New Jensen and Ostrowski Type Inequalities for General Lebesgue Integral with Applications

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Abstract. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for $f$-divergence measure are provided as well.

Keywords: Ostrowski’s inequality, Jensen’s inequality, $f$-Divergence measures.


1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set $\Omega$, a $\sigma$ – algebra $\mathcal{A}$ of parts of $\Omega$ and a countably additive and positive measure $\mu$ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_{\Omega} d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R}, \ f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(t)| \, d\mu(t) < \infty\}.$$ 

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w \, d\mu$ instead of $\int_{\Omega} w(t) \, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen’s integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:
Theorem 1.1. Let $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex function on $(m, M)$ and $f : \Omega \rightarrow [m, M]$ so that $\Phi \circ f$, $f$, $\Phi' \circ f$, $(\Phi' \circ f) : f \in L(\Omega, \mu)$. Then we have the inequality:

$$0 \leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left( \int_{\Omega} f d\mu \right)$$

(1.1)

$$\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu$$

$$\leq \frac{1}{2} \left[ \Phi' (M) - \Phi' (m) \right] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu.$$

In the case of discrete measure, we have:

Corollary 1.2. Let $\Phi : [m, M] \rightarrow \mathbb{R}$ be a differentiable convex function on $(m, M)$. If $x_i \in [m, M]$ and $w_i \geq 0$ ($i = 1, \ldots, n$) with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the counterpart of Jensen’s weighted discrete inequality:

$$0 \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi (\sum_{i=1}^{n} w_i x_i)$$

(1.2)

$$\leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \Phi' (x_i) w_j x_j$$

$$\leq \frac{1}{2} \left[ \Phi' (M) - \Phi' (m) \right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|.$$

Remark 1.3. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

If $f, g : \Omega \rightarrow \mathbb{R}$ are $\mu$-measurable functions and $f, g, fg \in L(\Omega, \mu)$, then we may consider the Čebyšev functional

$$T (f, g) := \int_{\Omega} fg d\mu - \int_{\Omega} f d\mu \int_{\Omega} g d\mu.$$  

(1.3)

The following result is known in the literature as the Grüss inequality

$$|T (f, g)| \leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta),$$

(1.4)

provided

$$-\infty < \gamma \leq f (t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g (t) \leq \Delta < \infty$$

(1.5)

for $\mu$–a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f (t) \leq \Gamma < \infty$ for $\mu$–a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz’s integral inequality, we have

$$\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \left[ \int_{\Omega} f^2 d\mu - \left( \int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} (\Gamma - \gamma).$$

(1.6)
On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

\[ 0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right) \]

\[ \leq \int_{\Omega} f \cdot (\Phi' \circ f) \, d\mu - \int_{\Omega} \Phi' \circ f \, d\mu \int_{\Omega} f \, d\mu \]

\[ \leq \frac{1}{2} \left( \Phi'(M) - \Phi'(m) \right) \int_{\Omega} \left| f - \int_{\Omega} f \, d\mu \right| \, d\mu \]

\[ \leq \frac{1}{2} \left( \Phi'(M) - \Phi'(m) \right) \left[ \int_{\Omega} f^2 \, d\mu - \left( \int_{\Omega} f \, d\mu \right)^2 \right]^{\frac{1}{2}} \]

\[ \leq \frac{1}{4} \left( \Phi'(M) - \Phi'(m) \right) \frac{(M - M)}{M - m}, \]

provided that \( \Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R} \) is a differentiable convex function on \( (m, M) \) and \( f : \Omega \rightarrow [m, M] \) so that \( \Phi \circ f, f' \circ f, f' \circ (\Phi' \circ f) \in L(\Omega, \mu) \), with \( \int_{\Omega} d\mu = 1 \).

The following reverse of the Jensen’s inequality also holds [33]:

**Theorem 1.4.** Let \( \Phi : I \rightarrow \mathbb{R} \) be a continuous convex function on the interval of real numbers \( I \) and \( m, M \in \mathbb{R}, m < M \) with \( [m, M] \subset \bar{I} \), where \( \bar{I} \) is the interior of \( I \). If \( f : \Omega \rightarrow \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds

\[-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega\]

and such that \( f, \Phi \circ f \in L(\Omega, \mu) \), then

\[ 0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right) \]

\[ \leq \left( M - \int_{\Omega} f \, d\mu \right) \left( \int_{\Omega} f \, d\mu - m \right) \frac{\Phi'(M) - \Phi'(m)}{M - m} \]

\[ \leq \frac{1}{4} (M - m) \left[ \Phi'_{-}(M) - \Phi'_{+}(m) \right] , \]

where \( \Phi'_{-} \) is the left and \( \Phi'_{+} \) is the right derivative of the convex function \( \Phi \).

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean \( \frac{1}{b-a} \int_{a}^{b} \Phi(t) \, dt \) and the value \( \Phi(x), x \in [a, b] \).

For various results related to Ostrowski’s inequality see [6]-[9], [15]-[41], [43] and the references therein.

**Theorem 1.5.** Let \( \Phi : [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\) such that \( \Phi' : (a, b) \rightarrow \mathbb{R} \) is bounded on \((a, b)\), i.e., \( \| \Phi' \|_{\infty} := \sup_{t \in (a, b)} |\Phi'(t)| < \)
Then
\[
\left| \Phi (x) - \frac{1}{b-a} \int_a^b f (t) \, dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \| \Phi' \|_\infty (b-a),
\]
(1.9)

for all \( x \in [a,b] \) and the constant \( \frac{1}{4} \) is the best possible.

Now, for \( \gamma, \Gamma \in \mathbb{C} \) and \([a,b]\) an interval of real numbers, define the sets of complex-valued functions [34]

\[ U_{[a,b]} (\gamma, \Gamma) := \{ f : [a,b] \to \mathbb{C} \mid \text{Re} \left[ (\Gamma - f(t)) \left( \overline{f(t)} - \gamma \right) \right] \geq 0 \text{ for almost every } t \in [a,b] \} \]

and

\[ \Delta_{[a,b]} (\gamma, \Gamma) := \left\{ f : [a,b] \to \mathbb{C} \mid \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a,b] \right\}. \]

The following representation result may be stated [34].

**Proposition 1.6.** For any \( \gamma, \Gamma \in \mathbb{C}, \gamma \neq \Gamma \), we have that \( U_{[a,b]} (\gamma, \Gamma) \) and \( \Delta_{[a,b]} (\gamma, \Gamma) \) are nonempty, convex and closed sets and

\[ U_{[a,b]} (\gamma, \Gamma) = \overline{\Delta_{[a,b]} (\gamma, \Gamma)}. \]  
(1.10)

On making use of the complex numbers field properties we can also state that:

**Corollary 1.7.** For any \( \gamma, \Gamma \in \mathbb{C}, \gamma \neq \Gamma \), we have that

\[
U_{[a,b]} (\gamma, \Gamma) = \{ f : [a,b] \to \mathbb{C} \mid (\text{Re} \Gamma - \text{Re} f(t)) (\text{Re} f(t) - \text{Re} \gamma) + (\text{Im} \Gamma - \text{Im} f(t)) (\text{Im} f(t) - \text{Im} \gamma) \geq 0 \text{ for a.e. } t \in [a,b] \}. \]  
(1.11)

Now, if we assume that \( \text{Re} (\Gamma) \geq \text{Re} (\gamma) \) and \( \text{Im} (\Gamma) \geq \text{Im} (\gamma) \), then we can define the following set of functions as well:

\[
\overline{S}_{[a,b]} (\gamma, \Gamma) := \{ f : [a,b] \to \mathbb{C} \mid \text{Re} (\Gamma) \geq \text{Re} f(t) \geq \text{Re} (\gamma) \} \quad \text{ and } \quad \text{Im} (\Gamma) \geq \text{Im} f(t) \geq \text{Im} (\gamma) \text{ for a.e. } t \in [a,b] \}. \]
(1.12)

One can easily observe that \( \overline{S}_{[a,b]} (\gamma, \Gamma) \) is closed, convex and

\[ \emptyset \neq \overline{S}_{[a,b]} (\gamma, \Gamma) \subseteq U_{[a,b]} (\gamma, \Gamma). \]  
(1.13)

The following result holds [34]:

**Theorem 1.8.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a,b] \subset I \), the interior of \( I \). For some \( \gamma, \Gamma \in \mathbb{C}, \gamma \neq \Gamma \), assume that \( \Phi' \in U_{[a,b]} (\gamma, \Gamma) (= \Delta_{[a,b]} (\gamma, \Gamma)) \). If \( g : \Omega \to [a,b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have the inequality

\[
\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma + \Gamma}{2} \left( \int_\Omega g d\mu - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_\Omega |g - x| d\mu \]
(1.14)
for any \( x \in [a, b] \).

In particular, we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{4} (b-a) |\Gamma - \gamma| \tag{1.15}
\]
and
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \leq \frac{1}{2} (b-a) |\Gamma - \gamma| \tag{1.16}
\]

Motivated by the above results, in this paper we provide more upper bounds for the quantity
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| - \lambda \left( \int_{\Omega} g d\mu - x \right), \quad x \in [a, b],
\]
under various assumptions on the absolutely continuous function \( \Phi \), which in the particular case of \( x = \int_{\Omega} g d\mu \) provides some results connected with Jensen’s inequality while in the case \( \lambda = 0 \) provides some generalizations of Ostrowski’s inequality. Applications for divergence measures are provided as well.

2. Some Identities

The following result holds \([34]\):

**Lemma 2.1.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset \bar{I} \), the interior of \( I \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, \ g \in L(\Omega, \mu) \), then we have the equality
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) - \lambda \left( \int_{\Omega} g d\mu - x \right), \quad x \in [a, b],
\]
for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \).

In particular, we have
\[
\int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) = \int_{\Omega} \left[ (g-x) \int_{0}^{1} \Phi'((1-s)x+sg) - \lambda \right] d\mu \tag{2.2}
\]
for any \( x \in [a, b] \).
Remark 2.2. With the assumptions of Lemma 2.1 we have
\[ \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \frac{a + b}{2} \right) = \int_{\Omega} \left[ \left( g - \frac{a + b}{2} \right) \int_{0}^{1} \Phi' \left( (1 - s) \frac{a + b}{2} + sg \right) \, ds \right] \, d\mu. \] (2.3)

Corollary 2.3. With the assumptions of Lemma 2.1 we have
\[ \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \int_{\Omega} g \, d\mu \right) = \int_{\Omega} \left[ \left( g - \int_{\Omega} g \, d\mu \right) \int_{0}^{1} \Phi' \left( (1 - s) \int_{\Omega} g \, d\mu + sg \right) \, ds \right] \, d\mu. \] (2.4)

Proof. We observe that since \( g : \Omega \to [a, b] \) and \( \int_{\Omega} d\mu = 1 \) then \( \int_{\Omega} g \, d\mu \in [a, b] \) and by taking \( x = \int_{\Omega} g \, d\mu \) in (2.2) we get (2.4). \( \square \)

Corollary 2.4. With the assumptions of Lemma 2.1 we have
\[ \int_{\Omega} \Phi \circ g \, d\mu - \frac{1}{b - a} \int_{a}^{b} \Phi (x) \, dx - \lambda \left( \int_{\Omega} g \, d\mu - \frac{a + b}{2} \right) = \int_{\Omega} \left\{ \frac{1}{b - a} \int_{a}^{b} \left[ (g - x) \int_{0}^{1} \Phi' \left( (1 - s) x + sg \right) \, ds \right] \, dx \right\} \, d\mu. \] (2.5)

Proof. Follows by integrating the identity (2.1) over \( x \in [a, b] \), dividing by \( b - a > 0 \) and using Fubini’s theorem. \( \square \)

Corollary 2.5. Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset \bar{I} \), the interior of \( I \). If \( g, h : \Omega \to [a, b] \) are Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, \Phi \circ h, g, h \in L(\Omega, \mu) \), then we have the equality
\[ \int_{\Omega} \Phi \circ g \, d\mu - \int_{\Omega} \Phi \circ h \, d\mu - \lambda \left( \int_{\Omega} g \, d\mu - \int_{\Omega} h \, d\mu \right) = \int_{\Omega} \int_{\Omega} \left[ (g(t) - h(\tau)) \int_{0}^{1} \Phi' \left( (1 - s) h(\tau) + sg(t) \right) - \lambda \right] \, ds \times d\mu(t) d\mu(\tau), \] (2.6)

for any \( \lambda \in \mathbb{C} \) and \( x \in [a, b] \).

In particular, we have
\[ \int_{\Omega} \Phi \circ g \, d\mu - \int_{\Omega} \Phi \circ h \, d\mu = \int_{\Omega} \int_{\Omega} \left[ (g(t) - h(\tau)) \int_{0}^{1} \Phi' \left( (1 - s) h(\tau) + sg(t) \right) \, ds \right] \, d\mu(t) d\mu(\tau), \] (2.7)

for any \( x \in [a, b] \).
Remark 2.6. The above inequality (2.6) can be extended for two measures as follows
\[
\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left( \int_{\Omega_1} gd\mu_1 - \int_{\Omega_2} hd\mu_2 \right)
\]
\[
= \int_{\Omega_1} \int_{\Omega_2} \left[ (g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) \, ds \right]
\]
\[
\times d\mu_1(t) \, d\mu_2(\tau),
\]
for any \( \lambda \in \mathbb{C} \) and \( x \in [a,b] \) and provided that \( \Phi \circ g, g \in L(\Omega_1, \mu_1) \) while \( \Phi \circ h, h \in L(\Omega_2, \mu_2) \).

Remark 2.7. If \( w \geq 0 \) \( \mu \)-almost everywhere \((\mu\text{-a.e.})\) on \( \Omega \) with \( \int_{\Omega} wd\mu > 0 \), then by replacing \( d\mu \) with \( \frac{w}{\int_{\Omega} wd\mu} \) in (2.1) we have the weighted equality
\[
\frac{1}{\int_{\Omega} wd\mu} \int_{\Omega} w (\Phi \circ g) d\mu - \Phi(x) - \Phi' \left( \frac{a+b}{2} \right) \left( \int_{\Omega} gd\mu - x \right)
\]
\[
\leq \frac{1}{2} b \max_{a} (\Phi') \int_{\Omega} |g - x| \, d\mu
\]
for any \( \lambda \in \mathbb{C} \) and \( x \in [a,b] \), provided \( \Phi \circ g, g \in L_w(\Omega, \mu) \) where
\[
L_w(\Omega, \mu) := \left\{ g \mid \int_{\Omega} |w| |g| \, d\mu < \infty \right\}.
\]
The other equalities have similar weighted versions. However the details are omitted.

3. INEQUALITIES FOR DERIVATIVES OF BOUNDED VARIATION

The following result holds:

**Theorem 3.1.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subset I\), the interior of \( I \) and with the property that the derivative \( \Phi' \) is of bounded variation on \([a, b]\). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have
\[
\left| \int_{\Omega} \Phi \circ gd\mu - \Phi(x) - \Phi' \left( \frac{a+b}{2} \right) \left( \int_{\Omega} gd\mu - x \right) \right|
\]
\[
\leq \frac{1}{2} \max_{a} (\Phi') \int_{\Omega} |g - x| \, d\mu
\]
for any \( x \in [a,b] \).

In particular, we have
\[
\left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \frac{a+b}{2} \right) \left( \int_{\Omega} gd\mu - \frac{a+b}{2} \right) \right|
\]
\[
\leq \frac{1}{2} \max_{a} (\Phi') \int_{\Omega} \left| g - \frac{a+b}{2} \right| \, d\mu \leq \frac{1}{2} (b-a) \max_{a} (\Phi')
\]
\[ \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| \leq \frac{b}{2} \left( \Phi' \right) \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \] \hfill (3.3) 

\[ \leq \frac{1}{2} \left( \Phi' \right) \left( \int_{\Omega} g^2 d\mu - \left( \int_{\Omega} g d\mu \right)^2 \right)^{1/2} \]

\[ \leq \frac{1}{4} (b - a) \left( \Phi' \right). \]

**Proof.** From the identity (2.1) we have

\[ \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) - \Phi' \left( \int_{\Omega} g d\mu - x \right) \]

\[ = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \left( \Phi' \left( (1 - s) x + sg \right) - \frac{\Phi' \left( a \right) + \Phi' \left( b \right)}{2} \right) ds \right] d\mu \]

for any \( x \in [a, b] \).

Taking the modulus in (3.4) we get

\[ \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu - x \right) \right| \]

\[ \leq \int_{\Omega} \left| (g - x) \int_{0}^{1} \left( \Phi' \left( (1 - s) x + sg \right) - \frac{\Phi' \left( a \right) + \Phi' \left( b \right)}{2} \right) ds \right| d\mu \]

\[ \leq \int_{\Omega} \left| g - x \right| \int_{0}^{1} \left| \Phi' \left( (1 - s) x + sg \right) - \frac{\Phi' \left( a \right) + \Phi' \left( b \right)}{2} \right| ds d\mu \]

for any \( x \in [a, b] \).

Since \( \Phi' \) is of bounded variation on \([a, b]\), then for any \( s \in [0, 1], x \in [a, b] \) and \( t \in \Omega \) we have

\[ \left| \Phi' \left( (1 - s) x + sg \left( t \right) \right) - \frac{\Phi' \left( a \right) + \Phi' \left( b \right)}{2} \right| \]

\[ = \frac{1}{2} \left| \Phi' \left( (1 - s) x + sg \left( t \right) \right) - \Phi' \left( (1 - s) x + sg \left( t \right) \right) - \Phi' \left( (1 - s) x + sg \left( t \right) \right) - \Phi' \left( b \right) \right| \]

\[ \leq \frac{1}{2} \left| \Phi' \left( (1 - s) x + sg \left( t \right) \right) - \Phi' \left( (1 - s) x + sg \left( t \right) \right) - \Phi' \left( (1 - s) x + sg \left( t \right) \right) \right| \]

\[ \leq \frac{1}{2} \left( \Phi' \right) \]

Then we have

\[ \int_{\Omega} \left| g - x \right| \int_{0}^{1} \left| \Phi' \left( (1 - s) x + sg \left( t \right) \right) - \frac{\Phi' \left( a \right) + \Phi' \left( b \right)}{2} \right| ds d\mu \]

\[ \leq \frac{1}{2} \left( \Phi' \right) \int_{\Omega} \left| g - x \right| d\mu \]
for any \( x \in [a, b] \).

Making use of (3.5) and (3.6) we deduce the desired result (3.1).

\[ \square \]

**Remark 3.2.** Let \( \Phi : I \rightarrow \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subset \hat{I}\), the interior of \( I \) and with the property that the derivative \( \Phi' \) is of bounded variation on \([a, b]\). If \( x_i \in [m, M] \) and \( w_i \geq 0 \) \((i = 1, \ldots, n)\) with \( W_n := \sum_{i=1}^{n} w_i = 1 \), then one has the weighted discrete inequality:

\[
\left| \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^{n} w_i x_i - x \right) \right| \leq \frac{1}{2} b \sqrt{\Phi'} \sum_{i=1}^{n} w_i |x_i - x| \tag{3.7}
\]

for any \( x \in [a, b] \).

In particular, we have

\[
\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi \left( \frac{a + b}{2} \right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^{n} w_i x_i - \frac{a + b}{2} \right) \leq \frac{1}{2} b \sqrt{\Phi'} \sum_{i=1}^{n} w_i \left| x_i - \frac{a + b}{2} \right| \leq \frac{1}{4} (b - a) \sqrt{\Phi'} \tag{3.8}
\]

and

\[
\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \leq \frac{1}{2} b \sqrt{\Phi'} \sum_{i=1}^{n} w_i \left| x_i - \sum_{i=1}^{n} w_i x_i \right| \leq \frac{1}{2} b \sqrt{\Phi'} \left( \sum_{j=1}^{n} w_j x_j^2 - \left( \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2} \leq \frac{1}{4} (b - a) \sqrt{\Phi'}. \tag{3.9}
\]

4. Inequalities for Lipschitzian Derivatives

The following result holds:

**Theorem 4.1.** Let \( \Phi : I \rightarrow \mathbb{C} \) be an absolutely continuous functions on \([a, b] \subset \hat{I}\), the interior of \( I \) and with the property that the derivative \( \Phi' \) is Lipschitzian with the constant \( K > 0 \) on \([a, b]\). If \( g : \Omega \rightarrow [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \left( \int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{2} K \left[ \sigma^2_{\mu}(g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right] \tag{4.1}
\]
for any $x \in [a, b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left(\int_\Omega (g - \int_\Omega g d\mu)^2 d\mu\right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2}.$$  

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi \left(\frac{a + b}{2}\right) - \Phi' \left(\frac{a + b}{2}\right) \left(\int_\Omega g d\mu - \frac{a + b}{2}\right)\right| \leq \frac{1}{2} K \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a + b}{2}\right)^2\right]$$

and

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi \left(\int_\Omega g d\mu\right)\right| \leq \frac{1}{2} K \sigma_\mu^2(g) \leq \frac{1}{8} K (b - a)^2. \quad (4.3)$$

**Proof.** From the identity (2.1) we have for $\lambda = \Phi' (x)$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left(\int_\Omega g d\mu - x\right)$$

$$= \int_\Omega \left[(g - x) \int_0^1 (\Phi' ((1 - s) x + sg) - \Phi' (x)) \, ds\right] d\mu$$

for any $x \in [a, b]$.

Taking the modulus in (4.4) we get

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left(\int_\Omega g d\mu - x\right)\right|$$

$$\leq \int_\Omega |g - x| \left|\int_0^1 (\Phi' ((1 - s) x + sg) - \Phi' (x)) \, ds\right| d\mu$$

$$\leq \int_\Omega \left[|g - x| \int_0^1 |(\Phi' ((1 - s) x + sg) - \Phi' (x))| \, ds\right] d\mu$$

$$\leq K \int_\Omega \left[|g - x| \int_0^1 s |g - x| \, ds\right] d\mu = \frac{1}{2} K \int_\Omega (g - x)^2 d\mu$$

for any $x \in [a, b]$. 

However, 
\[
\int_{\Omega} (g - x)^2 \, d\mu = \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu + \int_{\Omega} g \, d\mu - x \right)^2 \, d\mu \\
= \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu \right)^2 \, d\mu + 2 \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu \right) \left( \int_{\Omega} g \, d\mu - x \right) \, d\mu \\
+ \int_{\Omega} \left( \int_{\Omega} g \, d\mu - x \right)^2 \, d\mu \\
= \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu \right)^2 \, d\mu + \left( \int_{\Omega} g \, d\mu - x \right)^2
\]
for any \( x \in [a,b] \), and by (4.5) we get the desired result (4.1).

**Corollary 4.2.** Let \( \Phi : I \to \mathbb{C} \) be a twice differentiable functions on \( [a,b] \subset \mathbb{R} \) with \( \| \Phi'' \|_{[a,b],\infty} := \text{ess sup}_{t \in [a,b]} |\Phi''(t)| < \infty \). Then the inequalities (4.1)-(4.3) hold for \( K = \| \Phi'' \|_{[a,b],\infty} \).

**Remark 4.3.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a,b] \subset \mathbb{R} \) and with the property that the derivative \( \Phi' \) is Lipschitzian with the constant \( K > 0 \) on \( [a,b] \). If \( x_i \in [m,M] \) and \( w_i \geq 0 \) \( (i = 1, \ldots, n) \) with \( W_n := \sum_{i=1}^{n} w_i = 1 \), then one has the weighted discrete inequality:

\[
\left| \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi(x) \left( \sum_{i=1}^{n} w_ix_i - x \right) \right| \leq \frac{1}{2} K \left[ \sigma_w^2(x) + \left( \sum_{i=1}^{n} w_ix_i - x \right)^2 \right]^{1/2}
\]

for any \( x \in [a,b] \), where

\[
\sigma_w(x) := \left( \sum_{i=1}^{n} w_i \left( x_i - \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2} = \left( \sum_{i=1}^{n} w_i x_i^2 - \left( \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2}
\]

The following lemma may be stated:

**Lemma 4.4.** Let \( u : [a,b] \to \mathbb{R} \) and \( l, L \in \mathbb{R} \) with \( L > l \). The following statements are equivalent:

(i) The function \( u - \frac{t+L}{2} \cdot e \), where \( e(t) = t, t \in [a,b] \) is \( \frac{1}{2} (L - l) \) -Lipschitzian;

(ii) We have the inequalities

\[
l \leq \frac{u(t) - u(s)}{t-s} \leq L \quad \text{for each } t,s \in [a,b] \quad \text{with } t \neq s;
\]

(iii) We have the inequalities

\[
l (t-s) \leq u(t) - u(s) \leq L (t-s) \quad \text{for each } t,s \in [a,b] \quad \text{with } t > s.
\]
Following [53], we can introduce the definition of \((l,L)\)-Lipschitzian functions:

**Definition 4.5.** The function \(u : [a, b] \to \mathbb{R}\) which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be \((l,L)\)-Lipschitzian on \([a, b]\).

If \(L > 0\) and \(l = -L\), then \((-L,L)\)-Lipschitzian means \(L\)-Lipschitzian in the classical sense.

Utilising Lagrange’s mean value theorem, we can state the following result that provides examples of \((l,L)\)-Lipschitzian functions.

**Proposition 4.6.** Let \(u : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\) and differentiable on \((a,b)\). If \(-\infty < l = \inf_{t \in [a, b]} u'(t)\) and \(\sup_{t \in [a, b]} u'(t) = L < \infty\), then \(u\) is \((l,L)\)-Lipschitzian on \([a, b]\).

The following result holds.

**Corollary 4.7.** Let \(\Phi : I \to \mathbb{R}\) be an absolutely continuous functions on \([a,b] \subset \bar{I}\), with the property that the derivative \(\Phi'\) is \((l,L)\)-Lipschitzian on \([a, b]\), where \(l,L \in \mathbb{R}\) with \(L > l\). If \(g : \Omega \to [a,b]\) is Lebesgue \(\mu\)-measurable on \(\Omega\) and such that \(\Phi \circ g, g \in L(\Omega, \mu)\), then we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{4} \left( L + l \right) \sigma_{\mu}^2 (g) \tag{4.9}
\]

\[
= \frac{1}{4} \left( L + l \right) \left[ \sigma_{\mu}^2 (g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right]
\]

\[
\leq \frac{1}{4} \left( L - l \right) \left[ \sigma_{\mu}^2 (g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right]
\]

for any \(x \in [a, b]\).

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \frac{a+b}{2} \right) \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} \left( L + l \right) \sigma_{\mu}^2 (g) \tag{4.10}
\]

\[
\leq \frac{1}{4} \left( L - l \right) \left[ \sigma_{\mu}^2 (g) + \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right)^2 \right]
\]

and

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) - \frac{1}{4} \left( L + l \right) \sigma_{\mu}^2 (g) \right| \leq \frac{1}{4} \left( L - l \right) \sigma_{\mu}^2 (g) \tag{4.11}
\]

\[
\leq \frac{1}{16} \left( L - l \right) (b-a)^2.
\]
Proof. Consider the auxiliary function $\Psi : [a,b] \rightarrow \mathbb{R}$ given by

$$\Psi (x) = \Phi (x) - \frac{1}{4} (L + l) x^2.$$ 

We observe that $\Psi$ is differentiable and

$$\Psi' (x) = \Phi' (x) - \frac{1}{2} (L + l) x.$$ 

Since $\Phi'$ is $(l,L)$-Lipschitzian on $[a,b]$ it follows that $\Psi'$ is Lipschitzian with the constant $\frac{1}{2} (L - l)$, so we can apply Theorem 4.1 for $\Psi$, i.e. we have the inequality

$$\left| \int_{\Omega} (\Phi \circ g d\mu) - \Phi(x) \right| \leq \frac{1}{4} (L - l) \left[ \sigma_\mu^2 (g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right].$$ (4.12)

However

$$\int_{\Omega} (\Phi \circ g d\mu) - \Phi(x) \left( \int_{\Omega} g d\mu - x \right) = \int_{\Omega} (\Psi \circ g d\mu) - \Psi(x) \left( \int_{\Omega} g d\mu - x \right) - \frac{1}{4} (L + l) \left[ \int_{\Omega} g^2 d\mu - x^2 - 2x \left( \int_{\Omega} g d\mu - x \right) \right]$$

and by (4.12) we get the desired result (4.9). \hfill \Box

Remark 4.8. We observe that if the function $\Phi$ is twice differentiable on $\bar{I}$ and for $[a,b] \subset \bar{I}$ we have

$$-\infty < l \leq \Phi'' (x) \leq L < \infty$$

for any $x \in [a,b]$, then $\Phi'$ is $(l,L)$-Lipschitzian on $[a,b]$ and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

Theorem 4.9. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \bar{I}$, the interior of $I$ and with the property that the derivative $\Phi'$ is Lipschitzian with the constant $K > 0$ on $[a,b]$. If $g : \Omega \rightarrow [a,b]$ is Lebesgue $\mu$-measurable on
\[ \Omega \text{ and such that } \Phi \circ g, g \in L(\Omega, \mu), \text{ then we have} \]

\[ \left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right| \]

\[ \leq \frac{1}{2} K \left[ \left| x - \int_{\Omega} g d\mu \right| \left| \int_{\Omega} g - x d\mu \right| + \left| \int_{\Omega} g - \int_{\Omega} g d\mu d\mu \right| \left| \int_{\Omega} g - x d\mu \right| \right] \]

for any \( x \in [a, b] \), where

\[ \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} := \text{ess sup}_{t \in \Omega} \left| g(t) - \int_{\Omega} g d\mu \right| < \infty. \]

In particular, we have

\[ \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) \right| \]

\[ \leq \frac{1}{2} K \left[ \left| a + b \right| - \int_{\Omega} g d\mu \right| \left| \int_{\Omega} g - \frac{a + b}{2} \right| d\mu + \int_{\Omega} \left| g - \frac{a + b}{2} \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \]

\[ \leq \frac{1}{2} K \left[ \left| a + b \right| - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \left| \int_{\Omega} g - \frac{a + b}{2} \right| d\mu. \]

**Proof.** From the identity (2.1) we have for \( \lambda = \Phi' \left( \int_{\Omega} g d\mu \right) \) that

\[ \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \]

\[ = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \left( \Phi'((1 - s)x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right] d\mu \]

for any \( x \in [a, b] \).
Taking the modulus in (4.15) we get
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \left( \int_{\Omega} g d\mu \right) \right| > \Phi'(x) \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right)
\]
(4.16)
\[
\leq \int_{\Omega} |g - x| \left( \int_{0}^{1} \left( \Phi'((1-s)x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right) d\mu
\]
\[
\leq \int_{\Omega} \left| g - x \right| \left( \int_{0}^{1} \left( \Phi'((1-s)x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right) d\mu
\]
\[
\leq K \int_{\Omega} \left| g - x \right| \int_{0}^{1} \left( (1-s)x + sg - \int_{\Omega} g d\mu \right) ds d\mu
\]
\[
= K \int_{\Omega} \left| g - x \right| \int_{0}^{1} (1-s)x + sg - (1-s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right) ds d\mu
\]
:= B.

Using the triangle inequality we have for any \( t \in \Omega \)
\[
\int_{0}^{1} (1-s)x + sg(t) - (1-s) \int_{\Omega} g d\mu + s \int_{\Omega} g d\mu \right) ds
\]
\[
\leq \int_{0}^{1} (1-s)x - \int_{\Omega} g d\mu \right) ds + \int_{0}^{1} s \left| g(t) - \int_{\Omega} g d\mu \right| ds
\]
\[
= \frac{1}{2} \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g(t) - \int_{\Omega} g d\mu \right| \right]
\]
and then
\[
B \leq \frac{1}{2} K \int_{\Omega} |g - x| \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g(t) - \int_{\Omega} g d\mu \right| \right] d\mu
\]
(4.17)
\[
= \frac{1}{2} K \left[ \left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} |g - x| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right].
\]

Making use of (4.16) and (4.17) we deduce the desired result (4.13).

**Corollary 4.10.** Let \( \Phi : I \to \mathbb{R} \) be an absolutely continuous functions on \([a, b] \subset I \), with the property that the derivative \( \Phi' \) is \((l, L)\)-Lipschitzian on \([a, b]\), where \( l, L \in \mathbb{R} \) with \( L > l \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right|
\]
(4.18)
\[
- \frac{1}{4} (L + l) \left[ \sigma_{\mu}^{2}(g) - \left( x - \int_{\Omega} g d\mu \right)^{2} \right]
\]
\[
\leq \frac{1}{4} (L - l) \left[ \left| x - \int_{\Omega} g d\mu \right| \left( \int_{\Omega} |g - x| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right)
\]
\[
\leq \frac{1}{4} (L - l) \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g - \int_{\Omega} g d\mu \right| \Omega, \infty \right] \int_{\Omega} |g - x| d\mu \]
for any \( x \in [a, b] \).

In particular, we have

\[
\begin{aligned}
&\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) \right| \\
&\quad - \frac{1}{4} (L + l) \left[ \sigma^2_\mu (g) - \left( \frac{a + b}{2} - \int_{\Omega} g d\mu \right)^2 \right] \\
&\leq \frac{1}{4} (L - l) \left[ \left| \frac{a + b}{2} - \int_{\Omega} g d\mu \right| \left| \int_{\Omega} g - \frac{a + b}{2} d\mu \right| \\
&\quad + \int_{\Omega} \left| g - \frac{a + b}{2} \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\
&\leq \frac{1}{4} (L - l) \left[ \left| \frac{a + b}{2} - \int_{\Omega} g d\mu \right| + \left| \int_{\Omega} g d\mu \right| + \infty \right] \int_{\Omega} \left| g - \frac{a + b}{2} \right| d\mu.
\end{aligned}
\] (4.19)

5. Applications for \( f \)-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár \( f \)-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set \( \Omega \) and the \( \sigma \)-finite measure \( \mu \) are given. Consider the set of all probability densities on \( \mu \) to be \( \mathcal{P} := \{ p \mid p : \Omega \to \mathbb{R}, p(t) \geq 0, \int_{\Omega} p(t) d\mu(t) = 1 \} \).

The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

\[
D_{KL}(p, q) := \int_{\Omega} p(t) \ln \frac{p(t)}{q(t)} d\mu(t), \quad p, q \in \mathcal{P},
\] (5.1)

where \( \ln \) is to base \( e \).

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: variation distance \( D_v \), Hellinger distance \( D_H \) [45], \( \chi^2 \)-divergence \( D_{\chi^2} \), \( \alpha \)-divergence \( D_\alpha \), Bhattacharyya distance \( D_B \) [3], Harmonic distance \( D_{Ha} \), Jeffrey’s distance \( D_J \) [47],
triangular discrimination $D_{\Delta}$ [65], etc... They are defined as follows:

\[ D_v(p, q) := \int_{\Omega} |p(t) - q(t)| \, d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.2) \]

\[ D_H(p, q) := \int_{\Omega} \left| \sqrt{p(t)} - \sqrt{q(t)} \right| \, d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.3) \]

\[ D_{\chi^2}(p, q) := \int_{\Omega} \left( \frac{q(t)}{p(t)} \right)^2 - 1 \right) \, d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.4) \]

\[ D_{\alpha}(p, q) := \frac{4}{1 - \alpha^2} \left[ 1 - \int_{\Omega} \left( \frac{q(t)}{p(t)} \right)^{\frac{1-\alpha}{\alpha}} \, d\mu(t) \right], \quad p, q \in \mathcal{P}; \quad (5.5) \]

\[ D_B(p, q) := \int_{\Omega} \sqrt{p(t)q(t)} \, d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.6) \]

\[ D_{H\alpha}(p, q) := \int_{\Omega} \frac{2p(t)q(t)}{p(t) + q(t)} \, d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.7) \]

\[ D_J(p, q) := \int_{\Omega} \left[ p(t) - q(t) \right] \ln \left[ \frac{p(t)}{q(t)} \right] \, d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.8) \]

\[ D_{\Delta}(p, q) := \int_{\Omega} \left( \frac{p(t)}{p(t) + q(t)} \right)^2 \, d\mu(t), \quad p, q \in \mathcal{P}. \quad (5.9) \]

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár $f$-divergence is defined as follows [13]

\[ I_f(p, q) := \int_{\Omega} p(t) f \left[ \frac{q(t)}{p(t)} \right] \, d\mu(t), \quad p, q \in \mathcal{P}, \quad (5.10) \]

where $f$ is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár $f$-divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár $f$-divergence see [13], [14] and [68].

The following result holds:

**Proposition 5.1.** Let $f : (0, \infty) \to \mathbb{R}$ be a twice differentiable convex function with the property that $f(1) = 0$ and there exists the constants $\gamma, \Gamma$ so that

\[-\infty < \gamma \leq f(t) \leq \Gamma < \infty.\]

Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

\[ r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega. \quad (5.11) \]
If \( x \in [r, R] \), then we have the inequality
\[
\left| I_f (p, q) - f (x) - f' (x) (1 - x) - \frac{1}{4} (L + l) D_{\chi^2} (p, q) + (1 - x)^2 \right| \quad (5.12)
\]
\[
\leq \frac{1}{4} (L - l) D_{\chi^2} (p, q) + (1 - x)^2 .
\]
In particular, we have
\[
\left| I_f (p, q) - f \left( \frac{r + R}{2} \right) - f' \left( \frac{r + R}{2} \right) \left( 1 - \frac{r + R}{2} \right) \right| \quad (5.13)
\]
\[
- \frac{1}{4} (L + l) D_{\chi^2} (p, q) + \left( 1 - \frac{r + R}{2} \right)^2 \leq \frac{1}{4} (L - l) D_{\chi^2} (p, q) + \left( 1 - \frac{r + R}{2} \right)^2
\]
and
\[
\left| I_f (p, q) - \frac{1}{4} (L + l) D_{\chi^2} (p, q) \right| \leq \frac{1}{4} (L - l) D_{\chi^2} (p, q) . \quad (5.14)
\]
Proof. From (4.9) we have
\[
\left| \int_\Omega p (t) f \left( \frac{q (t)}{p (t)} \right) d\mu (t) - f (x) - f' (x) (1 - x) \right|
\]
\[
- \frac{1}{4} (L + l) \left| \int_\Omega p (t) \left( \frac{q (t)}{p (t)} \right)^2 d\mu (t) - 1 + (1 - x)^2 \right|
\]
\[
\leq \frac{1}{4} (L - l) \left| \int_\Omega p (t) \left( \frac{q (t)}{p (t)} \right)^2 d\mu (t) - 1 + (1 - x)^2 \right|
\]
for any \( x \in [r, R] \), which is equivalent to (5.12).

Utilising Corollary 4.10 we can state the following result as well:

**Proposition 5.2.** With the assumptions in Proposition 5.1, we have
\[
\left| I_f (p, q) - f (x) - f' (1) (1 - x) - \frac{1}{4} (L + l) D_{\chi^2} (p, q) - (1 - x)^2 \right| \quad (5.15)
\]
\[
\leq \frac{1}{4} (L - l) \left| x - 1 \right| \left| q - xp \right| d\mu + \int_\Omega \left| q - xp \right| \left| \frac{q}{p} - 1 \right| d\mu
\]
\[
\leq \frac{1}{4} (L - l) \left| x - 1 \right| + \left\| \frac{q}{p} - 1 \right\|_{\Omega, \infty} \int_\Omega \left| q - xp \right| d\mu
\]
for any \( x \in [r, R] \).

If we consider the convex function \( f : (0, \infty) \to \mathbb{R} \), \( f (t) = t \ln t \) then
\[
I_f (p, q) := \int_\Omega p (t) \frac{q (t)}{p (t)} \ln \frac{q (t)}{p (t)} d\mu (t) = \int_\Omega q (t) \ln \frac{q (t)}{p (t)} d\mu (t) = D_{KL} (q, p) .
\]
We have $f'(t) = \ln t + 1$ and $f''(t) = \frac{1}{t}$ and then we can choose $l = \frac{1}{R}$ and $L = \frac{1}{r}$. Applying the inequality (5.14) we get
\[
\left| D_{KL}(q,p) - \left( \frac{R + r}{4Rr} \right) D_{\chi^2}(p,q) \right| \leq \frac{R - r}{4Rr} D_{\chi^2}(p,q). \tag{5.16}
\]

If we consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = -\ln t$ then
\[
I_f(p,q) := -\int_{\Omega} p(t) \ln \left[ \frac{q(t)}{p(t)} \right] d\mu(t) = \int_{\Omega} p(t) \ln \left[ \frac{p(t)}{q(t)} \right] d\mu(t)
= D_{KL}(p,q).
\]
We have $f'(t) = -\frac{1}{t}$ and $f''(t) = \frac{1}{t^2}$ and then we can choose $l = \frac{1}{R^2}$ and $L = \frac{1}{r^2}$. Applying the inequality (5.14) we get
\[
\left| D_{KL}(p,q) - \frac{R^2 + r^2}{4R^2r^2} D_{\chi^2}(p,q) \right| \leq \frac{R^2 - r^2}{4R^2r^2} D_{\chi^2}(p,q). \tag{5.17}
\]

**ACKNOWLEDGMENTS**

The author would like to thank the referees for giving fruitful advices.

**REFERENCES**