - \int_{\Omega} f d\mu \int_{\Omega} g d\mu. \quad (1.3)$$

The following result is known in the literature as the Grüss inequality

$$|T (f, g)| \leq \frac{1}{4} \left( \Gamma - \gamma \right) \left( \Delta - \delta \right), \quad (1.4)$$

provided

$$-\infty < \gamma \leq f (t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g (t) \leq \Delta < \infty \quad (1.5)$$

for $\mu$-a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f (t) \leq \Gamma < \infty$ for $\mu$-a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz’s integral inequality, we have

$$\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \left[ \int_{\Omega} f^2 d\mu - \left( \int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} \left( \Gamma - \gamma \right). \quad (1.6)$$
On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

\[
0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right) \tag{1.7}
\]
\[
\leq \int_{\Omega} f \cdot \left( \Phi' \circ f \right) \, d\mu - \int_{\Omega} \Phi' \circ f \, d\mu \int_{\Omega} f \, d\mu
\]
\[
\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \left[ \int_{\Omega} f^2 \, d\mu - \left( \int_{\Omega} f \, d\mu \right)^2 \right]^{\frac{1}{2}}
\]
\[
\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] (M - m),
\]
provided that \( \Phi : [m, M] \subset \mathbb{R} \to \mathbb{R} \) is a differentiable convex function on \((m, M)\) and \( f : \Omega \to [m, M] \) so that \( \Phi \circ f, f, f \cdot (\Phi' \circ f) \in L(\Omega, \mu) \), with \( \int_{\Omega} f \, d\mu = 1 \).

The following reverse of the Jensen’s inequality also holds [33]:

**Theorem 1.4.** Let \( \Phi : I \to \mathbb{R} \) be a continuous convex function on the interval of real numbers \( I \) and \( m, M \in \mathbb{R}, \ M < m \) with \([m, M] \subset \overline{I}\), where \( \overline{I} \) is the interior of \( I \). If \( f : \Omega \to \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds

\(-\infty < m \leq f(t) \leq M < \infty \) for \( \mu \)-a.e. \( t \in \Omega \)

and such that \( f, \Phi \circ f \in L(\Omega, \mu) \), then

\[
0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right) \tag{1.8}
\]
\[
\leq \left( M - \int_{\Omega} f \, d\mu \right) \left( \int_{\Omega} f \, d\mu - m \right) \frac{\Phi'(M) - \Phi'(m)}{M - m}
\]
\[
\leq \frac{1}{4} (M - m) \left[ \Phi'_-(M) - \Phi'_+(m) \right],
\]
where \( \Phi'_- \) is the left and \( \Phi'_+ \) is the right derivative of the convex function \( \Phi \).

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean \( \frac{1}{b-a} \int_{a}^{b} \Phi (t) \, dt \) and the value \( \Phi(x) \), \( x \in [a, b] \).

For various results related to Ostrowski’s inequality see [6]-[9], [15]-[41], [43] and the references therein.

**Theorem 1.5.** Let \( \Phi : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\) such that \( \Phi' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\), i.e., \( \| \Phi' \|_{\infty} := \sup_{t \in (a,b)} |\Phi'(t)| < \)
Then
\[
|\Phi(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|\Phi'\|_{\infty} (b-a), \quad (1.9)
\]
for all \( x \in [a,b] \) and the constant \( \frac{1}{4} \) is the best possible.

Now, for \( \gamma, \Gamma \in \mathbb{C} \) and \([a,b]\) an interval of real numbers, define the sets of complex-valued functions [34]

\[
U_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a,b] \to \mathbb{C} \mid \text{Re} \left[ (\Gamma - f(t))(\bar{f}(t) - \bar{\gamma}) \right] \geq 0 \text{ for almost every } t \in [a,b] \right\}
\]

and

\[
\Delta_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a,b] \to \mathbb{C} \mid \left| f(t) - \gamma + \frac{\Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a,b] \right\}.
\]

The following representation result may be stated [34].

**Proposition 1.6.** For any \( \gamma, \Gamma \in \mathbb{C} \), \( \gamma \neq \Gamma \), we have that \( U_{[a,b]}(\gamma, \Gamma) \) and \( \Delta_{[a,b]}(\gamma, \Gamma) \) are nonempty, convex and closed sets and

\[
U_{[a,b]}(\gamma, \Gamma) = \Delta_{[a,b]}(\gamma, \Gamma).
\]

On making use of the complex numbers field properties we can also state that:

**Corollary 1.7.** For any \( \gamma, \Gamma \in \mathbb{C} \), \( \gamma \neq \Gamma \), we have that

\[
U_{[a,b]}(\gamma, \Gamma) = \{ f : [a,b] \to \mathbb{C} \mid (\text{Re} \Gamma - \text{Re} f(t))(\text{Re} f(t) - \text{Re} \gamma) + (\text{Im} \Gamma - \text{Im} f(t))(\text{Im} f(t) - \text{Im} \gamma) \geq 0 \text{ for a.e. } t \in [a,b] \}.
\]

Now, if we assume that \( \text{Re} \Gamma \geq \text{Re} \gamma \) and \( \text{Im} \Gamma \geq \text{Im} \gamma \), then we can define the following set of functions as well:

\[
\mathcal{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a,b] \to \mathbb{C} \mid \text{Re} \Gamma \geq \text{Re} f(t) \geq \text{Re} \gamma \}
\]

and \( \text{Im} \Gamma \geq \text{Im} f(t) \geq \text{Im} \gamma \) for a.e. \( t \in [a,b] \).

One can easily observe that \( \mathcal{S}_{[a,b]}(\gamma, \Gamma) \) is closed, convex and

\[
\emptyset \neq \mathcal{S}_{[a,b]}(\gamma, \Gamma) \subseteq U_{[a,b]}(\gamma, \Gamma).
\]

The following result holds [34]:

**Theorem 1.8.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \([a,b] \subset I \), the interior of \( I \). For some \( \gamma, \Gamma \in \mathbb{C} \), \( \gamma \neq \Gamma \), assume that \( \Phi' \in U_{[a,b]}(\gamma, \Gamma) (= \Delta_{[a,b]}(\gamma, \Gamma)) \). If \( g : \Omega \to [a,b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g \), \( g \in L(\Omega, \mu) \), then we have the inequality

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} |g - x| d\mu \quad (1.14)
\]
for any $x \in [a, b]$. In particular, we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \left| \int_{\Omega} g - \frac{a + b}{2} \right| d\mu \leq \frac{1}{4} (b - a) |\Gamma - \gamma| \tag{1.15}
\]
and
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \leq \frac{1}{2} |\Gamma - \gamma| \left( \int_{\Omega} g^2 d\mu - \left( \int_{\Omega} g d\mu \right)^2 \right)^{1/2} \leq \frac{1}{4} (b - a) |\Gamma - \gamma| . \tag{1.16}
\]

Motivated by the above results, in this paper we provide more upper bounds for the quantity
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \lambda \left( \int_{\Omega} g d\mu - x \right) \right| , \ x \in [a, b] ,
\]
under various assumptions on the absolutely continuous function $\Phi$, which in the particular case of $x = \int_{\Omega} g d\mu$ provides some results connected with Jensen’s inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski’s inequality. Applications for divergence measures are provided as well.

### 2. SOME IDENTITIES

The following result holds [34]:

**Lemma 2.1.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \bar{I}$, the interior of $I$. If $g : \Omega \to [a, b]$ is Lebesgue $\mu$-measurable on $\Omega$ and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have the equality
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \lambda \left( \int_{\Omega} g d\mu - x \right) \tag{2.1}
\]
for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi (x) = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \Phi' ((1 - s) x + sg) ds \right] d\mu , \tag{2.2}
\]
for any $x \in [a, b]$. 


Remark 2.2. With the assumptions of Lemma 2.1 we have
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) 
= \int_{\Omega} \left[ \left( g - \frac{a + b}{2} \right) \int_{0}^{1} \Phi' \left( (1 - s) \frac{a + b}{2} + sg \right) ds \right] d\mu.
\] (2.3)

Corollary 2.3. With the assumptions of Lemma 2.1 we have
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) 
= \int_{\Omega} \left[ \left( g - \int_{\Omega} g d\mu \right) \int_{0}^{1} \Phi' \left( (1 - s) \int_{\Omega} g d\mu + sg \right) ds \right] d\mu.
\] (2.4)

Proof. We observe that since \( g : \Omega \rightarrow [a, b] \) and \( \int_{\Omega} d\mu = 1 \) then \( \int_{\Omega} g d\mu \in [a, b] \) and by taking \( x = \int_{\Omega} g d\mu \) in (2.2) we get (2.4). \( \square \)

Corollary 2.4. With the assumptions of Lemma 2.1 we have
\[
\int_{\Omega} \Phi \circ g d\mu - \frac{1}{b - a} \int_{a}^{b} \Phi(x) dx - \lambda \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) 
= \int_{\Omega} \left\{ \frac{1}{b - a} \int_{a}^{b} \left[ (g - x) \int_{0}^{1} \Phi'((1 - s)x + sg) ds \right] dx \right\} d\mu.
\] (2.5)

Proof. Follows by integrating the identity (2.1) over \( x \in [a, b] \), dividing by \( b - a > 0 \) and using Fubini’s theorem. \( \square \)

Corollary 2.5. Let \( \Phi : I \rightarrow \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset \mathring{I} \), the interior of \( I \). If \( g, h : \Omega \rightarrow [a, b] \) are Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, \Phi \circ h, g, h \in L(\Omega, \mu) \), then we have the equality
\[
\int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu - \lambda \left( \int_{\Omega} g d\mu - \int_{\Omega} h d\mu \right) 
= \int_{\Omega} \int_{\Omega} \left[ \left( g(t) - h(\tau) \right) \int_{0}^{1} \Phi' \left( (1 - s) h(\tau) + sg(t) \right) ds \right] d\mu(t) d\mu(\tau)
\times d\mu(t) d\mu(\tau)
\] (2.6)

for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \).

In particular, we have
\[
\int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu 
= \int_{\Omega} \int_{\Omega} \left[ \left( g(t) - h(\tau) \right) \int_{0}^{1} \Phi' \left( (1 - s) h(\tau) + sg(t) \right) ds \right] d\mu(t) d\mu(\tau).
\] (2.7)

for any \( x \in [a, b] \).
Remark 2.6. The above inequality (2.6) can be extended for two measures as follows

\[
\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left( \int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right)
\]

(2.8)

\[
= \int_{\Omega_1} \int_{\Omega_2} \left[ (g(t) - h(\tau)) \int_0^1 \left( \Phi^\prime ((1 - s) h(\tau) + sg(t)) - \lambda \right) ds \right]
\]

\[
\times d\mu_1(t) d\mu_2(\tau),
\]

for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \) and provided that \( \Phi \circ g, g \in L(\Omega_1, \mu_1) \) while \( \Phi \circ h, h \in L(\Omega_2, \mu_2) \).

Remark 2.7. If \( w \geq 0 \) \( \mu \)-almost everywhere (\( \mu \)-a.e.) on \( \Omega \) with \( \int_{\Omega} w d\mu > 0 \), then by replacing \( d\mu \) with \( \frac{w d\mu}{\int_{\Omega} w d\mu} \) in (2.1) we have the weighted equality

\[
\frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w \left( \Phi \circ g \right) d\mu - \Phi(x) - \Phi^\prime(a) + \Phi^\prime(b)
\]

(3.1)

\[
\leq \frac{1}{2} \left( \Phi^\prime \right) \int_{\Omega} \left| g - x \right| d\mu
\]

for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \), provided \( \Phi \circ g, g \in L_w(\Omega, \mu) \) where

\[
L_w(\Omega, \mu) := \left\{ g \mid \int_{\Omega} w \left| g \right| d\mu < \infty \right\}.
\]

The other equalities have similar weighted versions. However the details are omitted.

3. INEQUALITIES FOR DERIVATIVES OF BOUNDED VARIATION

The following result holds:

**Theorem 3.1.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset I \), the interior of \( I \) and with the property that the derivative \( \Phi^\prime \) is of bounded variation on \( [a, b] \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) + \frac{\Phi^\prime(a) + \Phi^\prime(b)}{2} \left( \int_{\Omega} g d\mu - x \right) \right|
\]

(3.1)

\[
\leq \frac{1}{2} \left( \Phi^\prime \right) \int_{\Omega} \left| g - x \right| d\mu
\]

for any \( x \in [a, b] \).

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) + \frac{\Phi^\prime(a) + \Phi^\prime(b)}{2} \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) \right|
\]

(3.2)

\[
\leq \frac{1}{2} \left( \Phi^\prime \right) \int_{\Omega} \left| g - \frac{a + b}{2} \right| d\mu \leq \frac{1}{2} (b - a) \left( \Phi^\prime \right)
\]
and
\[
\left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \int_{\Omega} g \, d\mu \right) \right| \leq \frac{1}{2} \sqrt{\Phi' \left( \int_{\Omega} g^2 \, d\mu \right)} \left| \int_{\Omega} g \, d\mu \right| d\mu \tag{3.3}
\]
\[
\leq \frac{1}{2} \sqrt{\Phi' \left( \int_{\Omega} g^2 \, d\mu \right)} \left( \int_{\Omega} g^2 \, d\mu \right)^{1/2}
\]
\[
\leq \frac{1}{4} (b-a) \sqrt{\Phi'}.
\]

**Proof.** From the identity (2.1) we have
\[
\int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \int_{\Omega} g \, d\mu \right) - \Phi' \left( a \right) + \Phi' \left( b \right)
\]
\[
= \int_{\Omega} \left[ (g-x) \int_{0}^{1} \left( \Phi' ((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) ds \right] d\mu
\]
for any \( x \in [a,b] \).

Taking the modulus in (3.4) we get
\[
\left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \int_{\Omega} g \, d\mu \right) - \Phi' \left( a \right) + \Phi' \left( b \right) \right| \leq \int_{\Omega} \left| (g-x) \int_{0}^{1} \left( \Phi' ((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) ds \right| d\mu
\]
for any \( x \in [a,b] \).

Since \( \Phi' \) is of bounded variation on \([a,b]\), then for any \( s \in [0,1], x \in [a,b] \) and \( t \in \Omega \) we have
\[
\left| \Phi' ((1-s)x + sg(t)) - \frac{\Phi'(a) + \Phi'(b)}{2} \right|
\]
\[
= \frac{1}{2} \left| \Phi' ((1-s)x + sg(t)) - \Phi'(a) + \Phi' ((1-s)x + sg(t)) - \Phi'(b) \right|
\]
\[
\leq \frac{1}{2} \left| \Phi' ((1-s)x + sg(t)) - \Phi'(a) \right| + \left| \Phi'(b) - \Phi' ((1-s)x + sg(t)) \right|
\]
\[
\leq \frac{1}{2} \sqrt{\Phi'}.
\]
Then we have
\[
\int_{\Omega} |g-x| \int_{0}^{1} \left| \Phi' ((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds \, d\mu \tag{3.6}
\]
\[
\leq \frac{1}{2} \sqrt{\Phi'} \int_{\Omega} |g-x| \, d\mu
\]
for any \( x \in [a, b] \).

Making use of (3.5) and (3.6) we deduce the desired result (3.1). \( \square \)

Remark 3.2. Let \( \Phi : I \rightarrow \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subset \hat{I} \), the interior of \( I \) and with the property that the derivative \( \Phi' \) is of bounded variation on \([a, b] \). If \( x_i \in [m, M] \) and \( w_i \geq 0 \) \( (i = 1, \ldots, n) \) with \( W_n := \sum_{i=1}^{n} w_i = 1 \), then one has the weighted discrete inequality:

\[
\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi (x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^{n} w_i x_i - x \right) \right| (3.7)
\]

\[
\leq \frac{1}{2} \left( \Phi' \right) \sum_{i=1}^{n} w_i |x_i - x|
\]

for any \( x \in [a, b] \).

In particular, we have

\[
\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \frac{a + b}{2} \right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^{n} w_i x_i - \frac{a + b}{2} \right) \right| (3.8)
\]

\[
\leq \frac{1}{2} \left( \Phi' \right) \sum_{i=1}^{n} w_i \left| x_i - \frac{a + b}{2} \right| \leq \frac{1}{4} (b - a) \sqrt{\left( \Phi' \right)}
\]

and

\[
\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \right| \leq \frac{1}{2} \left( \Phi' \right) \sum_{i=1}^{n} w_i \left| x_i - \sum_{i=1}^{n} w_i x_i \right| (3.9)
\]

\[
\leq \frac{1}{2} \left( \Phi' \right) \left( \sum_{j=1}^{n} w_j x_j^2 - \left( \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2}
\]

\[
\leq \frac{1}{4} (b - a) \sqrt{\left( \Phi' \right)}.
\]

4. Inequalities for Lipschitzian Derivatives

The following result holds:

Theorem 4.1. Let \( \Phi : I \rightarrow \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subset \hat{I} \), the interior of \( I \) and with the property that the derivative \( \Phi' \) is Lipschitzian with the constant \( K > 0 \) on \([a, b] \). If \( g : \Omega \rightarrow [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L (\Omega, \mu) \), then we have

\[
\left| \int_{\Omega} \Phi \circ g \mu - \Phi (x) \left( \int_{\Omega} g \mu - x \right) \right| (4.1)
\]

\[
\leq \frac{1}{2} K \left[ \sigma^2 (g) + \left( \int_{\Omega} g \mu - x \right)^2 \right]
\]
for any $x \in [a,b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$
\sigma_\mu(g) := \left( \int_{\Omega} \left( g - \int_{\Omega} gd\mu \right)^2 d\mu \right)^{1/2} = \left( \int_{\Omega} g^2 d\mu - \left( \int_{\Omega} gd\mu \right)^2 \right)^{1/2}.
$$

In particular, we have

$$
\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) \right| \leq \frac{1}{2} K \left[ \sigma_\mu^2(g) + \left( \int_{\Omega} gd\mu - \frac{a+b}{2} \right)^2 \right] \tag{4.2}
$$

and

$$
\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) \right| \leq \frac{1}{2} K \sigma_\mu^2(g) \leq \frac{1}{8} K (b-a)^2. \tag{4.3}
$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$
\int_{\Omega} \Phi \circ gd\mu = \Phi \left( \int_{\Omega} gd\mu \right) - \Phi'(x) \left( \int_{\Omega} gd\mu - x \right) \tag{4.4}
$$

$$
= \int_{\Omega} \left[ (g-x) \int_0^1 (\Phi'((1-s)x+sg) - \Phi'(x)) \, ds \right] d\mu
$$

for any $x \in [a,b]$.

Taking the modulus in (4.4) we get

$$
\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) \right| \leq \int_{\Omega} |g-x| \left| \int_0^1 (\Phi'((1-s)x+sg) - \Phi'(x)) \, ds \right| d\mu
$$

$$
\leq \int_{\Omega} |g-x| \int_0^1 \left| (\Phi'((1-s)x+sg) - \Phi'(x)) \right| ds \, d\mu
$$

$$
\leq K \int_{\Omega} \left| g-x \right| \int_0^1 s |g-x| \, ds \, d\mu = \frac{1}{2} K \int_{\Omega} (g-x)^2 \, d\mu
$$

for any $x \in [a,b]$. 

However,
\[
\int_{\Omega} (g - x)^2 \, d\mu \\
= \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu + \int_{\Omega} g \, d\mu - x \right)^2 \, d\mu \\
= \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu \right)^2 \, d\mu + 2 \int_{\Omega} g \left( \int_{\Omega} g \, d\mu - x \right) \, d\mu \\
+ \int_{\Omega} \left( \int_{\Omega} g \, d\mu - x \right)^2 \, d\mu \\
= \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu \right)^2 \, d\mu + \left( \int_{\Omega} g \, d\mu - x \right)^2
\]
for any \( x \in [a, b] \), and by (4.5) we get the desired result (4.1). \( \square \)

**Corollary 4.2.** Let \( \Phi : I \to \mathbb{C} \) be a twice differentiable functions on \( [a, b] \subset \bar{I} \) with \( \| \Phi'' \|_{[a,b],\infty} := \text{ess sup}_{t \in [a,b]} |\Phi''(t)| < \infty \). Then the inequalities (4.1)-(4.3) hold for \( K = \| \Phi'' \|_{[a,b],\infty} \).

**Remark 4.3.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset \bar{I} \) and with the property that the derivative \( \Phi' \) is Lipschitzian with the constant \( K > 0 \) on \( [a, b] \). If \( x_i \in [m, M] \) and \( w_i \geq 0 \) \( (i = 1, \ldots, n) \) with \( W_n := \sum_{i=1}^{n} w_i = 1 \), then one has the weighted discrete inequality:

\[
\left| \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi(x) \left( \sum_{i=1}^{n} w_i x_i - x \right) \right| \leq \frac{1}{2} K \left[ \sigma_w^2(x) + \left( \sum_{i=1}^{n} w_i x_i - x \right)^2 \right]
\]

for any \( x \in [a, b] \), where

\[
\sigma_w(x) := \left( \sum_{i=1}^{n} w_i \left( x_i - \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2} = \left( \sum_{i=1}^{n} w_i x_i^2 - \left( \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2}.
\]

The following lemma may be stated:

**Lemma 4.4.** Let \( u : [a, b] \to \mathbb{R} \) and \( l, L \in \mathbb{R} \) with \( L > l \). The following statements are equivalent:

(i) The function \( u - \frac{L + l}{2} e \), where \( e(t) = t \), \( t \in [a, b] \) is \( \frac{1}{2} (L - l) \) -Lipschitzian;

(ii) We have the inequalities

\[
l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each} \quad t, s \in [a, b] \quad \text{with} \quad t \neq s;
\]

(iii) We have the inequalities

\[
l (t - s) \leq u(t) - u(s) \leq L (t - s) \quad \text{for each} \quad t, s \in [a, b] \quad \text{with} \quad t > s.
\]
Following [53], we can introduce the definition of \((l, L)\)-Lipschitzian functions:

**Definition 4.5.** The function \(u : [a, b] \to \mathbb{R}\) which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be \((l, L)\)-Lipschitzian on \([a, b]\).

If \(L > 0\) and \(l = -L\), then \((-L, L)\)-Lipschitzian means \(L\)-Lipschitzian in the classical sense.

Utilising Lagrange’s mean value theorem, we can state the following result that provides examples of \((l, L)\)-Lipschitzian functions.

**Proposition 4.6.** Let \(u : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \(-\infty < l = \inf_{t \in [a, b]} u'(t)\) and \(\sup_{t \in [a, b]} u'(t) = L < \infty\), then \(u\) is \((l, L)\)-Lipschitzian on \([a, b]\).

The following result holds.

**Corollary 4.7.** Let \(\Phi : I \to \mathbb{R}\) be an absolutely continuous functions on \([a, b] \subset \bar{I}\), with the property that the derivative \(\Phi'\) is \((l, L)\)-Lipschitzian on \([a, b]\), where \(l, L \in \mathbb{R}\) with \(L > l\). If \(g : \Omega \to [a, b]\) is Lebesgue \(\mu\)-measurable on \(\Omega\) and such that \(\Phi \circ g, g \in L(\Omega, \mu)\), then we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi'(x) \left( \int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{4} (L + l) \left[ \sigma^2_{\mu}(g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right] (4.9)
\]

for any \(x \in [a, b]\).

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) - \Phi' \left( \frac{a + b}{2} \right) \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) \right| \leq \frac{1}{4} (L + l) \left[ \sigma^2_{\mu}(g) + \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right)^2 \right] (4.10)
\]

and

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) - \frac{1}{4} (L + l) \sigma^2_{\mu}(g) \right| \leq \frac{1}{4} (L - l) \sigma^2_{\mu}(g) (4.11)
\]

\[
\leq \frac{1}{16} (L - l) (b - a)^2.
\]
**Proof.** Consider the auxiliary function $\Psi : [a, b] \to \mathbb{R}$ given by

$$\Psi (x) = \Phi (x) - \frac{1}{4} (L + l) x^2.$$  

We observe that $\Psi$ is differentiable and

$$\Psi'(x) = \Phi'(x) - \frac{1}{2} (L + l) x.$$  

Since $\Phi'$ is $(l, L)$-Lipschitzian on $[a, b]$ it follows that $\Psi'$ is Lipschitzian with the constant $\frac{1}{2} (L - l)$, so we can apply Theorem 4.1 for $\Psi$, i.e. we have the inequality

$$\left| \int_{\Omega} \Psi \circ g \, d\mu - \Psi (x) \left( \int_{\Omega} g \, d\mu - x \right) \right| \leq \frac{1}{4} (L - l) \left[ \sigma_\mu^2 (g) + \left( \int_{\Omega} g \, d\mu - x \right)^2 \right].$$  

However

$$\int_{\Omega} \Psi \circ g \, d\mu - \Psi (x) \left( \int_{\Omega} g \, d\mu - x \right)$$

$$= \int_{\Omega} \Phi \circ g \, d\mu - \Phi (x) \left( \int_{\Omega} g \, d\mu - x \right)$$

$$- \frac{1}{4} (L + l) \left[ \int_{\Omega} g^2 \, d\mu - x^2 - 2x \left( \int_{\Omega} g \, d\mu - x \right) \right]$$

$$= \int_{\Omega} \Phi \circ g \, d\mu - \Phi (x) \left( \int_{\Omega} g \, d\mu - x \right)$$

$$- \frac{1}{4} (L + l) \left[ \sigma_\mu^2 (g) + \left( \int_{\Omega} g \, d\mu - x \right)^2 \right]$$

and by (4.12) we get the desired result (4.9). \qed

**Remark 4.8.** We observe that if the function $\Phi$ is twice differentiable on $I$ and for $[a, b] \subset \bar{I}$ we have

$$-\infty < l \leq \Phi'' (x) \leq L < \infty$$

for any $x \in [a, b]$, then $\Phi'$ is $(l, L)$-Lipschitzian on $[a, b]$ and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

**Theorem 4.9.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \bar{I}$, the interior of $I$ and with the property that the derivative $\Phi'$ is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue $\mu$-measurable on
\( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have

\[
\left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \int_{\Omega} g \, d\mu \right) \left( \int_{\Omega} g \, d\mu - x \right) \right| \leq \frac{1}{2} K \left( \| g \|_{\Omega, \infty}^2 + \| g - \int_{\Omega} g \, d\mu \|_{\Omega, \infty} \right) \int_{\Omega} |g - x| \, d\mu
\]

for any \( x \in [a, b] \), where

\[
\| g - \int_{\Omega} g \, d\mu \|_{\Omega, \infty} := \text{ess sup}_{t \in \Omega} |g(t) - \int_{\Omega} g \, d\mu| < \infty.
\]

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \frac{a + b}{2} \right) - \Phi' \left( \int_{\Omega} g \, d\mu \right) \left( \int_{\Omega} g \, d\mu - \frac{a + b}{2} \right) \right| \leq \frac{1}{2} K \left[ \left( \frac{a + b}{2} - \int_{\Omega} g \, d\mu \right) \| g - \frac{a + b}{2} \|_{\Omega, \infty} \right] \int_{\Omega} |g - \frac{a + b}{2}| \, d\mu.
\]

Proof. From the identity (2.1) we have for \( \lambda = \Phi' \left( \int_{\Omega} g \, d\mu \right) \) that

\[
\int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( x \right) - \Phi' \left( \int_{\Omega} g \, d\mu \right) \left( \int_{\Omega} g \, d\mu - x \right) = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \Phi' \left( (1 - s) x + sg \right) - \Phi' \left( \int_{\Omega} g \, d\mu \right) \right] \, ds \, d\mu
\]

for any \( x \in [a, b] \).
Taking the modulus in (4.15) we get
\[ \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) \left( \int_\Omega g d\mu \right) \left( \int_\Omega g d\mu - x \right) \right| \] (4.16)
\[ \leq \int_\Omega |g - x| \left| \left( \Phi'( (1 - s) x + sg ) - \Phi'( \left( \int_\Omega g d\mu \right) ) \right) ds \right| d\mu \]
\[ \leq \int_\Omega \left[ |g - x| \int_0^1 \left( \Phi'( (1 - s) x + sg ) - \Phi'( \left( \int_\Omega g d\mu \right) ) \right) ds \right] d\mu \]
\[ \leq K \int_\Omega \left[ |g - x| \int_0^1 \left( (1 - s) x + sg - \int_\Omega g d\mu \right) ds \right] d\mu \]
\[ = K \int_\Omega \left[ |g - x| \int_0^1 \left( (1 - s) x + sg - (1 - s) \int_\Omega g d\mu - s \int_\Omega g d\mu \right) ds \right] d\mu \]
\[ := B. \]

Using the triangle inequality we have for any \( t \in \Omega \)
\[ \int_0^1 \left| (1 - s) x + sg(t) - (1 - s) \int_\Omega g d\mu - s \int_\Omega g d\mu \right| ds \]
\[ \leq \int_0^1 \left| (1 - s) x - \int_\Omega g d\mu \right| ds + \int_0^1 \left| g(t) - \int_\Omega g d\mu \right| ds \]
\[ = \frac{1}{2} \left[ \left| x - \int_\Omega g d\mu \right| + \left| g(t) - \int_\Omega g d\mu \right| \right] \]

and then
\[ B \leq \frac{1}{2} K \int_\Omega |g - x| \left[ \left| x - \int_\Omega g d\mu \right| + \left| g(t) - \int_\Omega g d\mu \right| \right] d\mu \] (4.17)
\[ = \frac{1}{2} K \left[ \left| x - \int_\Omega g d\mu \right| \int_\Omega |g - x| d\mu + \int_\Omega |g - x| d\mu \right]. \]

Making use of (4.16) and (4.17) we deduce the desired result (4.13). □

**Corollary 4.10.** Let \( \Phi : I \to \mathbb{R} \) be an absolutely continuous functions on \( [a, b] \subset I \), with the property that the derivative \( \Phi' \) is \((l, L)\)-Lipschitzian on \( [a, b] \), where \( l, L \in \mathbb{R} \) with \( L > l \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have
\[ \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) \left( \int_\Omega g d\mu \right) \left( \int_\Omega g d\mu - x \right) \right| \] (4.18)
\[ \leq \frac{1}{4} (L - l) \left[ \left| x - \int_\Omega g d\mu \right| \int_\Omega |g - x| d\mu + \int_\Omega |g - x| d\mu \right] \]
\[ \leq \frac{1}{4} (L - l) \left[ \left| x - \int_\Omega g d\mu \right| + \left| g - \int_\Omega g d\mu \right| \right] \int_\Omega |g - x| d\mu \]
for any \( x \in [a, b] \).

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \right| \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \\
- \frac{1}{4} (L + l) \left[ \sigma^2_{\mu}(g) - \left( \frac{a+b}{2} - \int_{\Omega} g d\mu \right)^2 \right] \\
\leq \frac{1}{4} (L - l) \left[ \left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\
+ \int_{\Omega} \left| g - \frac{a+b}{2} \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \\
\leq \frac{1}{4} (L - l) \left[ \left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| + \left| g - \int_{\Omega} g d\mu \right| \right] \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu.
\] (4.19)

5. APPLICATIONS FOR \( f \)-DIVERGENCE

One of the important issues in many applications of Probability Theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár \( f \)-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set \( \Omega \) and the \( \sigma \)-finite measure \( \mu \) are given. Consider the set of all probability densities on \( \mu \) to be \( \mathcal{P} := \{ p | p : \Omega \rightarrow \mathbb{R}, p(t) \geq 0, \int_{\Omega} p(t) d\mu(t) = 1 \} \).

The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

\[
D_{KL}(p, q) := \int_{\Omega} p(t) \ln \left( \frac{p(t)}{q(t)} \right) d\mu(t), \quad p, q \in \mathcal{P},
\] (5.1)

where \( \ln \) is to base \( e \).

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: variation distance \( D_v \), Hellinger distance \( D_H \) [45], \( \chi^2 \)-divergence \( D_{\chi^2} \), \( \alpha \)-divergence \( D_\alpha \), Bhattacharyya distance \( D_B \) [3], Harmonic distance \( D_{Ha} \), Jeffreys’ distance \( D_J \) [47],
triangular discrimination $D_{\Delta}$ [65], etc... They are defined as follows:

\[
D_v \left( p, q \right) := \int_{\Omega} \left| p(t) - q(t) \right| d\mu(t), \ p, q \in \mathcal{P};
\]

(5.2)

\[
D_H \left( p, q \right) := \int_{\Omega} \left| \sqrt{p(t)} - \sqrt{q(t)} \right| d\mu(t), \ p, q \in \mathcal{P};
\]

(5.3)

\[
D_{\chi^2} \left( p, q \right) := \int_{\Omega} \left( \frac{q(t)}{p(t)} \right)^2 - 1 \right] d\mu(t), \ p, q \in \mathcal{P};
\]

(5.4)

\[
D_{\alpha} \left( p, q \right) := \frac{4}{1-\alpha^2} \left[ 1 - \int_{\Omega} \left[ p(t) \frac{1-\alpha}{p(t)} q(t) \frac{1+\alpha}{q(t)} d\mu(t) \right] \right], \ p, q \in \mathcal{P};
\]

(5.5)

\[
D_B \left( p, q \right) := \int_{\Omega} \sqrt{p(t)q(t)} d\mu(t), \ p, q \in \mathcal{P};
\]

(5.6)

\[
D_{Ha} \left( p, q \right) := \int_{\Omega} \frac{2p(t)q(t)}{p(t)+q(t)} d\mu(t), \ p, q \in \mathcal{P};
\]

(5.7)

\[
D_f \left( p, q \right) := \int_{\Omega} \left[ p(t) - q(t) \right] \ln \left[ \frac{p(t)}{q(t)} \right] d\mu(t), \ p, q \in \mathcal{P};
\]

(5.8)

\[
D_{\Delta} \left( p, q \right) := \int_{\Omega} \frac{\left[ p(t) - q(t) \right]^2}{p(t) + q(t)} d\mu(t), \ p, q \in \mathcal{P}.
\]

(5.9)

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár $f$-divergence is defined as follows [13]

\[
I_f \left( p, q \right) := \int_{\Omega} p(t) f \left[ \frac{q(t)}{p(t)} \right] d\mu(t), \ p, q \in \mathcal{P},
\]

(5.10)

where $f$ is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár $f$-divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár $f$-divergence see [13], [14] and [68].

The following result holds:

**Proposition 5.1.** Let $f : (0, \infty) \to \mathbb{R}$ be a twice differentiable convex function with the property that $f (1) = 0$ and there exists the constants $\gamma, \Gamma$ so that

\[-\infty < \gamma \leq f(t) \leq \Gamma < \infty.\]

Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

\[r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega.
\]

(5.11)
If \( x \in [r, R] \), then we have the inequality
\[
\left| I_f (p, q) - f (x) - f' (x) (1 - x) - \frac{1}{4} (L + l) D^{2} (p, q) + (1 - x)^2 \right| (5.12)
\]
\[
\leq \frac{1}{4} (L - l) D^{2} (p, q) + (1 - x)^2 .
\]
In particular, we have
\[
\left| I_f (p, q) - f \left( \frac{r + R}{2} \right) - f' \left( \frac{r + R}{2} \right) (1 - \frac{r + R}{2}) \right|
\]
\[
- \frac{1}{4} (L + l) D^{2} (p, q) + \left( 1 - \frac{r + R}{2} \right)^2 \leq \frac{1}{4} (L - l) D^{2} (p, q) + \left( 1 - \frac{r + R}{2} \right)^2
\]
and
\[
\left| I_f (p, q) - \frac{1}{4} (L + l) D^{2} (p, q) \right| \leq \frac{1}{4} (L - l) D^{2} (p, q).
\] (5.14)

**Proof.** From (4.9) we have
\[
\left| \int_{\Omega} p (t) f \left( \frac{q (t)}{p (t)} \right) d\mu (t) - f (x) - f' (x) (1 - x) \right|
\]
\[
- \frac{1}{4} (L + l) \left| \int_{\Omega} p (t) \left( \frac{q (t)}{p (t)} \right)^2 d\mu (t) - 1 + (1 - x)^2 \right|
\]
\[
\leq \frac{1}{4} (L - l) \left| \int_{\Omega} p (t) \left( \frac{q (t)}{p (t)} \right)^2 d\mu (t) - 1 + (1 - x)^2 \right|
\]
for any \( x \in [r, R] \), which is equivalent to (5.12). \( \square \)

Using Corollary 4.10 we can state the following result as well:

**Proposition 5.2.** With the assumptions in Proposition 5.1, we have
\[
\left| I_f (p, q) - f (x) - f' (1) (1 - x) - \frac{1}{4} (L + l) D^{2} (p, q) - (1 - x)^2 \right| (5.15)
\]
\[
\leq \frac{1}{4} (L - l) \left| \left[ x - 1 \right] \int_{\Omega} |q - xp| d\mu + \int_{\Omega} |q - xp| \left| \frac{q}{p} - 1 \right| d\mu \right|
\]
\[
\leq \frac{1}{4} (L - l) \left[ x - 1 \right] + \left| \left[ \frac{q}{p} - 1 \right] \right|_{\Omega, \infty} \int_{\Omega} |q - xp| d\mu
\]
for any \( x \in [r, R] \).

If we consider the convex function \( f : (0, \infty) \to \mathbb{R}, f (t) = t \ln t \) then
\[
I_f (p, q) := \int_{\Omega} p (t) \frac{q (t)}{p (t)} \ln \left[ \frac{q (t)}{p (t)} \right] d\mu (t) = \int_{\Omega} q (t) \ln \left[ \frac{q (t)}{p (t)} \right] d\mu (t)
\]
\[
= D_{KL} (q, p).
\]
We have $f'(t) = \ln t + 1$ and $f''(t) = \frac{1}{t}$ and then we can choose $l = \frac{1}{R}$ and $L = \frac{1}{r}$. Applying the inequality (5.14) we get

$$
|D_{KL}(q,p) - \left(\frac{R + r}{4rR}\right)\chi^2(p,q)| \leq \frac{R - r}{4rR}D_{\chi^2}(p,q).
$$

(5.16)

If we consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = -\ln t$ then

$$
I_f(p,q) := -\int_{\Omega} p(t) \ln \left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_{\Omega} p(t) \ln \left[\frac{p(t)}{q(t)}\right] d\mu(t) = D_{KL}(p,q).
$$

We have $f'(t) = -\frac{1}{t}$ and $f''(t) = \frac{1}{t^2}$ and then we can choose $l = \frac{1}{R^2}$ and $L = \frac{1}{r^2}$. Applying the inequality (5.14) we get

$$
|D_{KL}(p,q) - \frac{R^2 + r^2}{4R^2r^2}D_{\chi^2}(p,q)| \leq \frac{R^2 - r^2}{4R^2r^2}D_{\chi^2}(p,q).
$$

(5.17)

ACKNOWLEDGMENTS

The author would like to thank the referees for giving fruitful advices.

REFERENCES

23. S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ where $f$ is of Hölder type and $u$ is of bounded variation and applications, *J. KSIAM*, 5(1), (2001), 35-45.