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New Jensen and Ostrowski Type Inequalities for General Lebesgue Integral with Applications

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Abstract. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for f-divergence measure are provided as well.

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1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:

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Theorem 1.1. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : \Omega \to [m,M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega,\mu)$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \int_\Omega f \cdot (\Phi' \circ f)\, d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \\ &\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu. \end{aligned} \tag{1.1}$$

In the case of discrete measure, we have:

Corollary 1.2. *Let $\Phi : [m,M] \to \mathbb{R}$ be a differentiable convex function on (m,M). If $x_i \in [m,M]$ and $w_i \geq 0$ $(i = 1,\dots,n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$\begin{aligned} 0 &\leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \\ &\leq \sum_{i=1}^n w_i \Phi'(x_i)\, x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\ &\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|. \end{aligned} \tag{1.2}$$

Remark 1.3. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f,\ g,\ fg \in L(\Omega,\mu)$, then we may consider the *Čebyšev functional*

$$T(f,g) := \int_\Omega fg d\mu - \int_\Omega f d\mu \int_\Omega g d\mu. \tag{1.3}$$

The following result is known in the literature as the *Grüss inequality*

$$|T(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

provided

$$-\infty < \gamma \leq f(t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(t) \leq \Delta < \infty \tag{1.5}$$

for $\mu -$ a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f(t) \leq \Gamma < \infty$ for $\mu -$ a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$\int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu \leq \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{2}(\Gamma - \gamma). \tag{1.6}$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$\begin{aligned} 0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \int_\Omega f \cdot (\Phi' \circ f)\, d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \\ &\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu \\ &\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\left[\Phi'(M) - \Phi'(m)\right](M - m), \end{aligned} \tag{1.7}$$

provided that $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ f \cdot (\Phi' \circ f) \in L(\Omega, \mu)$, with $\int_\Omega d\mu = 1$.

The following reverse of the Jensen's inequality also holds [33]:

Theorem 1.4. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $f,\ \Phi \circ f \in L(\Omega, \mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \left(M - \int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{1.8}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b \Phi(t)\, dt$ and the value $\Phi(x)$, $x \in [a, b]$.

For various results related to Ostrowski's inequality see [6]-[9], [15]-[41], [43] and the references therein.

Theorem 1.5. *Let $\Phi : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $\Phi' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|\Phi'\|_\infty := \sup_{t \in (a,b)} |\Phi'(t)| <$*

∞. *Then*

$$\left|\Phi(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|\Phi'\|_\infty(b-a),\tag{1.9}$$

for all $x\in[a,b]$ *and the constant* $\frac{1}{4}$ *is the best possible.*

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [34]

$$\begin{aligned}&\bar{U}_{[a,b]}(\gamma,\Gamma)\\&:=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0\text{ for almost every }t\in[a,b]\right\}\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated [34].

Proposition 1.6. *For any* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma).\tag{1.10}$$

On making use of the complex numbers field properties we can also state that:

Corollary 1.7. *For any* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$,*we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma)=\{f:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma)\\&+(\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma)\geq 0\ \textit{for a.e. } t\in[a,b]\}.\end{aligned}\tag{1.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma)\geq\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\geq\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}\ |\ &\operatorname{Re}(\Gamma)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\gamma)\\&\text{and }\operatorname{Im}(\Gamma)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\gamma)\ \text{ for a.e. } t\in[a,b]\}.\end{aligned}\tag{1.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma).\tag{1.13}$$

The following result holds [34]:

Theorem 1.8. *Let* $\Phi:I\to\mathbb{C}$ *be an absolutely continuous functions on* $[a,b]\subset\mathring{I}$, *the interior of* I. *For some* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *assume that* $\Phi'\in\bar{U}_{[a,b]}(\gamma,\Gamma)\left(=\bar{\Delta}_{[a,b]}(\gamma,\Gamma)\right)$. *If* $g:\Omega\to[a,b]$ *is Lebesgue* μ*-measurable on* Ω *and such that* $\Phi\circ g$, $g\in L(\Omega,\mu)$, *then we have the inequality*

$$\left|\int_\Omega\Phi\circ g\,d\mu-\Phi(x)-\frac{\gamma+\Gamma}{2}\left(\int_\Omega g\,d\mu-x\right)\right|\leq\frac{1}{2}|\Gamma-\gamma|\int_\Omega|g-x|\,d\mu\tag{1.14}$$

for any $x \in [a, b]$.

In particular, we have

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \le \frac{1}{2}\left|\Gamma - \gamma\right| \int_\Omega \left| g - \frac{a+b}{2}\right| d\mu \le \frac{1}{4}(b-a)\left|\Gamma - \gamma\right| \tag{1.15}$$

and

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \le \frac{1}{2}\left|\Gamma - \gamma\right| \int_\Omega \left| g - \int_\Omega g d\mu\right| d\mu \le \frac{1}{2}\left|\Gamma - \gamma\right| \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2} \le \frac{1}{4}(b-a)\left|\Gamma - \gamma\right|. \tag{1.16}$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right)\right|, \ x \in [a, b],$$

under various assumptions on the absolutely continuous function Φ, which in the particular case of $x = \int_\Omega g d\mu$ provides some results connected with Jensen's inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. Some Identities

The following result holds [34]:

Lemma 2.1. *Let* $\Phi : I \to \mathbb{C}$ *be an absolutely continuous functions on* $[a, b] \subset \mathring{I}$, *the interior of* I. *If* $g : \Omega \to [a, b]$ *is Lebesgue* μ*-measurable on* Ω *and such that* $\Phi \circ g$, $g \in L(\Omega, \mu)$, *then we have the equality*

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right) = \int_\Omega \left[(g - x)\int_0^1 \left(\Phi'((1-s)x + sg) - \lambda\right) ds\right] d\mu \tag{2.1}$$

for any $\lambda \in \mathbb{C}$ *and* $x \in [a, b]$.

In particular, we have

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) = \int_\Omega \left[(g - x)\int_0^1 \Phi'((1-s)x + sg) ds\right] d\mu, \tag{2.2}$$

for any $x \in [a, b]$.

Remark 2.2. With the assumptions of Lemma 2.1 we have

$$\begin{aligned}&\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\frac{a+b}{2}\right) \qquad (2.3)\\&=\int_{\Omega}\left[\left(g-\frac{a+b}{2}\right)\int_{0}^{1}\Phi'\left((1-s)\frac{a+b}{2}+sg\right)ds\right]d\mu.\end{aligned}$$

Corollary 2.3. *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned}&\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\int_{\Omega}gd\mu\right) \qquad (2.4)\\&=\int_{\Omega}\left[\left(g-\int_{\Omega}gd\mu\right)\int_{0}^{1}\Phi'\left((1-s)\int_{\Omega}gd\mu+sg\right)ds\right]d\mu.\end{aligned}$$

Proof. We observe that since $g:\Omega\to[a,b]$ and $\int_{\Omega}d\mu=1$ then $\int_{\Omega}gd\mu\in[a,b]$ and by taking $x=\int_{\Omega}gd\mu$ in (2.2) we get (2.4). □

Corollary 2.4. *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned}&\int_{\Omega}\Phi\circ gd\mu-\frac{1}{b-a}\int_{a}^{b}\Phi(x)\,dx-\lambda\left(\int_{\Omega}gd\mu-\frac{a+b}{2}\right) \qquad (2.5)\\&=\int_{\Omega}\left\{\frac{1}{b-a}\int_{a}^{b}\left[(g-x)\int_{0}^{1}\left(\Phi'((1-s)x+sg)-\lambda\right)ds\right]dx\right\}d\mu.\end{aligned}$$

Proof. Follows by integrating the identity (2.1) over $x\in[a,b]$, dividing by $b-a>0$ and using Fubini's theorem. □

Corollary 2.5. *Let $\Phi:I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I. If $g,h:\Omega\to[a,b]$ are Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $\Phi\circ h$, g, $h\in L(\Omega,\mu)$, then we have the equality*

$$\begin{aligned}&\int_{\Omega}\Phi\circ gd\mu-\int_{\Omega}\Phi\circ hd\mu-\lambda\left(\int_{\Omega}gd\mu-\int_{\Omega}hd\mu\right) \qquad (2.6)\\&=\int_{\Omega}\int_{\Omega}\left[(g(t)-h(\tau))\int_{0}^{1}\left(\Phi'((1-s)h(\tau)+sg(t))-\lambda\right)ds\right]\\&\times d\mu(t)\,d\mu(\tau)\end{aligned}$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

In particular, we have

$$\begin{aligned}&\int_{\Omega}\Phi\circ gd\mu-\int_{\Omega}\Phi\circ hd\mu \qquad (2.7)\\&=\int_{\Omega}\int_{\Omega}\left[(g(t)-h(\tau))\int_{0}^{1}\Phi'((1-s)h(\tau)+sg(t))\,ds\right]d\mu(t)\,d\mu(\tau),\end{aligned}$$

for any $x\in[a,b]$.

Remark 2.6. The above inequality (2.6) can be extended for two measures as follows

$$\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left(\int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right) \tag{2.8}$$
$$= \int_{\Omega_1} \int_{\Omega_2} \left[(g(t) - h(\tau)) \int_0^1 (\Phi'((1-s) h(\tau) + s g(t)) - \lambda) ds \right]$$
$$\times d\mu_1(t) d\mu_2(\tau),$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$ and provided that $\Phi \circ g,\ g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

Remark 2.7. If $w \geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega w d\mu > 0$, then by replacing $d\mu$ with $\frac{w d\mu}{\int_\Omega w d\mu}$ in (2.1) we have the weighted equality

$$\frac{1}{\int_\Omega w d\mu} \int_\Omega w (\Phi \circ g) d\mu - \Phi(x) - \lambda \left(\frac{1}{\int_\Omega w d\mu} \int_\Omega w g d\mu - x \right) \tag{2.9}$$
$$= \frac{1}{\int_\Omega w d\mu} \int_\Omega w \cdot \left[(g - x) \int_0^1 (\Phi'((1-s) x + s g) - \lambda) ds \right] d\mu$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$, provided $\Phi \circ g,\ g \in L_w(\Omega, \mu)$ where

$$L_w(\Omega, \mu) := \left\{ g | \ \int_\Omega w |g| d\mu < \infty \right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

3. Inequalities for Derivatives of Bounded Variation

The following result holds:

Theorem 3.1. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\ g \in L(\Omega, \mu)$, then we have*

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_\Omega g d\mu - x \right) \right| \tag{3.1}$$
$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \int_\Omega |g - x| d\mu$$

for any $x \in [a,b]$.

In particular, we have

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2} \right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) \right| \tag{3.2}$$
$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \int_\Omega \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{2} (b-a) \bigvee_a^b (\Phi')$$

and

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\int_{\Omega}gd\mu\right)\right|\leq\frac{1}{2}\bigvee_{a}^{b}\left(\Phi'\right)\int_{\Omega}\left|g-\int_{\Omega}gd\mu\right|d\mu \tag{3.3}$$
$$\leq\frac{1}{2}\bigvee_{a}^{b}\left(\Phi'\right)\left(\int_{\Omega}g^{2}d\mu-\left(\int_{\Omega}gd\mu\right)^{2}\right)^{1/2}$$
$$\leq\frac{1}{4}\left(b-a\right)\bigvee_{a}^{b}\left(\Phi'\right).$$

Proof. From the identity (2.1) we have

$$\int_{\Omega}\Phi\circ gd\mu-\Phi\left(x\right)-\frac{\Phi'\left(a\right)+\Phi'\left(b\right)}{2}\left(\int_{\Omega}gd\mu-x\right) \tag{3.4}$$
$$=\int_{\Omega}\left[\left(g-x\right)\int_{0}^{1}\left(\Phi'\left(\left(1-s\right)x+sg\right)-\frac{\Phi'\left(a\right)+\Phi'\left(b\right)}{2}\right)ds\right]d\mu$$

for any $x\in[a,b]$.

Taking the modulus in (3.4) we get

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(x\right)-\frac{\Phi'\left(a\right)+\Phi'\left(b\right)}{2}\left(\int_{\Omega}gd\mu-x\right)\right| \tag{3.5}$$
$$\leq\int_{\Omega}\left|\left(g-x\right)\int_{0}^{1}\left(\Phi'\left(\left(1-s\right)x+sg\right)-\frac{\Phi'\left(a\right)+\Phi'\left(b\right)}{2}\right)\right|dsd\mu$$
$$\leq\int_{\Omega}\left|g-x\right|\int_{0}^{1}\left|\Phi'\left(\left(1-s\right)x+sg\right)-\frac{\Phi'\left(a\right)+\Phi'\left(b\right)}{2}\right|dsd\mu$$

for any $x\in[a,b]$.

Since Φ' is of bounded variation on $[a,b]$, then for any $s\in[0,1]$, $x\in[a,b]$ and $t\in\Omega$ we have

$$\left|\Phi'\left(\left(1-s\right)x+sg\left(t\right)\right)-\frac{\Phi'\left(a\right)+\Phi'\left(b\right)}{2}\right|$$
$$=\frac{1}{2}\left|\Phi'\left(\left(1-s\right)x+sg\left(t\right)\right)-\Phi'\left(a\right)+\Phi'\left(\left(1-s\right)x+sg\left(t\right)\right)-\Phi'\left(b\right)\right|$$
$$\leq\frac{1}{2}\left[\left|\Phi'\left(\left(1-s\right)x+sg\left(t\right)\right)-\Phi'\left(a\right)\right|+\left|\Phi'\left(b\right)-\Phi'\left(\left(1-s\right)x+sg\left(t\right)\right)\right|\right]$$
$$\leq\frac{1}{2}\bigvee_{a}^{b}\left(\Phi'\right).$$

Then we have

$$\int_{\Omega}\left|g-x\right|\int_{0}^{1}\left|\Phi'\left(\left(1-s\right)x+sg\right)-\frac{\Phi'\left(a\right)+\Phi'\left(b\right)}{2}\right|dsd\mu \tag{3.6}$$
$$\leq\frac{1}{2}\bigvee_{a}^{b}\left(\Phi'\right)\int_{\Omega}\left|g-x\right|d\mu$$

for any $x \in [a, b]$.

Making use of (3.5) and (3.6) we deduce the desired result (3.1). $\square$

Remark 3.2. Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a, b]$. If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the weighted discrete inequality:

$$\begin{aligned} &\left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\sum_{i=1}^n w_i x_i - x \right) \right| \\ &\leq \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^n w_i |x_i - x| \end{aligned} \tag{3.7}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} &\left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\sum_{i=1}^n w_i x_i - \frac{a+b}{2} \right) \right| \\ &\leq \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^n w_i \left| x_i - \frac{a+b}{2} \right| \leq \frac{1}{4} (b-a) \bigvee_a^b (\Phi') \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i \right) \right| &\leq \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^n w_i \left| x_i - \sum_{i=1}^n w_i x_i \right| \\ &\leq \frac{1}{2} \bigvee_a^b (\Phi') \left(\sum_{j=1}^n w_i x_j^2 - \left(\sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2} \\ &\leq \frac{1}{4} (b-a) \bigvee_a^b (\Phi'). \end{aligned} \tag{3.9}$$

4. Inequalities for Lipschitzian Derivatives

The following result holds:

Theorem 4.1. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have*

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_\Omega g d\mu - x \right) \right| \\ &\leq \frac{1}{2} K \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x \right)^2 \right] \end{aligned} \tag{4.1}$$

for any $x \in [a,b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu\right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2}.$$

In particular, we have

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\ &\le \frac{1}{2}K\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2}\right)^2\right] \end{aligned} \tag{4.2}$$

and

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \le \frac{1}{2}K\sigma_\mu^2(g) \le \frac{1}{8}K(b-a)^2. \tag{4.3}$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) \\ &= \int_\Omega \left[(g-x)\int_0^1 (\Phi'((1-s)x + sg) - \Phi'(x))\, ds\right] d\mu \end{aligned} \tag{4.4}$$

for any $x \in [a,b]$.

Taking the modulus in (4.4) we get

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\right| \\ &\le \int_\Omega |g-x| \left|\int_0^1 (\Phi'((1-s)x + sg) - \Phi'(x))\, ds\right| d\mu \\ &\le \int_\Omega \left[|g-x| \int_0^1 |(\Phi'((1-s)x + sg) - \Phi'(x))|\, ds\right] d\mu \\ &\le K\int_\Omega \left[|g-x|\int_0^1 s|g-x|\, ds\right] d\mu = \frac{1}{2}K\int_\Omega (g-x)^2 d\mu \end{aligned} \tag{4.5}$$

for any $x \in [a,b]$.

However,

$$\begin{aligned}
&\int_\Omega (g-x)^2 \, d\mu \\
&= \int_\Omega \left(g - \int_\Omega g d\mu + \int_\Omega g d\mu - x\right)^2 d\mu \\
&= \int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu + 2\int_\Omega \left(g - \int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) d\mu \\
&+ \int_\Omega \left(\int_\Omega g d\mu - x\right)^2 d\mu \\
&= \int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu + \left(\int_\Omega g d\mu - x\right)^2
\end{aligned}$$

for any $x \in [a,b]$, and by (4.5) we get the desired result (4.1). $\square$

Corollary 4.2. *Let $\Phi : I \to \mathbb{C}$ be a twice differentiable functions on $[a,b] \subset \mathring{I}$ with $\|\Phi''\|_{[a,b],\infty} := ess\sup_{t\in[a,b]} |\Phi''(t)| < \infty$. Then the inequalities (4.1)-(4.3) hold for $K = \|\Phi''\|_{[a,b],\infty}$.*

Remark 4.3. Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$ and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a,b]$. If $x_i \in [m,M]$ and $w_i \geq 0$ $(i = 1,\ldots,n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the weighted discrete inequality:

$$\begin{aligned}
&\left|\sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) - \Phi'(x)\left(\sum_{i=1}^n w_i x_i - x\right)\right| \\
&\leq \frac{1}{2} K \left[\sigma_w^2(\mathbf{x}) + \left(\sum_{i=1}^n w_i x_i - x\right)^2\right]
\end{aligned} \tag{4.6}$$

for any $x \in [a,b]$, where

$$\sigma_w(\mathbf{x}) := \left(\sum_{i=1}^n w_i \left(x_i - \sum_{k=1}^n w_k x_k\right)^2\right)^{1/2} = \left(\sum_{i=1}^n w_i x_i^2 - \left(\sum_{k=1}^n w_k x_k\right)^2\right)^{1/2}.$$

The following lemma may be stated:

Lemma 4.4. *Let $u : [a,b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) *The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(L-l)$ −Lipschitzian;*

(ii) *We have the inequalities*

$$l \leq \frac{u(t) - u(s)}{t-s} \leq L \quad \text{for each} \;\; t,s \in [a,b] \quad \text{with} \;\; t \neq s; \tag{4.7}$$

(iii) *We have the inequalities*

$$l(t-s) \leq u(t) - u(s) \leq L(t-s) \quad \text{for each} \;\; t,s \in [a,b] \quad \text{with} \;\; t > s. \tag{4.8}$$

Following [53], we can introduce the definition of (l, L)-Lipschitzian functions:

Definition 4.5. The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be (l, L)-Lipschitzian on $[a, b]$.

If $L > 0$ and $l = -L$, then $(-L, L)$ −Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l, L)-Lipschitzian functions.

Proposition 4.6. *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $-\infty < l = \inf_{t\in[a,b]} u'(t)$ and $\sup_{t\in[a,b]} u'(t) = L < \infty$, then u is (l, L)-Lipschitzian on $[a, b]$.*

The following result holds.

Corollary 4.7. *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, with the property that the derivative Φ' is (l, L)-Lipschitzian on $[a, b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have*

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\right. \\ &\left.-\frac{1}{4}(L + l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]\right| \\ &\le \frac{1}{4}(L - l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right] \end{aligned} \tag{4.9}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right. \\ &\left.-\frac{1}{4}(L + l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2}\right)^2\right]\right| \\ &\le \frac{1}{4}(L - l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2}\right)^2\right] \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right) - \frac{1}{4}(L + l)\sigma_\mu^2(g)\right| &\le \frac{1}{4}(L - l)\sigma_\mu^2(g) \\ &\le \frac{1}{16}(L - l)(b - a)^2. \end{aligned} \tag{4.11}$$

Proof. Consider the auxiliary function $\Psi : [a,b] \to \mathbb{R}$ given by

$$\Psi(x) = \Phi(x) - \frac{1}{4}(L+l)x^2.$$

We observe that Ψ is differentiable and

$$\Psi'(x) = \Phi'(x) - \frac{1}{2}(L+l)x.$$

Since Φ' is (l,L)-Lipschitzian on $[a,b]$ it follows that Ψ' is Lipschitzian with the constant $\frac{1}{2}(L-l)$, so we can apply Theorem 4.1 for Ψ, i.e. we have the inequality

$$\begin{aligned}&\left|\int_\Omega \Psi \circ g d\mu - \Psi(x) - \Psi'(x)\left(\int_\Omega g d\mu - x\right)\right| \\ &\leq \frac{1}{4}(L-l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right].\end{aligned} \tag{4.12}$$

However

$$\begin{aligned}&\int_\Omega \Psi \circ g d\mu - \Psi(x) - \Psi'(x)\left(\int_\Omega g d\mu - x\right) \\ &= \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) \\ &- \frac{1}{4}(L+l)\left[\int_\Omega g^2 d\mu - x^2 - 2x\left(\int_\Omega g d\mu - x\right)\right] \\ &= \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) \\ &- \frac{1}{4}(L+l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]\end{aligned}$$

and by (4.12) we get the desired result (4.9). □

Remark 4.8. We observe that if the function Φ is twice differentiable on $\mathring{I}$ and for $[a,b] \subset \mathring{I}$ we have

$$-\infty < l \leq \Phi''(x) \leq L < \infty \text{ for any } x \in [a,b],$$

then Φ' is (l,L)-Lipschitzian on $[a,b]$ and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

Theorem 4.9. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on*

Ω and such that $\Phi \circ g,\ g \in L(\Omega, \mu)$, then we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \tag{4.13}$$
$$\leq \frac{1}{2} K \left[\left|x - \int_\Omega g d\mu\right| \int_\Omega |g - x| \, d\mu + \int_\Omega |g - x| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2} K \left[\left|x - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega |g - x| \, d\mu$$

for any $x \in [a, b]$, where

$$\left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty} := ess \sup_{t \in \Omega} \left|g(t) - \int_\Omega g d\mu\right| < \infty.$$

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \tag{4.14}$$
$$\leq \frac{1}{2} K \left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right.$$
$$\left. + \int_\Omega \left|g - \frac{a+b}{2}\right| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2} K \left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'\left(\int_\Omega g d\mu\right)$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) \tag{4.15}$$
$$= \int_\Omega \left[(g - x) \int_0^1 \left(\Phi'((1 - s) x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right] d\mu$$

for any $x \in [a, b]$.

Taking the modulus in (4.15) we get

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \qquad (4.16)\\
&\le \int_\Omega |g-x| \left|\int_0^1 \left(\Phi'((1-s)x+sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right| d\mu\\
&\le \int_\Omega \left[|g-x| \int_0^1 \left|\left(\Phi'((1-s)x+sg) - \Phi'\left(\int_\Omega g d\mu\right)\right)\right| ds\right] d\mu\\
&\le K\int_\Omega \left[|g-x| \int_0^1 \left|(1-s)x+sg - \int_\Omega g d\mu\right| ds\right] d\mu\\
&= K\int_\Omega \left[|g-x| \int_0^1 \left|(1-s)x+sg - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds\right] d\mu\\
&:= B.
\end{aligned}$$

Using the triangle inequality we have for any $t\in\Omega$

$$\begin{aligned}
&\int_0^1 \left|(1-s)x+sg(t) - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds\\
&\le \int_0^1 (1-s)\left|x - \int_\Omega g d\mu\right| ds + \int_0^1 s\left|g(t) - \int_\Omega g d\mu\right| ds\\
&= \frac{1}{2}\left[\left|x - \int_\Omega g d\mu\right| + \left|g(t) - \int_\Omega g d\mu\right|\right]
\end{aligned}$$

and then

$$\begin{aligned}
B &\le \frac{1}{2}K\int_\Omega |g-x| \left[\left|x - \int_\Omega g d\mu\right| + \left|g(t) - \int_\Omega g d\mu\right|\right] d\mu \qquad (4.17)\\
&= \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right| \int_\Omega |g-x| d\mu + \int_\Omega |g-x|\left|g - \int_\Omega g d\mu\right| d\mu\right].
\end{aligned}$$

Making use of (4.16) and (4.17) we deduce the desired result (4.13). □

Corollary 4.10. *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, with the property that the derivative Φ' is (l,L)-Lipschitzian on $[a,b]$, where $l,L\in\mathbb{R}$ with $L>l$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ g\in L(\Omega,\mu)$, then we have*

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right. \qquad (4.18)\\
&\left.-\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(x - \int_\Omega g d\mu\right)^2\right]\right|\\
&\le \frac{1}{4}(L-l)\left[\left|x - \int_\Omega g d\mu\right| \int_\Omega |g-x| d\mu + \int_\Omega |g-x|\left|g - \int_\Omega g d\mu\right| d\mu\right]\\
&\le \frac{1}{4}(L-l)\left[\left|x - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega |g-x| d\mu
\end{aligned}$$

for any $x \in [a,b]$.

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right. \tag{4.19}$$
$$\left. -\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(\frac{a+b}{2} - \int_\Omega g d\mu\right)^2\right]\right|$$
$$\leq \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right.$$
$$\left. + \int_\Omega \left|g - \frac{a+b}{2}\right| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.$$

5. Applications for f-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p | p : \Omega \to \mathbb{R},\ p(t) \geq 0,\ \int_\Omega p(t)\, d\mu(t) = 1\right\}$. The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t) \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [45], χ^2*-divergence* D_{χ^2}, α*-divergence* D_α, *Bhattacharyya distance* D_B [3], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [47], *triangular discrimination* D_Δ [65], etc... They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(t) - q(t)|\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.2}$$

$$D_H(p,q) := \int_\Omega \left|\sqrt{p(t)} - \sqrt{q(t)}\right| d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.3}$$

$$D_{\chi^2}(p,q) := \int_\Omega p(t) \left[\left(\frac{q(t)}{p(t)}\right)^2 - 1\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t)\right], \quad p,q \in \mathcal{P}; \tag{5.5}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(t)\, q(t)}\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.6}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(t)\, q(t)}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.7}$$

$$D_J(p,q) := \int_\Omega [p(t) - q(t)] \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.8}$$

$$D_\Delta(p,q) := \int_\Omega \frac{[p(t) - q(t)]^2}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}. \tag{5.9}$$

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár f-divergence is defined as follows [13]

$$I_f(p,q) := \int_\Omega p(t)\, f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár f-divergence see [13], [14] and [68].

The following result holds:

Proposition 5.1. *Let $f : (0,\infty) \to \mathbb{R}$ be a twice differentiable convex function with the property that $f(1) = 0$ and there exists the constants γ, Γ so that*

$$-\infty < \gamma \le f(t) \le \Gamma < \infty.$$

Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega. \tag{5.11}$$

If $x \in [r, R]$, *then we have the inequality*

$$\left| I_f(p,q) - f(x) - f'(x)(1-x) - \frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) + (1-x)^2\right]\right| \le \frac{1}{4}(L-l)\left[D_{\chi^2}(p,q) + (1-x)^2\right]. \tag{5.12}$$

In particular, we have

$$\left| I_f(p,q) - f\left(\frac{r+R}{2}\right) - f'\left(\frac{r+R}{2}\right)\left(1 - \frac{r+R}{2}\right) - \frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) + \left(1-\frac{r+R}{2}\right)^2\right]\right| \le \frac{1}{4}(L-l)\left[D_{\chi^2}(p,q) + \left(1-\frac{r+R}{2}\right)^2\right] \tag{5.13}$$

and

$$\left| I_f(p,q) - \frac{1}{4}(L+l)D_{\chi^2}(p,q)\right| \le \frac{1}{4}(L-l)D_{\chi^2}(p,q). \tag{5.14}$$

Proof. From (4.9) we have

$$\left|\int_\Omega p(t) f\left(\frac{q(t)}{p(t)}\right) d\mu(t) - f(x) - f'(x)(1-x) - \frac{1}{4}(L+l)\left[\int_\Omega p(t)\left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2\right]\right| \le \frac{1}{4}(L-l)\left[\int_\Omega p(t)\left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2\right]$$

for any $x \in [r, R]$, which is equivalent to (5.12). □

Utilising Corollary 4.10 we can state the following result as well:

Proposition 5.2. *With the assumptions in Proposition 5.1, we have*

$$\begin{aligned} &\left| I_f(p,q) - f(x) - f'(1)(1-x) - \frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) - (1-x)^2\right]\right| \\ &\le \frac{1}{4}(L-l)\left[|x-1|\int_\Omega |q-xp|\,d\mu + \int_\Omega |q-xp|\left|\frac{q}{p}-1\right| d\mu\right] \\ &\le \frac{1}{4}(L-l)\left[|x-1| + \left\|\frac{q}{p}-1\right\|_{\Omega,\infty}\right]\int_\Omega |q-xp|\,d\mu \end{aligned} \tag{5.15}$$

for any $x \in [r, R]$.

If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t\ln t$ then

$$\begin{aligned} I_f(p,q) &:= \int_\Omega p(t)\frac{q(t)}{p(t)}\ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega q(t)\ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) \\ &= D_{KL}(q,p). \end{aligned}$$

We have $f'(t) = \ln t + 1$ and $f''(t) = \frac{1}{t}$ and then we can choose $l = \frac{1}{R}$ and $L = \frac{1}{r}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(q,p) - \left(\frac{R+r}{4rR} \right) D_{\chi^2}(p,q) \right| \leq \frac{R-r}{4rR} D_{\chi^2}(p,q). \tag{5.16}$$

If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$ then

$$\begin{aligned} I_f(p,q) &:= -\int_\Omega p(t) \ln \left[\frac{q(t)}{p(t)} \right] d\mu(t) = \int_\Omega p(t) \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t) \\ &= D_{KL}(p,q). \end{aligned}$$

We have $f'(t) = -\frac{1}{t}$ and $f''(t) = \frac{1}{t^2}$ and then we can choose $l = \frac{1}{R^2}$ and $L = \frac{1}{r^2}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(p,q) - \frac{R^2+r^2}{4R^2r^2} D_{\chi^2}(p,q) \right| \leq \frac{R^2-r^2}{4R^2r^2} D_{\chi^2}(p,q). \tag{5.17}$$

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