New Jensen and Ostrowski Type Inequalities for General Lebesgue Integral with Applications

S. S. Dragomir\textsuperscript{a,b}
\textsuperscript{a}Mathematics, College of Engineering & Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia.
\textsuperscript{b}School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.

E-mail: sever.dragomir@vu.edu.au

Abstract. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for \(f\)-divergence measure are provided as well.

Keywords: Ostrowski’s inequality, Jensen’s inequality, \(f\)-Divergence measures.


1. Introduction

Let \((\Omega, \mathcal{A}, \mu)\) be a measurable space consisting of a set \(\Omega\), a \(\sigma\)–algebra \(\mathcal{A}\) of parts of \(\Omega\) and a countably additive and positive measure \(\mu\) on \(\mathcal{A}\) with values in \(\mathbb{R} \cup \{\infty\}\). Assume, for simplicity, that \(\int_{\Omega} d\mu = 1\). Consider the Lebesgue space

\[ L(\Omega, \mu) := \{ f : \Omega \rightarrow \mathbb{R}, f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(t)| d\mu(t) < \infty \}. \]

For simplicity of notation we write everywhere in the sequel \(\int_{\Omega} w(t) d\mu(t)\) instead of \(\int_{\Omega} w(t) d\mu(t)\).

In order to provide a reverse of the celebrated Jensen’s integral inequality for convex functions, S.S. Dragomir obtained in 2002 \cite{29} the following result:
Theorem 1.1. Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on $(m,M)$ and $f : \Omega \to [m,M]$ so that $\Phi \circ f, f, (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:

$$0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right)$$

$$\leq \int_{\Omega} f \cdot (\Phi' \circ f) \, d\mu - \int_{\Omega} \Phi' \circ f \, d\mu \int_{\Omega} f \, d\mu$$

$$\leq \frac{1}{2} \left( \Phi' (M) - \Phi' (m) \right) \int_{\Omega} \left| f - \int_{\Omega} f \, d\mu \right| \, d\mu.$$

In the case of discrete measure, we have:

Corollary 1.2. Let $\Phi : [m,M] \to \mathbb{R}$ be a differentiable convex function on $(m,M)$. If $x_i \in [m,M]$ and $w_i \geq 0$ ($i = 1, \ldots, n$) with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the counterpart of Jensen’s weighted discrete inequality:

$$0 \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right)$$

$$\leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i$$

$$\leq \frac{1}{2} \left( \Phi' (M) - \Phi' (m) \right) \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|.$$

Remark 1.3. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

As we assume that $-\infty < f (t) \leq \Gamma < \infty$, $-\infty < \delta \leq g (t) \leq \Delta < \infty$ for $\mu - \text{a.e. } t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz’s integral inequality, we have

$$\frac{1}{4} \left( \Gamma - \gamma \right) \left( \Delta - \delta \right).$$

provided

$$\frac{1}{4} \left( \Gamma - \gamma \right) \left( \Delta - \delta \right).$$

for $\mu - \text{a.e. } t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f (t) \leq \Gamma < \infty$ for $\mu - \text{a.e. } t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz’s integral inequality, we have

$$\int_{\Omega} \left| f - \int_{\Omega} f \, d\mu \right| \, d\mu \leq \left[ \int_{\Omega} f^2 \, d\mu - \left( \int_{\Omega} f \, d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} \left( \Gamma - \gamma \right).$$

(1.6)
On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

\[
0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right) \quad (1.7)
\]

\[
\leq \int_{\Omega} f \cdot (\Phi' \circ f) \, d\mu - \int_{\Omega} \Phi' \circ f \, d\mu \int_{\Omega} f \, d\mu
\]

\[
\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \left[ \int_{\Omega} f^2 \, d\mu - \left( \int_{\Omega} f \, d\mu \right)^2 \right]^{\frac{1}{2}}
\]

\[
\leq \frac{1}{4} [\Phi'(M) - \Phi'(m)] (M - m),
\]

provided that \( \Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R} \) is a differentiable convex function on \( (m, M) \) and \( f : \Omega \rightarrow [m, M] \) so that \( \Phi \circ f, \Phi' \circ f, f \cdot (\Phi' \circ f) \in L(\Omega, \mu) \), with \( \int_{\Omega} d\mu = 1 \).

The following reverse of the Jensen’s inequality also holds [33]:

**Theorem 1.4.** Let \( \Phi : I \rightarrow \mathbb{R} \) be a continuous convex function on the interval of real numbers \( I \) and \( m, M \in \mathbb{R} \), \( m < M \) with \( [m, M] \subset \bar{I} \), where \( \bar{I} \) is the interior of \( I \). If \( f : \Omega \rightarrow \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds

\[-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu \text{-a.e. } t \in \Omega\]

and such that \( f, \Phi \circ f \in L(\Omega, \mu) \), then

\[
0 \leq \int_{\Omega} \Phi \circ f \, d\mu - \Phi \left( \int_{\Omega} f \, d\mu \right) \quad (1.8)
\]

\[
\leq \left( M - \int_{\Omega} f \, d\mu \right) \left( \int_{\Omega} f \, d\mu - m \right) \frac{\Phi'(M) - \Phi'(m)}{M - m}
\]

\[
\leq \frac{1}{4} (M - m) [\Phi'_-(M) - \Phi'_+(m)],
\]

where \( \Phi'_- \) is the left and \( \Phi'_+ \) is the right derivative of the convex function \( \Phi \).

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean \( \frac{1}{b-a} \int_{a}^{b} \Phi(t) \, dt \) and the value \( \Phi(x) \), \( x \in [a, b] \).

For various results related to Ostrowski’s inequality see [6]-[9], [15]-[41], [43] and the references therein.

**Theorem 1.5.** Let \( \Phi : [a, b] \rightarrow \mathbb{R} \) be continuous on \( [a, b] \) and differentiable on \( (a, b) \) such that \( \Phi' : (a, b) \rightarrow \mathbb{R} \) is bounded on \( (a, b) \), i.e., \( \|\Phi'\|_{\infty} := \sup_{t \in (a, b)} |\Phi'(t)| < \)
Then
\[ |\Phi(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|\Phi'\|_{\infty} (b-a), \quad (1.9) \]
for all \( x \in [a, b] \) and the constant \( \frac{1}{4} \) is the best possible.

Now, for \( \gamma, \Gamma \in \mathbb{C} \) and \([a, b]\) an interval of real numbers, define the sets of complex-valued functions [34]
\[ U_{[a, b]}(\gamma, \Gamma) := \left\{ f : [a, b] \to \mathbb{C} \mid \text{Re} \left( (\Gamma - f(t)) \left( \overline{f(t)} - \gamma \right) \right) \geq 0 \text{ for almost every } t \in [a, b] \right\} \]
and
\[ \Delta_{[a, b]}(\gamma, \Gamma) := \left\{ f : [a, b] \to \mathbb{C} \mid \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a, b] \right\}. \]

The following representation result may be stated [34].

**Proposition 1.6.** For any \( \gamma, \Gamma \in \mathbb{C} \), \( \gamma \neq \Gamma \), we have that \( U_{[a, b]}(\gamma, \Gamma) \) and \( \Delta_{[a, b]}(\gamma, \Gamma) \) are nonempty, convex and closed sets and
\[ U_{[a, b]}(\gamma, \Gamma) = \Delta_{[a, b]}(\gamma, \Gamma) \quad (1.10) \]

On making use of the complex numbers field properties we can also state that:

**Corollary 1.7.** For any \( \gamma, \Gamma \in \mathbb{C} \), \( \gamma \neq \Gamma \), we have that
\[ U_{[a, b]}(\gamma, \Gamma) = \{ f : [a, b] \to \mathbb{C} \mid \text{Re} \left( (\Gamma - \text{Re} f(t)) \left( \text{Re} f(t) - \text{Re} \gamma \right) \right) + (\text{Im} \Gamma - \text{Im} f(t)) \left( \text{Im} f(t) - \text{Im} \gamma \right) \geq 0 \text{ for a.e. } t \in [a, b] \} \quad (1.11) \]

Now, if we assume that \( \text{Re} (\Gamma) \geq \text{Re} (\gamma) \) and \( \text{Im} (\Gamma) \geq \text{Im} (\gamma) \), then we can define the following set of functions as well:
\[ S_{[a, b]}(\gamma, \Gamma) := \{ f : [a, b] \to \mathbb{C} \mid \text{Re} (\Gamma) \geq \text{Re} f(t) \geq \text{Re} (\gamma) \quad (1.12) \]
and \( \text{Im} (\Gamma) \geq \text{Im} f(t) \geq \text{Im} (\gamma) \) for a.e. \( t \in [a, b] \).

One can easily observe that \( S_{[a, b]}(\gamma, \Gamma) \) is closed, convex and
\[ \emptyset \neq S_{[a, b]}(\gamma, \Gamma) \subseteq U_{[a, b]}(\gamma, \Gamma) \quad (1.13) \]

The following result holds [34]:

**Theorem 1.8.** Let \( \Phi : I \to \mathbb{C} \) be an absolutely continuous functions on \( [a, b] \subset \bar{I} \), the interior of \( I \). For some \( \gamma, \Gamma \in \mathbb{C} \), \( \gamma \neq \Gamma \), assume that \( \Phi' \in U_{[a, b]}(\gamma, \Gamma) (= \Delta_{[a, b]}(\gamma, \Gamma)) \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L(\Omega, \mu) \), then we have the inequality
\[ \left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi(x) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g \, d\mu - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} |g - x| \, d\mu \quad (1.14) \]
for any $x \in [a, b]$. In particular, we have

$$\left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \frac{a+b}{2} \right) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g \, d\mu - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \frac{a+b}{2} \right| \, d\mu \leq \frac{1}{4} (b-a) |\Gamma - \gamma|$$

(1.15)

and

$$\left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi \left( \int_{\Omega} g \, d\mu \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g \, d\mu \right| \, d\mu \leq \frac{1}{2} |\Gamma - \gamma| \left( \int_{\Omega} g^2 \, d\mu - \left( \int_{\Omega} g \, d\mu \right)^2 \right)^{1/2} \leq \frac{1}{4} (b-a) |\Gamma - \gamma|.$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi (x) - \lambda \left( \int_{\Omega} g \, d\mu - x \right) \right|, \ x \in [a, b],$$

under various assumptions on the absolutely continuous function $\Phi$, which in the particular case of $x = \int_{\Omega} g \, d\mu$ provides some results connected with Jensen’s inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski’s inequality. Applications for divergence measures are provided as well.

2. Some Identities

The following result holds [34]:

**Lemma 2.1.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset I$, the interior of $I$. If $g : \Omega \to [a, b]$ is Lebesgue $\mu$-measurable on $\Omega$ and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have the equality

$$\int_{\Omega} \Phi \circ g \, d\mu - \Phi (x) = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \Phi'((1-s)x + sg) \, ds \right] \, d\mu$$

(2.1)

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$\int_{\Omega} \Phi \circ g \, d\mu - \Phi (x) = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \Phi'((1-s)x + sg) \, ds \right] \, d\mu,$$

(2.2)

for any $x \in [a, b]$. 

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Remark 2.2. With the assumptions of Lemma 2.1 we have

\[ \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \frac{a+b}{2} \right) \]

\[ = \int_{\Omega} \left[ \left( g - \frac{a+b}{2} \right) \int_{0}^{1} \Phi' \left( 1 - s \right) \frac{a+b}{2} + sg \right] ds \] \tag{2.3} d\mu.

Corollary 2.3. With the assumptions of Lemma 2.1 we have

\[ \int_{\Omega} \Phi \circ gd\mu - \Phi \left( \int_{\Omega} gd\mu \right) \]

\[ = \int_{\Omega} \left[ \left( g - \int_{\Omega} gd\mu \right) \int_{0}^{1} \Phi' \left( 1 - s \right) \int_{\Omega} gd\mu + sg \right] ds \] \tag{2.4} d\mu.

Proof. We observe that since \( g : \Omega \rightarrow [a,b] \) and \( \int_{\Omega} d\mu = 1 \) then \( \int_{\Omega} gd\mu \in [a,b] \) and by taking \( x = \int_{\Omega} gd\mu \) in (2.2) we get (2.4). \( \square \)

Corollary 2.4. With the assumptions of Lemma 2.1 we have

\[ \int_{\Omega} \Phi \circ gd\mu - 1 - \frac{1}{b-a} \int_{a}^{b} \Phi (x) dx - \lambda \left( \int_{\Omega} gd\mu - \frac{a+b}{2} \right) \]

\[ = \int_{\Omega} \left\{ \frac{1}{b-a} \int_{a}^{b} \left( g - x \right) \int_{0}^{1} \Phi' \left( 1 - s \right) x + sg \right\} ds \] \tag{2.5} d\mu.

Proof. Follows by integrating the identity (2.1) over \( x \in [a,b] \), dividing by \( b-a > 0 \) and using Fubini’s theorem. \( \square \)

Corollary 2.5. Let \( \Phi : I \rightarrow \mathbb{C} \) be an absolutely continuous functions on \([a,b] \subset \mathring{I}, \text{ the interior of } I \). If \( g, h : \Omega \rightarrow [a,b] \) are Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, \Phi \circ h, g, h \in L(\Omega, \mu) \), then we have the equality

\[ \int_{\Omega} \Phi \circ gd\mu - \int_{\Omega} \Phi \circ hd\mu - \lambda \left( \int_{\Omega} gd\mu - \frac{a+b}{2} \right) \]

\[ = \int_{\Omega} \int_{\Omega} \left[ \left( g(t) - h(\tau) \right) \int_{0}^{1} \Phi' \left( 1 - s \right) h(\tau) + sg(t) \right] ds \] \tag{2.6} d\mu(t) d\mu(\tau).

for any \( \lambda \in \mathbb{C} \) and \( x \in [a,b] \).

In particular, we have

\[ \int_{\Omega} \Phi \circ gd\mu - \int_{\Omega} \Phi \circ hd\mu \]

\[ = \int_{\Omega} \int_{\Omega} \left[ \left( g(t) - h(\tau) \right) \int_{0}^{1} \Phi' \left( 1 - s \right) h(\tau) + sg(t) \right] d\mu(t) d\mu(\tau), \] \tag{2.7}

for any \( x \in [a,b] \).
Remark 2.6. The above inequality (2.6) can be extended for two measures as follows
\[
\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left( \int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right)
\]
\[
= \int_{\Omega_1} \int_{\Omega_2} \left[ \left( g(t) - h(\tau) \right) \int_0^1 \Phi' \left( (1 - s) h(\tau) + sg(t) \right) - \lambda \right] ds
\]
\[
\times d\mu_1(t) d\mu_2(\tau),
\]
for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$ and provided that $\Phi \circ g, g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

Remark 2.7. If $w \geq 0$ $\mu$-almost everywhere ($\mu$-a.e.) on $\Omega$ with $\int_{\Omega} wd\mu > 0$, then by replacing $d\mu$ with $\frac{wd\mu}{\int_{\Omega} wd\mu}$ in (2.1) we have the weighted equality
\[
\frac{1}{\int_{\Omega} wd\mu} \int_{\Omega} w(\Phi \circ g) d\mu - \Phi(x) - \Phi'(a) + \Phi'(b)
\]
\[
\leq \frac{1}{2} \left( b - a \right) \int_{\Omega} g - a + b d\mu
\]
for any $x \in [a,b]$. In particular, we have
\[
\frac{1}{\int_{\Omega} wd\mu} \int_{\Omega} w g d\mu - \Phi(x) - \Phi'(a) + \Phi'(b)
\]
\[
\leq \frac{1}{2} \left( b - a \right) \int_{\Omega} \left| g - a + b \right| d\mu \leq \frac{1}{2} \left( b - a \right) \int_a^b (\Phi')
\]
for any $x \in [a,b]$.

3. Inequalities for Derivatives of Bounded Variation

The following result holds:

**Theorem 3.1.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset I$, the interior of $I$ and with the property that the derivative $\Phi'$ is of bounded variation on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue $\mu$-measurable on $\Omega$ and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - x \right) \right|
\]
\[
\leq \frac{1}{2} \left( b - a \right) \int_{\Omega} \left| g - a + b \right| d\mu
\]
for any $x \in [a,b]$. In particular, we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - \frac{a + b}{2} \right) \right|
\]
\[
\leq \frac{1}{2} \left( b - a \right) \int_{\Omega} \left| g - \frac{a + b}{2} \right| d\mu \leq \frac{1}{2} \left( b - a \right) \int_a^b (\Phi')
\]
and
\[ \left| \int_\Omega \Phi \circ g d\mu - \Phi \left( \int_\Omega g d\mu \right) \right| \leq \frac{1}{2} \left( \int_a^b (\Phi') \left( \int_\Omega g^2 d\mu - \left( \int_\Omega g d\mu \right)^2 \right)^{1/2} \right. \]
\[ \left. \leq \frac{1}{2} \int_a^b (\Phi') \left( \int_\Omega g^2 d\mu - \left( \int_\Omega g d\mu \right)^2 \right)^{1/2} \right) \]
\[ \leq \frac{1}{4} (b-a) b \int_a^b (\Phi') \]
\[ \leq \frac{1}{4} (b-a) \int_a^b (\Phi') \]

Proof. From the identity (2.1) we have
\[ \int_\Omega \Phi \circ g d\mu - \Phi (x) - \Phi' (a) + \Phi' (b) \]
\[ \leq \frac{1}{2} \left( \int_\Omega g^2 d\mu - \left( \int_\Omega g d\mu \right)^2 \right)^{1/2} \]
\[ \leq \frac{1}{4} (b-a) \int_a^b (\Phi') \]
\[ \leq \frac{1}{4} (b-a) \int_a^b (\Phi') \]
for any \( x \in [a,b] \).

Taking the modulus in (3.4) we get
\[ \left| \int_\Omega \Phi \circ g d\mu - \Phi (x) - \Phi' (a) + \Phi' (b) \right| \]
\[ \leq \frac{1}{2} \left( \int_\Omega g^2 d\mu - \left( \int_\Omega g d\mu \right)^2 \right)^{1/2} \]
\[ \leq \frac{1}{4} (b-a) \int_a^b (\Phi') \]
for any \( x \in [a,b] \).

Since \( \Phi' \) is of bounded variation on \([a,b]\), then for any \( s \in [0,1] \), \( x \in [a,b] \) and \( t \in \Omega \) we have
\[ \left| \Phi' ((1-s) x + sg (t)) - \Phi' (a) + \Phi' ((1-s) x + sg (t)) \right| \]
\[ = \frac{1}{2} \Phi' ((1-s) x + sg (t)) - \Phi' ((1-s) x + sg (t)) \]
\[ \leq \frac{1}{2} \left| \Phi' ((1-s) x + sg (t)) - \Phi' (a) + \Phi' ((1-s) x + sg (t)) - \Phi' (b) \right| \]
\[ \leq \frac{1}{2} \left| \Phi' ((1-s) x + sg (t)) - \Phi' (a) \right| + \left| \Phi' (b) - \Phi' ((1-s) x + sg (t)) \right| \]
\[ \leq \frac{1}{2} \int_a^b (\Phi') \]

Then we have
\[ \int_\Omega |g-x| \int_0^1 \left| \Phi' ((1-s) x + sg (t)) - \Phi' (a) + \Phi' (b) \right| d\mu \]
\[ \leq \frac{1}{2} \int_a^b (\Phi') \int_\Omega |g-x| d\mu \]
for any $x \in [a, b]$.

Making use of (3.5) and (3.6) we deduce the desired result (3.1).

□

Remark 3.2. Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of $I$ and with the property that the derivative $\Phi'$ is of bounded variation on $[a, b]$. If $x_i \in [m, M]$ and $w_i \geq 0$ ($i = 1, \ldots, n$) with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the weighted discrete inequality:

$$\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^{n} w_i x_i - x \right) \right| \leq \frac{1}{2} \left( \Phi' \right) \left( \sum_{i=1}^{n} w_i |x_i - x| \right)$$

(3.7)

for any $x \in [a, b]$.

In particular, we have

$$\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \frac{a + b}{2} \right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^{n} w_i x_i - \frac{a + b}{2} \right) \right| \leq \frac{1}{2} \left( \Phi' \right) \left( \sum_{i=1}^{n} w_i |x_i - \frac{a + b}{2}| \right) \leq \frac{1}{4} (b - a) \left( \Phi' \right)$$

(3.8)

and

$$\left| \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \right| \leq \frac{1}{2} \left( \Phi' \right) \left( \sum_{i=1}^{n} w_i |x_i - \sum_{i=1}^{n} w_i x_i| \right) \leq \frac{1}{2} \left( \Phi' \right) \left( \sum_{j=1}^{n} w_j x_j^2 - \left( \sum_{k=1}^{n} w_k x_k \right)^2 \right)^{1/2} \leq \frac{1}{4} (b - a) \left( \Phi' \right)$$

(3.9)

4. Inequalities for Lipschitzian Derivatives

The following result holds:

**Theorem 4.1.** Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of $I$ and with the property that the derivative $\Phi'$ is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue $\mu$-measurable on $\Omega$ and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left( \int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{2} K \left[ \sigma^2_\mu (g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right]$$

(4.1)
for any $x \in [a,b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left( \int_\Omega \left( g - \int_\Omega g \, d\mu \right)^2 \, d\mu \right)^{1/2} = \left( \int_\Omega g^2 \, d\mu - \left( \int_\Omega g \, d\mu \right)^2 \right)^{1/2}.$$

In particular, we have

$$\left| \int_\Omega \Phi \circ gd\mu - \Phi \left( \frac{a + b}{2} \right) - \Phi' \left( \frac{a + b}{2} \right) \left( \int_\Omega gd\mu - \frac{a + b}{2} \right) \right| \leq \frac{1}{2} K \left[ \sigma_\mu^2(g) + \left( \int_\Omega gd\mu - \frac{a + b}{2} \right)^2 \right]$$

and

$$\left| \int_\Omega \Phi \circ gd\mu - \Phi \left( \int_\Omega gd\mu \right) \right| \leq \frac{1}{2} K \sigma_\mu^2(g) \leq \frac{1}{8} K (b-a)^2. \quad (4.3)$$

**Proof.** From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$\int_\Omega \Phi \circ gd\mu - \Phi (x) - \Phi'(x) \left( \int_\Omega gd\mu - x \right) = \int_\Omega \left[(g-x) \int_0^1 (\Phi'((1-s)x+sg) - \Phi'(x)) \, ds \right] \, d\mu$$

for any $x \in [a,b]$.

Taking the modulus in (4.4) we get

$$\left| \int_\Omega \Phi \circ gd\mu - \Phi (x) - \Phi'(x) \left( \int_\Omega gd\mu - x \right) \right| \leq \int_\Omega |g-x| \left| \int_0^1 (\Phi'((1-s)x+sg) - \Phi'(x)) \, ds \right| \, d\mu$$

$$\leq \int_\Omega |g-x| \int_0^1 |(\Phi'((1-s)x+sg) - \Phi'(x))| \, ds \, d\mu$$

$$\leq K \int_\Omega \left[ |g-x| \int_0^1 s |g-x| \, ds \right] \, d\mu = \frac{1}{2} K \int_\Omega (g-x)^2 \, d\mu$$

for any $x \in [a,b]$. 

However,
\[
\int_{\Omega} (g - x)^2 \, d\mu = \int_{\Omega} \left( g - \int_{\Omega} g \, d\mu + \int_{\Omega} g \, d\mu - x \right)^2 \, d\mu \\
+ \int_{\Omega} \left( \int_{\Omega} g \, d\mu - x \right)^2 \, d\mu
\]
for any \(x \in [a,b]\), and by (4.5) we get the desired result (4.1). □

**Corollary 4.2.** Let \(\Phi : I \to \mathbb{C}\) be a twice differentiable functions on \([a,b] \subset \hat{I}\) with \(\|\Phi''\|_{[a,b],\infty} := \text{ess sup}_{t \in [a,b]} |\Phi''(t)| < \infty\). Then the inequalities (4.1)-(4.3) hold for \(K = \|\Phi''\|_{[a,b],\infty}\).

**Remark 4.3.** Let \(\Phi : I \to \mathbb{C}\) be an absolutely continuous functions on \([a,b] \subset \hat{I}\) and with the property that the derivative \(\Phi'\) is Lipschitzian with the constant \(K > 0\) on \([a,b]\). If \(x_i \in [m,M]\) and \(w_i \geq 0\) \((i = 1,\ldots,n)\) with \(W_n := \sum_{i=1}^n w_i = 1\), then one has the weighted discrete inequality:
\[
\left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) \left( \sum_{i=1}^n w_i x_i - x \right) \right| \leq \frac{1}{2} K \left[ \sigma_w^2(x) + \left( \sum_{i=1}^n w_i x_i - x \right)^2 \right]^{1/2}
\]
for any \(x \in [a,b]\), where
\[
\sigma_w(x) := \left( \sum_{i=1}^n \left( x_i - \sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2} = \left( \sum_{i=1}^n \left( w_i x_i^2 - \left( \sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2}
\]

The following lemma may be stated:

**Lemma 4.4.** Let \(u : [a,b] \to \mathbb{R}\) and \(l, L \in \mathbb{R}\) with \(L > l\). The following statements are equivalent:

(i) The function \(u - \frac{L + l}{2} e\), where \(e(t) = t, t \in [a,b]\) is \(\frac{1}{2} (L - l)\) -Lipschitzian;

(ii) We have the inequalities
\[
l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each } t, s \in [a,b] \quad \text{with } t \neq s; \quad (4.7)
\]

(iii) We have the inequalities
\[
l (t-s) \leq u(t) - u(s) \leq L (t-s) \quad \text{for each } t, s \in [a,b] \quad \text{with } t > s. \quad (4.8)
\]
Following [53], we can introduce the definition of \((l,L)\)-Lipschitzian functions:

**Definition 4.5.** The function \(u : [a,b] \to \mathbb{R}\) which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be \((l,L)\)-Lipschitzian on \([a,b]\).

If \(L > 0\) and \(l = -L\), then \((-L,L)\)-Lipschitzian means \(L\)-Lipschitzian in the classical sense.

Utilising *Lagrange’s mean value theorem*, we can state the following result that provides examples of \((l,L)\)-Lipschitzian functions.

**Proposition 4.6.** Let \(u : [a,b] \to \mathbb{R}\) be continuous on \([a,b]\) and differentiable on \((a,b)\). If \(-\infty < l = \inf_{t \in [a,b]} u'(t)\) and \(\sup_{t \in [a,b]} u'(t) = L < \infty\), then \(u\) is \((l,L)\)-Lipschitzian on \([a,b]\).

The following result holds.

**Corollary 4.7.** Let \(\Phi : I \to \mathbb{R}\) be an absolutely continuous functions on \([a,b] \subset \mathring{I}\), with the property that the derivative \(\Phi'\) is \((l,L)\)-Lipschitzian on \([a,b]\), where \(l, L \in \mathbb{R}\) with \(L > l\). If \(g : \Omega \to [a,b]\) is Lebesgue \(\mu\)-measurable on \(\Omega\) and such that \(\Phi \circ g, g \in L(\Omega, \mu)\), then we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \frac{a+b}{2} \right) \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| 
\leq \frac{1}{4} (L-l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right)^2 \right] 
\tag{4.10}
\]

for any \(x \in [a,b]\).

In particular, we have

\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \frac{a+b}{2} \right) \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| 
\leq \frac{1}{4} (L-l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right)^2 \right] 
\leq \frac{1}{16} (L-l) (b-a)^2 .
\tag{4.11}
\]
Proof. Consider the auxiliary function \( \Psi : [a,b] \rightarrow \mathbb{R} \) given by
\[
\Psi (x) = \Phi (x) - \frac{1}{4} (L + l) x^2.
\]
We observe that \( \Psi \) is differentiable and
\[
\Psi' (x) = \Phi' (x) - \frac{1}{2} (L + l) x.
\]
Since \( \Phi' \) is \((l, L)\)-Lipschitzian on \([a,b]\) it follows that \( \Psi' \) is Lipschitzian with the constant \( \frac{1}{2} (L - l) \), so we can apply Theorem 4.1 for \( \Psi \), i.e. we have the inequality
\[
\left| \int_\Omega \Psi \circ g d\mu - \Psi (x) - \Psi' (x) \left( \int_\Omega g d\mu - x \right) \right| \leq \frac{1}{4} (L - l) \left[ \sigma_\mu^2 (g) + \left( \int_\Omega g d\mu - x \right)^2 \right].
\] (4.12)

However
\[
\int_\Omega \Psi \circ g d\mu - \Psi (x) - \Psi' (x) \left( \int_\Omega g d\mu - x \right)
\]
\[
= \int_\Omega \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left( \int_\Omega g d\mu - x \right)
\]
\[
- \frac{1}{4} (L + l) \left[ \int_\Omega g^2 d\mu - x^2 - 2x \left( \int_\Omega g d\mu - x \right) \right]
\]
\[
= \int_\Omega \Phi \circ g d\mu - \Phi (x) - \Phi' (x) \left( \int_\Omega g d\mu - x \right)
\]
\[
- \frac{1}{4} (L + l) \left[ \sigma_\mu^2 (g) + \left( \int_\Omega g d\mu - x \right)^2 \right]
\]
and by (4.12) we get the desired result (4.9). \( \square \)

Remark 4.8. We observe that if the function \( \Phi \) is twice differentiable on \( \bar{I} \) and for \([a,b] \subset \bar{I} \) we have
\[-\infty < l \leq \Phi'' (x) \leq L < \infty \text{ for any } x \in [a,b],\]
then \( \Phi' \) is \((l, L)\)-Lipschitzian on \([a,b]\) and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

Theorem 4.9. Let \( \Phi : I \rightarrow \mathbb{C} \) be an absolutely continuous functions on \([a,b] \subset \bar{I}, \) the interior of \( I \) and with the property that the derivative \( \Phi' \) is Lipschitzian with the constant \( K > 0 \) on \([a,b]\). If \( g : \Omega \rightarrow [a,b] \) is Lebesgue \( \mu \)-measurable on
\[ \Omega \text{ and such that } \Phi \circ g, g \in L(\Omega, \mu), \text{ then we have} \]

\[ \left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi(x) - \Phi' \left( \int_{\Omega} g \, d\mu \right) \left( \int_{\Omega} g \, d\mu - x \right) \right| \leq \frac{1}{2} K \left[ x - \int_{\Omega} g \, d\mu \right] \int_{\Omega} |g - x| \, d\mu + \int_{\Omega} |g - x| \left( \int_{\Omega} g \, d\mu \right) \, d\mu \]

\[ \leq \frac{1}{2} K \left[ x - \int_{\Omega} g \, d\mu \right] \int_{\Omega} |g - x| \, d\mu \]

for any \( x \in [a, b] \), where

\[ \left\| g - \int_{\Omega} g \, d\mu \right\|_{\Omega, \infty} := \sup_{t \in \Omega} |g(t) - \int_{\Omega} g \, d\mu| < \infty. \]

In particular, we have

\[ \left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi\left( \frac{a + b}{2} \right) - \Phi' \left( \int_{\Omega} g \, d\mu \right) \left( \int_{\Omega} g \, d\mu - \frac{a + b}{2} \right) \right| \leq \frac{1}{2} K \left[ \left( \frac{a + b}{2} - \int_{\Omega} g \, d\mu \right) \int_{\Omega} |g - \frac{a + b}{2}| \, d\mu 
\right. 
\]

\[ + \int_{\Omega} |g - \frac{a + b}{2}| \left( \int_{\Omega} g \, d\mu \right) \, d\mu \]

\[ \leq \frac{1}{2} K \left[ \left( \frac{a + b}{2} - \int_{\Omega} g \, d\mu \right) + \left( \int_{\Omega} g \, d\mu \right) \right] \int_{\Omega} \left| g - \frac{a + b}{2} \right| \, d\mu. \]

Proof. From the identity (2.1) we have for \( \lambda = \Phi' \left( \int_{\Omega} g \, d\mu \right) \) that

\[ \int_{\Omega} \Phi \circ g \, d\mu - \Phi(x) - \Phi' \left( \int_{\Omega} g \, d\mu \right) \left( \int_{\Omega} g \, d\mu - x \right) \]

\[ = \int_{\Omega} \left[ (g - x) \int_{0}^{1} \Phi'((1 - s)x + sg) - \Phi' \left( \int_{\Omega} g \, d\mu \right) \, ds \right] \, d\mu \]

for any \( x \in [a, b] \).
Taking the modulus in (4.15) we get
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right| \tag{4.16}
\]
\[
\leq \int_{\Omega} |g - x| \left| \int_{0}^{1} \left( \Phi' ((1 - s) x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right| d\mu
\]
\[
\leq \int_{\Omega} \left[ |g - x| \left| \int_{0}^{1} (\Phi' ((1 - s) x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) ) \right| ds \right] d\mu
\]
\[
\leq K \int_{\Omega} \left[ |g - x| \int_{0}^{1} (1 - s) x + sg - \int_{\Omega} g d\mu \right] ds \right] d\mu
\]
\[
= K \int_{\Omega} \left[ |g - x| \int_{0}^{1} (1 - s) x + sg - (1 - s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right] ds \right] d\mu
\]
\[
:= B.
\]
Using the triangle inequality we have for any \( t \in \Omega \)
\[
\int_{0}^{1} \left| (1 - s) x + sg (t) - (1 - s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right| ds
\]
\[
\leq \int_{0}^{1} (1 - s) \left| x - \int_{\Omega} g d\mu \right| ds + \int_{0}^{1} s \left| g (t) - \int_{\Omega} g d\mu \right| ds
\]
\[
= \frac{1}{2} \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g (t) - \int_{\Omega} g d\mu \right| \right]
\]
and then
\[
B \leq \frac{1}{2} K \int_{\Omega} |g - x| \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g (t) - \int_{\Omega} g d\mu \right| \right] d\mu \tag{4.17}
\]
\[
\frac{1}{2} K \left[ \left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} |g - x| d\mu + \int_{\Omega} |g - x| \int_{\Omega} |g - x| d\mu \right] .
\]
Making use of (4.16) and (4.17) we deduce the desired result (4.13). \( \square \)

**Corollary 4.10.** Let \( \Phi : I \to \mathbb{R} \) be an absolutely continuous functions on \( [a, b] \subset I \), with the property that the derivative \( \Phi' \) is \( (l, L) \)-Lipschitzian on \( [a, b] \), where \( l, L \in \mathbb{R} \) with \( L > l \). If \( g : \Omega \to [a, b] \) is Lebesgue \( \mu \)-measurable on \( \Omega \) and such that \( \Phi \circ g, g \in L (\Omega, \mu) \), then we have
\[
\left| \int_{\Omega} \Phi \circ g d\mu - \Phi (x) \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right| \tag{4.18}
\]
\[
- \frac{1}{4} (L + l) \left[ \sigma_{\mu}^{2} (g) - \left( x - \int_{\Omega} g d\mu \right)^2 \right]
\]
\[
\leq \frac{1}{4} (L - l) \left[ \left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} |g - x| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right]
\]
\[
\leq \frac{1}{4} (L - l) \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g - \int_{\Omega} g d\mu \right| \int_{\Omega} |g - x| d\mu \right]
\]
for any \( x \in [a, b] \).

In particular, we have

\[
\left| \int_\Omega \Phi \circ g d\mu - \Phi \left( \frac{a + b}{2} \right) - \Phi' \left( \int_\Omega g d\mu \right) \left( \int_\Omega g d\mu - \frac{a + b}{2} \right) \right|
\]

\[
\leq \frac{1}{4} (L - l) \left[ \left| \sigma_\mu^2 (g) - \left( \frac{a + b}{2} - \int_\Omega g d\mu \right)^2 \right| + \int_\Omega \left| g - \frac{a + b}{2} \right| d\mu \right]
\]

\[
\leq \frac{1}{4} (L - l) \left[ \left| \frac{a + b}{2} - \int_\Omega g d\mu \right| + \left| \frac{a + b}{2} - \int_\Omega \left\| g - \int_\Omega g d\mu \right\|_{\Omega, \infty} \right| \right]
\]

\[
\int_\Omega \left| g - \frac{a + b}{2} \right| d\mu.
\]

5. Applications for \( f \)-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár \( f \)-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set \( \Omega \) and the \( \sigma \)-finite measure \( \mu \) are given. Consider the set of all probability densities on \( \mu \) to be \( \mathcal{P} := \{ p | p : \Omega \rightarrow \mathbb{R}, p(t) \geq 0, \int_\Omega p(t) d\mu(t) = 1 \} \).

The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

\[
D_{KL}(p, q) := \int_\Omega p(t) \ln \left( \frac{p(t)}{q(t)} \right) d\mu(t), \quad p, q \in \mathcal{P},
\]

where \( \ln \) is to base \( e \).

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: variation distance \( D_v \), Hellinger distance \( D_H \) [45], \( \chi^2 \)-divergence \( D_{\chi^2} \), \( \alpha \)-divergence \( D_{\alpha} \), Bhattacharyya distance \( D_B \) [3], Harmonic distance \( D_H^a \), Jeffrey’s distance \( D_J \) [47],
triangular discrimination $D_\Delta$ [65], etc... They are defined as follows:

$$D_v (p,q) := \int_\Omega |p(t) - q(t)| \, d\mu(t), \ p,q \in \mathcal{P}; \quad (5.2)$$

$$D_H (p,q) := \int_\Omega \left| \sqrt{p(t)} - \sqrt{q(t)} \right| \, d\mu(t), \ p,q \in \mathcal{P}; \quad (5.3)$$

$$D_{\chi^2} (p,q) := \int_\Omega \left( q(t) \right) - \left( p(t) \right) d\mu(t), \ p,q \in \mathcal{P}; \quad (5.4)$$

$$D_\alpha (p,q) := \frac{4}{1-\alpha^2} \left[ 1 - \int_\Omega \left( \frac{p(t)^{\frac{1+\alpha}{2}} - q(t)^{\frac{1+\alpha}{2}}}{\frac{1+\alpha}{2}} \right) \, d\mu(t) \right], \ p,q \in \mathcal{P}; \quad (5.5)$$

$$D_B (p,q) := \int_\Omega \sqrt{p(t)q(t)} \, d\mu(t), \ p,q \in \mathcal{P}; \quad (5.6)$$

$$D_{Ha} (p,q) := \int_\Omega 2p(t)q(t) \left( p(t) + q(t) \right) \, d\mu(t), \ p,q \in \mathcal{P}; \quad (5.7)$$

$$D_J (p,q) := \int_\Omega \left[ p(t) - q(t) \right] \ln \left( \frac{p(t)}{q(t)} \right) \, d\mu(t), \ p,q \in \mathcal{P}; \quad (5.8)$$

$$D_\Delta (p,q) := \int_\Omega \frac{(p(t) - q(t))^2}{p(t) + q(t)} \, d\mu(t), \ p,q \in \mathcal{P}. \quad (5.9)$$

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár $f$-divergence is defined as follows [13]

$$I_f (p,q) := \int_\Omega p(t) f \left( \frac{q(t)}{p(t)} \right) \, d\mu(t), \ p,q \in \mathcal{P}, \quad (5.10)$$

where $f$ is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár $f$-divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár $f$-divergence see [13], [14] and [68].

The following result holds:

**Proposition 5.1.** Let $f : (0, \infty) \to \mathbb{R}$ be a twice differentiable convex function with the property that $f(1) = 0$ and there exists the constants $\gamma, \Gamma$ so that

$$-\infty < \gamma \leq f(t) \leq \Gamma < \infty.$$

Assume that $p,q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

$$r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega. \quad (5.11)$$
If \( x \in [r, R] \), then we have the inequality
\[
|I_f(p, q) - f(x) - f'(x)(1 - x)| \leq \frac{1}{4} (L + l) \left[ D_{\chi^2}(p, q) + (1 - x)^2 \right].
\] (5.12)

In particular, we have
\[
|I_f(p, q) - f(x) - f'(x)(1 - x)| - \frac{1}{4} (L + l) \left[ D_{\chi^2}(p, q) + (1 - x)^2 \right] 
\leq \frac{1}{4} (L - l) \left[ D_{\chi^2}(p, q) + (1 - x)^2 \right].
\] (5.13)

and
\[
|I_f(p, q) - \frac{1}{4} (L + l) D_{\chi^2}(p, q)| \leq \frac{1}{4} (L - l) D_{\chi^2}(p, q).
\] (5.14)

Proof. From (4.9) we have
\[
\left| \int_\Omega p(t) f \left( \frac{q(t)}{p(t)} \right) d\mu(t) - f(x) - f'(x)(1 - x) \right| 
\leq \frac{1}{4} (L + l) \left[ \int_\Omega p(t) \left( \frac{q(t)}{p(t)} \right)^2 d\mu(t) - 1 + (1 - x)^2 \right].
\]
for any \( x \in [r, R] \), which is equivalent to (5.12). \( \Box \)

Utilising Corollary 4.10 we can state the following result as well:

**Proposition 5.2.** With the assumptions in Proposition 5.1, we have
\[
|I_f(p, q) - f(x) - f'(1)(1 - x)| \leq \frac{1}{4} (L + l) \left[ D_{\chi^2}(p, q) - (1 - x)^2 \right]
\] (5.15)

and
\[
|I_f(p, q) - \frac{1}{4} (L + l) D_{\chi^2}(p, q)| \leq \frac{1}{4} (L - l) D_{\chi^2}(p, q).
\]
for any \( x \in [r, R] \).

If we consider the convex function \( f : (0, \infty) \to \mathbb{R}, \ f(t) = t \ln t \) then
\[
I_f(p, q) := \int_\Omega p(t) q(t) \ln \left[ \frac{q(t)}{p(t)} \right] d\mu(t) = \int_\Omega q(t) \ln \left[ \frac{q(t)}{p(t)} \right] d\mu(t) = D_{KL}(q, p).
\]
We have $f^{'}(t) = \ln t + 1$ and $f^{''}(t) = \frac{1}{t}$ and then we can choose $l = \frac{1}{R}$ and $L = \frac{1}{r}$. Applying the inequality (5.14) we get
\begin{equation}
\left| D_{KL}(q, p) - \left( \frac{R + r}{4Rr} \right) D_{\chi^2}(p, q) \right| \leq \frac{R - r}{4Rr} D_{\chi^2}(p, q). \tag{5.16}
\end{equation}

If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = -\ln t$ then
\begin{align*}
I_f(p, q) &:= -\int_\Omega p(t) \ln \left[ \frac{q(t)}{p(t)} \right] d\mu(t) = \int_\Omega p(t) \ln \left[ \frac{p(t)}{q(t)} \right] d\mu(t) \\
&= D_{KL}(p, q).
\end{align*}
We have $f^{'}(t) = -\frac{1}{t}$ and $f^{''}(t) = \frac{1}{t^2}$ and then we can choose $l = \frac{1}{R^2}$ and $L = \frac{1}{r^2}$. Applying the inequality (5.14) we get
\begin{equation}
\left| D_{KL}(p, q) - \frac{R^2 + r^2}{4R^2r^2} D_{\chi^2}(p, q) \right| \leq \frac{R^2 - r^2}{4R^2r^2} D_{\chi^2}(p, q). \tag{5.17}
\end{equation}

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