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New Jensen and Ostrowski Type Inequalities for General Lebesgue Integral with Applications

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Abstract. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for f-divergence measure are provided as well.

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1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the Lebesgue space

$$L(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:

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Theorem 1.1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:*

$$\begin{aligned} 0 &\le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \qquad (1.1)\\ &\le \int_\Omega f \cdot (\Phi' \circ f)\, d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \\ &\le \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu. \end{aligned}$$

In the case of discrete measure, we have:

Corollary 1.2. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$\begin{aligned} 0 &\le \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \qquad (1.2)\\ &\le \sum_{i=1}^n w_i \Phi'(x_i)\, x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\ &\le \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|. \end{aligned}$$

Remark 1.3. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f,\ g,\ fg \in L(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T(f, g) := \int_\Omega fg d\mu - \int_\Omega f d\mu \int_\Omega g d\mu. \qquad (1.3)$$

The following result is known in the literature as the *Grüss inequality*

$$|T(f, g)| \le \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \qquad (1.4)$$

provided

$$-\infty < \gamma \le f(t) \le \Gamma < \infty, \quad -\infty < \delta \le g(t) \le \Delta < \infty \qquad (1.5)$$

for $\mu -$ a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \le f(t) \le \Gamma < \infty$ for $\mu -$ a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$\int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu \le \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \le \frac{1}{2}(\Gamma - \gamma). \qquad (1.6)$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$\begin{aligned} 0 &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \quad (1.7)\\ &\le \int_\Omega f\cdot(\Phi'\circ f)\,d\mu - \int_\Omega \Phi'\circ f d\mu \int_\Omega f d\mu \\ &\le \frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\int_\Omega \left| f-\int_\Omega f d\mu\right| d\mu \\ &\le \frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}\left[\Phi'(M)-\Phi'(m)\right](M-m), \end{aligned}$$

provided that $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ f\cdot(\Phi'\circ f)\in L(\Omega,\mu)$, with $\int_\Omega d\mu=1$.

The following reverse of the Jensen's inequality also holds [33]:

Theorem 1.4. *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I. If $f:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\le f(t)\le M<\infty \text{ for } \mu\text{-a.e. } t\in\Omega$$

and such that $f,\ \Phi\circ f\in L(\Omega,\mu)$, then

$$\begin{aligned} 0 &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \quad (1.8)\\ &\le \left(M-\int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m} \\ &\le \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right], \end{aligned}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b \Phi(t)\,dt$ and the value $\Phi(x)$, $x\in[a,b]$.

For various results related to Ostrowski's inequality see [6]-[9], [15]-[41], [43] and the references therein.

Theorem 1.5. *Let $\Phi:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $\Phi':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|\Phi'\|_\infty:=\sup_{t\in(a,b)}|\Phi'(t)|<$*

∞. *Then*

$$\left|\Phi(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|\Phi'\|_\infty(b-a), \tag{1.9}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [34]

$$\bar{U}_{[a,b]}(\gamma,\Gamma)$$
$$:=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\ge 0\text{ for almost every }t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\le\frac{1}{2}|\Gamma-\gamma|\ \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated [34].

Proposition 1.6. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{1.10}$$

On making use of the complex numbers field properties we can also state that:

Corollary 1.7. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$,we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma)=\{f:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma)\\&+(\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma)\ge 0\ \textit{for a.e. } t\in[a,b]\}.\end{aligned} \tag{1.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma)\ge\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\ge\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}\ |\ &\operatorname{Re}(\Gamma)\ge\operatorname{Re}f(t)\ge\operatorname{Re}(\gamma)\\&\text{and }\operatorname{Im}(\Gamma)\ge\operatorname{Im}f(t)\ge\operatorname{Im}(\gamma)\ \text{ for a.e. } t\in[a,b]\}.\end{aligned} \tag{1.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\ne\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma). \tag{1.13}$$

The following result holds [34]:

Theorem 1.8. *Let $\Phi:I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I. For some $\gamma,\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$, assume that $\Phi'\in\bar{U}_{[a,b]}(\gamma,\Gamma)\left(=\bar{\Delta}_{[a,b]}(\gamma,\Gamma)\right)$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $g\in L(\Omega,\mu)$, then we have the inequality*

$$\left|\int_\Omega\Phi\circ g\,d\mu-\Phi(x)-\frac{\gamma+\Gamma}{2}\left(\int_\Omega g\,d\mu-x\right)\right|\le\frac{1}{2}|\Gamma-\gamma|\int_\Omega|g-x|\,d\mu \tag{1.14}$$

for any $x \in [a,b]$.

In particular, we have

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \tag{1.15}$$
$$\leq \frac{1}{2}\left|\Gamma-\gamma\right| \int_\Omega \left| g - \frac{a+b}{2}\right| d\mu \leq \frac{1}{4}(b-a)\left|\Gamma-\gamma\right|$$

and

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right| \int_\Omega \left| g - \int_\Omega g d\mu\right| d\mu \tag{1.16}$$
$$\leq \frac{1}{2}\left|\Gamma-\gamma\right| \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2}$$
$$\leq \frac{1}{4}(b-a)\left|\Gamma-\gamma\right|.$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right)\right|, \; x \in [a,b],$$

under various assumptions on the absolutely continuous function Φ, which in the particular case of $x = \int_\Omega g d\mu$ provides some results connected with Jensen's inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. Some Identities

The following result holds [34]:

Lemma 2.1. *Let* $\Phi : I \to \mathbb{C}$ *be an absolutely continuous functions on* $[a,b] \subset \mathring{I}$, *the interior of* I. *If* $g : \Omega \to [a,b]$ *is Lebesgue* μ*-measurable on* Ω *and such that* $\Phi \circ g,\ g \in L(\Omega,\mu)$, *then we have the equality*

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right) \tag{2.1}$$
$$= \int_\Omega \left[(g-x)\int_0^1 \left(\Phi'((1-s)x+sg) - \lambda\right) ds\right] d\mu$$

for any $\lambda \in \mathbb{C}$ *and* $x \in [a,b]$.

In particular, we have

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) = \int_\Omega \left[(g-x)\int_0^1 \Phi'((1-s)x+sg)\, ds\right] d\mu, \tag{2.2}$$

for any $x \in [a,b]$.

Remark 2.2. With the assumptions of Lemma 2.1 we have

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) \qquad (2.3)\\ &= \int_\Omega \left[\left(g - \frac{a+b}{2}\right)\int_0^1 \Phi'\left((1-s)\frac{a+b}{2} + sg\right) ds\right] d\mu. \end{aligned}$$

Corollary 2.3. *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right) \qquad (2.4)\\ &= \int_\Omega \left[\left(g - \int_\Omega g d\mu\right)\int_0^1 \Phi'\left((1-s)\int_\Omega g d\mu + sg\right) ds\right] d\mu. \end{aligned}$$

Proof. We observe that since $g : \Omega \to [a,b]$ and $\int_\Omega d\mu = 1$ then $\int_\Omega g d\mu \in [a,b]$ and by taking $x = \int_\Omega g d\mu$ in (2.2) we get (2.4). □

Corollary 2.4. *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \frac{1}{b-a}\int_a^b \Phi(x)\,dx - \lambda\left(\int_\Omega g d\mu - \frac{a+b}{2}\right) \qquad (2.5)\\ &= \int_\Omega \left\{\frac{1}{b-a}\int_a^b \left[(g-x)\int_0^1 \left(\Phi'((1-s)x + sg) - \lambda\right) ds\right] dx\right\} d\mu. \end{aligned}$$

Proof. Follows by integrating the identity (2.1) over $x \in [a,b]$, dividing by $b-a>0$ and using Fubini's theorem. □

Corollary 2.5. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g, h : \Omega \to [a,b]$ are Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $\Phi \circ h$, g, $h \in L(\Omega,\mu)$, then we have the equality*

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \int_\Omega \Phi \circ h d\mu - \lambda\left(\int_\Omega g d\mu - \int_\Omega h d\mu\right) \qquad (2.6)\\ &= \int_\Omega \int_\Omega \left[(g(t) - h(\tau))\int_0^1 \left(\Phi'((1-s)h(\tau) + sg(t)) - \lambda\right) ds\right]\\ &\times d\mu(t)\,d\mu(\tau) \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \int_\Omega \Phi \circ h d\mu \qquad (2.7)\\ &= \int_\Omega \int_\Omega \left[(g(t) - h(\tau))\int_0^1 \Phi'((1-s)h(\tau) + sg(t))\,ds\right] d\mu(t)\,d\mu(\tau), \end{aligned}$$

for any $x \in [a,b]$.

Remark 2.6. The above inequality (2.6) can be extended for two measures as follows

$$\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left(\int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right) \tag{2.8}$$
$$= \int_{\Omega_1} \int_{\Omega_2} \left[(g(t) - h(\tau)) \int_0^1 (\Phi'((1-s) h(\tau) + s g(t)) - \lambda) \, ds \right]$$
$$\times d\mu_1(t) \, d\mu_2(\tau),$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$ and provided that $\Phi \circ g, \ g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

Remark 2.7. If $w \geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega w d\mu > 0$, then by replacing $d\mu$ with $\frac{w d\mu}{\int_\Omega w d\mu}$ in (2.1) we have the weighted equality

$$\frac{1}{\int_\Omega w d\mu} \int_\Omega w (\Phi \circ g) \, d\mu - \Phi(x) - \lambda \left(\frac{1}{\int_\Omega w d\mu} \int_\Omega w g d\mu - x \right) \tag{2.9}$$
$$= \frac{1}{\int_\Omega w d\mu} \int_\Omega w \cdot \left[(g - x) \int_0^1 (\Phi'((1-s) x + s g) - \lambda) \, ds \right] d\mu$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$, provided $\Phi \circ g, \ g \in L_w(\Omega, \mu)$ where

$$L_w(\Omega, \mu) := \left\{ g | \ \int_\Omega w |g| \, d\mu < \infty \right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

3. Inequalities for Derivatives of Bounded Variation

The following result holds:

Theorem 3.1. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g, \ g \in L(\Omega, \mu)$, then we have*

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_\Omega g d\mu - x \right) \right| \tag{3.1}$$
$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \int_\Omega |g - x| \, d\mu$$

for any $x \in [a, b]$.

In particular, we have

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2} \right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) \right| \tag{3.2}$$
$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \int_\Omega \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{2} (b - a) \bigvee_a^b (\Phi')$$

and

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \le \frac{1}{2}\bigvee_a^b (\Phi') \int_\Omega \left|g - \int_\Omega g d\mu\right| d\mu \tag{3.3}$$
$$\le \frac{1}{2}\bigvee_a^b (\Phi') \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2}$$
$$\le \frac{1}{4}(b-a)\bigvee_a^b (\Phi').$$

Proof. From the identity (2.1) we have

$$\int_\Omega \Phi\circ g d\mu - \Phi(x) - \frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega g d\mu - x\right) \tag{3.4}$$
$$= \int_\Omega \left[(g-x)\int_0^1 \left(\Phi'((1-s)x+sg) - \frac{\Phi'(a)+\Phi'(b)}{2}\right) ds\right] d\mu$$

for any $x\in[a,b]$.

Taking the modulus in (3.4) we get

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi(x) - \frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega g d\mu - x\right)\right| \tag{3.5}$$
$$\le \int_\Omega \left|(g-x)\int_0^1 \left(\Phi'((1-s)x+sg) - \frac{\Phi'(a)+\Phi'(b)}{2}\right)\right| ds d\mu$$
$$\le \int_\Omega |g-x| \int_0^1 \left|\Phi'((1-s)x+sg) - \frac{\Phi'(a)+\Phi'(b)}{2}\right| ds d\mu$$

for any $x\in[a,b]$.

Since Φ' is of bounded variation on $[a,b]$, then for any $s\in[0,1]$, $x\in[a,b]$ and $t\in\Omega$ we have

$$\left|\Phi'((1-s)x+sg(t)) - \frac{\Phi'(a)+\Phi'(b)}{2}\right|$$
$$= \frac{1}{2}\left|\Phi'((1-s)x+sg(t)) - \Phi'(a) + \Phi'((1-s)x+sg(t)) - \Phi'(b)\right|$$
$$\le \frac{1}{2}\left[\left|\Phi'((1-s)x+sg(t)) - \Phi'(a)\right| + \left|\Phi'(b) - \Phi'((1-s)x+sg(t))\right|\right]$$
$$\le \frac{1}{2}\bigvee_a^b (\Phi').$$

Then we have

$$\int_\Omega |g-x| \int_0^1 \left|\Phi'((1-s)x+sg) - \frac{\Phi'(a)+\Phi'(b)}{2}\right| ds d\mu \tag{3.6}$$
$$\le \frac{1}{2}\bigvee_a^b (\Phi') \int_\Omega |g-x| d\mu$$

for any $x \in [a, b]$.

Making use of (3.5) and (3.6) we deduce the desired result (3.1). □

Remark 3.2. Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a, b]$. If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the weighted discrete inequality:

$$\left|\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2}\left(\sum_{i=1}^{n} w_i x_i - x\right)\right| \tag{3.7}$$
$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^{n} w_i |x_i - x|$$

for any $x \in [a, b]$.

In particular, we have

$$\left|\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a) + \Phi'(b)}{2}\left(\sum_{i=1}^{n} w_i x_i - \frac{a+b}{2}\right)\right| \tag{3.8}$$
$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^{n} w_i \left|x_i - \frac{a+b}{2}\right| \leq \frac{1}{4}(b-a) \bigvee_a^b (\Phi')$$

and

$$\left|\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right)\right| \leq \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^{n} w_i \left|x_i - \sum_{i=1}^{n} w_i x_i\right| \tag{3.9}$$
$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \left(\sum_{j=1}^{n} w_i x_j^2 - \left(\sum_{k=1}^{n} w_k x_k\right)^2\right)^{1/2}$$
$$\leq \frac{1}{4}(b-a) \bigvee_a^b (\Phi').$$

4. Inequalities for Lipschitzian Derivatives

The following result holds:

Theorem 4.1. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\right| \tag{4.1}$$
$$\leq \frac{1}{2} K \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]$$

for any $x \in [a,b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu\right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2}.$$

In particular, we have

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{2}K\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2}\right)^2\right] \end{aligned} \tag{4.2}$$

and

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \leq \frac{1}{2}K\sigma_\mu^2(g) \leq \frac{1}{8}K(b-a)^2. \tag{4.3}$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$\begin{aligned} &\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) \\ &= \int_\Omega \left[(g-x)\int_0^1 \left(\Phi'((1-s)x + sg) - \Phi'(x)\right) ds\right] d\mu \end{aligned} \tag{4.4}$$

for any $x \in [a,b]$.

Taking the modulus in (4.4) we get

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\right| \\ &\leq \int_\Omega |g-x| \left|\int_0^1 \left(\Phi'((1-s)x + sg) - \Phi'(x)\right) ds\right| d\mu \\ &\leq \int_\Omega \left[|g-x| \int_0^1 \left|\left(\Phi'((1-s)x + sg) - \Phi'(x)\right)\right| ds\right] d\mu \\ &\leq K\int_\Omega \left[|g-x| \int_0^1 s|g-x| ds\right] d\mu = \frac{1}{2}K\int_\Omega (g-x)^2 d\mu \end{aligned} \tag{4.5}$$

for any $x \in [a,b]$.

However,

$$\begin{aligned}
&\int_{\Omega}(g-x)^2\,d\mu\\
&=\int_{\Omega}\left(g-\int_{\Omega}gd\mu+\int_{\Omega}gd\mu-x\right)^2 d\mu\\
&=\int_{\Omega}\left(g-\int_{\Omega}gd\mu\right)^2 d\mu+2\int_{\Omega}\left(g-\int_{\Omega}gd\mu\right)\left(\int_{\Omega}gd\mu-x\right)d\mu\\
&+\int_{\Omega}\left(\int_{\Omega}gd\mu-x\right)^2 d\mu\\
&=\int_{\Omega}\left(g-\int_{\Omega}gd\mu\right)^2 d\mu+\left(\int_{\Omega}gd\mu-x\right)^2
\end{aligned}$$

for any $x\in[a,b]$, and by (4.5) we get the desired result (4.1). $\square$

Corollary 4.2. *Let $\Phi:I\to\mathbb{C}$ be a twice differentiable functions on $[a,b]\subset\mathring{I}$ with $\|\Phi''\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}|\Phi''(t)|<\infty$. Then the inequalities (4.1)-(4.3) hold for $K=\|\Phi''\|_{[a,b],\infty}$.*

Remark 4.3. Let $\Phi:I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$ and with the property that the derivative Φ' is Lipschitzian with the constant $K>0$ on $[a,b]$. If $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\ldots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$, then one has the weighted discrete inequality:

$$\begin{aligned}
&\left|\sum_{i=1}^n w_i\Phi(x_i)-\Phi(x)-\Phi'(x)\left(\sum_{i=1}^n w_ix_i-x\right)\right| \qquad (4.6)\\
&\leq\frac{1}{2}K\left[\sigma_w^2(\mathbf{x})+\left(\sum_{i=1}^n w_ix_i-x\right)^2\right]
\end{aligned}$$

for any $x\in[a,b]$, where

$$\sigma_w(\mathbf{x}):=\left(\sum_{i=1}^n w_i\left(x_i-\sum_{k=1}^n w_kx_k\right)^2\right)^{1/2}=\left(\sum_{i=1}^n w_ix_i^2-\left(\sum_{k=1}^n w_kx_k\right)^2\right)^{1/2}.$$

The following lemma may be stated:

Lemma 4.4. *Let $u:[a,b]\to\mathbb{R}$ and $l,L\in\mathbb{R}$ with $L>l$. The following statements are equivalent:*

(i) *The function $u-\frac{l+L}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$ is $\frac{1}{2}(L-l)-$Lipschitzian;*

(ii) *We have the inequalities*

$$l\leq\frac{u(t)-u(s)}{t-s}\leq L\quad\textit{for each}\ \ t,s\in[a,b]\quad\textit{with}\ \ t\neq s; \qquad (4.7)$$

(iii) *We have the inequalities*

$$l(t-s)\leq u(t)-u(s)\leq L(t-s)\quad\textit{for each}\ \ t,s\in[a,b]\quad\textit{with}\ \ t>s. \qquad (4.8)$$

Following [53], we can introduce the definition of (l, L)-Lipschitzian functions:

Definition 4.5. The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be (l, L)-Lipschitzian on $[a, b]$.

If $L > 0$ and $l = -L$, then $(-L, L)$ −Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l, L)-Lipschitzian functions.

Proposition 4.6. *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $-\infty < l = \inf_{t\in[a,b]} u'(t)$ and $\sup_{t\in[a,b]} u'(t) = L < \infty$, then u is (l, L)-Lipschitzian on $[a, b]$.*

The following result holds.

Corollary 4.7. *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, with the property that the derivative Φ' is (l, L)-Lipschitzian on $[a, b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have*

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_\Omega g d\mu - x \right) \right. \\ &\left. -\frac{1}{4}(L + l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x \right)^2 \right] \right| \\ &\leq \frac{1}{4}(L - l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x \right)^2 \right] \end{aligned} \tag{4.9}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right) \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) \right. \\ &\left. -\frac{1}{4}(L + l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \right| \\ &\leq \frac{1}{4}(L - l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu \right) - \frac{1}{4}(L + l)\sigma_\mu^2(g) \right| &\leq \frac{1}{4}(L - l)\sigma_\mu^2(g) \\ &\leq \frac{1}{16}(L - l)(b - a)^2. \end{aligned} \tag{4.11}$$

Proof. Consider the auxiliary function $\Psi : [a, b] \to \mathbb{R}$ given by

$$\Psi(x) = \Phi(x) - \frac{1}{4}(L + l)x^2.$$

We observe that Ψ is differentiable and

$$\Psi'(x) = \Phi'(x) - \frac{1}{2}(L + l)x.$$

Since Φ' is (l, L)-Lipschitzian on $[a, b]$ it follows that Ψ' is Lipschitzian with the constant $\frac{1}{2}(L - l)$, so we can apply Theorem 4.1 for Ψ, i.e. we have the inequality

$$\left|\int_\Omega \Psi \circ g d\mu - \Psi(x) - \Psi'(x)\left(\int_\Omega g d\mu - x\right)\right| \tag{4.12}$$
$$\le \frac{1}{4}(L - l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right].$$

However

$$\begin{aligned}
&\int_\Omega \Psi \circ g d\mu - \Psi(x) - \Psi'(x)\left(\int_\Omega g d\mu - x\right)\\
&= \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\\
&- \frac{1}{4}(L + l)\left[\int_\Omega g^2 d\mu - x^2 - 2x\left(\int_\Omega g d\mu - x\right)\right]\\
&= \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\\
&- \frac{1}{4}(L + l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]
\end{aligned}$$

and by (4.12) we get the desired result (4.9). □

Remark 4.8. We observe that if the function Φ is twice differentiable on $\mathring{I}$ and for $[a, b] \subset \mathring{I}$ we have

$$-\infty < l \le \Phi''(x) \le L < \infty \text{ for any } x \in [a, b],$$

then Φ' is (l, L)-Lipschitzian on $[a, b]$ and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

Theorem 4.9. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on*

Ω and such that $\Phi \circ g,\ g \in L(\Omega, \mu)$, then we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \tag{4.13}$$
$$\leq \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right|\int_\Omega |g - x| d\mu + \int_\Omega |g - x|\left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right]\int_\Omega |g - x| d\mu$$

for any $x \in [a, b]$, where

$$\left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty} := ess\sup_{t\in\Omega}\left|g(t) - \int_\Omega g d\mu\right| < \infty.$$

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \tag{4.14}$$
$$\leq \frac{1}{2}K\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right|\int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right.$$
$$\left.+ \int_\Omega \left|g - \frac{a+b}{2}\right|\left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2}K\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right]\int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'\left(\int_\Omega g d\mu\right)$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) \tag{4.15}$$
$$= \int_\Omega \left[(g - x)\int_0^1 \left(\Phi'((1-s)x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right] d\mu$$

for any $x \in [a, b]$.

Taking the modulus in (4.15) we get

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \qquad (4.16)\\
&\leq \int_\Omega |g-x|\left|\int_0^1 \left(\Phi'((1-s)x+sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right| d\mu\\
&\leq \int_\Omega \left[|g-x|\int_0^1 \left|\left(\Phi'((1-s)x+sg) - \Phi'\left(\int_\Omega g d\mu\right)\right)\right| ds\right] d\mu\\
&\leq K\int_\Omega \left[|g-x|\int_0^1 \left|(1-s)x+sg - \int_\Omega g d\mu\right| ds\right] d\mu\\
&= K\int_\Omega \left[|g-x|\int_0^1 \left|(1-s)x+sg - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds\right] d\mu\\
&:= B.
\end{aligned}$$

Using the triangle inequality we have for any $t\in\Omega$

$$\begin{aligned}
&\int_0^1 \left|(1-s)x+sg(t) - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds\\
&\leq \int_0^1 (1-s)\left|x-\int_\Omega g d\mu\right| ds + \int_0^1 s\left|g(t) - \int_\Omega g d\mu\right| ds\\
&= \frac{1}{2}\left[\left|x-\int_\Omega g d\mu\right| + \left|g(t)-\int_\Omega g d\mu\right|\right]
\end{aligned}$$

and then

$$\begin{aligned}
B &\leq \frac{1}{2}K\int_\Omega |g-x|\left[\left|x-\int_\Omega g d\mu\right| + \left|g(t)-\int_\Omega g d\mu\right|\right] d\mu \qquad (4.17)\\
&= \frac{1}{2}K\left[\left|x-\int_\Omega g d\mu\right|\int_\Omega |g-x| d\mu + \int_\Omega |g-x|\left|g-\int_\Omega g d\mu\right| d\mu\right].
\end{aligned}$$

Making use of (4.16) and (4.17) we deduce the desired result (4.13). □

Corollary 4.10. *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, with the property that the derivative Φ' is (l,L)-Lipschitzian on $[a,b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ g \in L(\Omega,\mu)$, then we have*

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right. \qquad (4.18)\\
&\left.-\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(x-\int_\Omega g d\mu\right)^2\right]\right|\\
&\leq \frac{1}{4}(L-l)\left[\left|x-\int_\Omega g d\mu\right|\int_\Omega |g-x| d\mu + \int_\Omega |g-x|\left|g-\int_\Omega g d\mu\right| d\mu\right]\\
&\leq \frac{1}{4}(L-l)\left[\left|x-\int_\Omega g d\mu\right| + \left\|g-\int_\Omega g d\mu\right\|_{\Omega,\infty}\right]\int_\Omega |g-x| d\mu
\end{aligned}$$

for any $x \in [a, b]$.

In particular, we have

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right) \right. \tag{4.19}$$
$$\left. -\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(\frac{a+b}{2} - \int_\Omega g d\mu\right)^2\right] \right|$$
$$\leq \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right.$$
$$\left. + \int_\Omega \left|g - \frac{a+b}{2}\right| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.$$

5. Applications for f-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p | p : \Omega \to \mathbb{R},\ p(t) \geq 0,\ \int_\Omega p(t)\, d\mu(t) = 1\right\}$. The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

$$D_{KL}(p, q) := \int_\Omega p(t) \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p, q \in \mathcal{P}, \tag{5.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [45], χ^2*-divergence* D_{χ^2}, α*-divergence* D_α, *Bhattacharyya distance* D_B [3], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [47], *triangular discrimination* D_Δ [65], etc... They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(t) - q(t)|\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.2}$$

$$D_H(p,q) := \int_\Omega \left|\sqrt{p(t)} - \sqrt{q(t)}\right| d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.3}$$

$$D_{\chi^2}(p,q) := \int_\Omega p(t)\left[\left(\frac{q(t)}{p(t)}\right)^2 - 1\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t)\right], \quad p,q \in \mathcal{P}; \tag{5.5}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(t)\,q(t)}\,d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.6}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(t)\,q(t)}{p(t)+q(t)} d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.7}$$

$$D_J(p,q) := \int_\Omega [p(t) - q(t)] \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.8}$$

$$D_\Delta(p,q) := \int_\Omega \frac{[p(t) - q(t)]^2}{p(t)+q(t)} d\mu(t), \quad p,q \in \mathcal{P}. \tag{5.9}$$

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár f-divergence is defined as follows [13]

$$I_f(p,q) := \int_\Omega p(t)\, f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár f-divergence see [13], [14] and [68].

The following result holds:

Proposition 5.1. *Let $f : (0,\infty) \to \mathbb{R}$ be a twice differentiable convex function with the property that $f(1) = 0$ and there exists the constants γ, Γ so that*

$$-\infty < \gamma \le f(t) \le \Gamma < \infty.$$

Assume that $p,q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega. \tag{5.11}$$

If $x \in [r, R]$, then we have the inequality

$$\left| I_f(p,q) - f(x) - f'(x)(1-x) - \frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) + (1-x)^2\right]\right| \leq \frac{1}{4}(L-l)\left[D_{\chi^2}(p,q) + (1-x)^2\right]. \tag{5.12}$$

In particular, we have

$$\begin{aligned} &\left| I_f(p,q) - f\left(\frac{r+R}{2}\right) - f'\left(\frac{r+R}{2}\right)\left(1 - \frac{r+R}{2}\right)\right. \\ &\left. -\frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) + \left(1 - \frac{r+R}{2}\right)^2\right]\right| \\ &\leq \frac{1}{4}(L-l)\left[D_{\chi^2}(p,q) + \left(1 - \frac{r+R}{2}\right)^2\right] \end{aligned} \tag{5.13}$$

and

$$\left| I_f(p,q) - \frac{1}{4}(L+l) D_{\chi^2}(p,q)\right| \leq \frac{1}{4}(L-l) D_{\chi^2}(p,q). \tag{5.14}$$

Proof. From (4.9) we have

$$\begin{aligned} &\left|\int_\Omega p(t) f\left(\frac{q(t)}{p(t)}\right) d\mu(t) - f(x) - f'(x)(1-x)\right. \\ &\left. -\frac{1}{4}(L+l)\left[\int_\Omega p(t)\left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2\right]\right| \\ &\leq \frac{1}{4}(L-l)\left[\int_\Omega p(t)\left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2\right] \end{aligned}$$

for any $x \in [r, R]$, which is equivalent to (5.12). □

Utilising Corollary 4.10 we can state the following result as well:

Proposition 5.2. *With the assumptions in Proposition 5.1, we have*

$$\begin{aligned} &\left| I_f(p,q) - f(x) - f'(1)(1-x) - \frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) - (1-x)^2\right]\right| \\ &\leq \frac{1}{4}(L-l)\left[|x-1|\int_\Omega |q - xp|\, d\mu + \int_\Omega |q - xp|\left|\frac{q}{p} - 1\right| d\mu\right] \\ &\leq \frac{1}{4}(L-l)\left[|x-1| + \left\|\frac{q}{p} - 1\right\|_{\Omega,\infty}\right]\int_\Omega |q - xp|\, d\mu \end{aligned} \tag{5.15}$$

for any $x \in [r, R]$.

If we consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t \ln t$ then

$$\begin{aligned} I_f(p,q) &:= \int_\Omega p(t)\frac{q(t)}{p(t)}\ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega q(t)\ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) \\ &= D_{KL}(q,p). \end{aligned}$$

We have $f'(t) = \ln t + 1$ and $f''(t) = \frac{1}{t}$ and then we can choose $l = \frac{1}{R}$ and $L = \frac{1}{r}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(q,p) - \left(\frac{R+r}{4rR}\right) D_{\chi^2}(p,q) \right| \leq \frac{R-r}{4rR} D_{\chi^2}(p,q). \tag{5.16}$$

If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$ then

$$\begin{aligned} I_f(p,q) &:= -\int_\Omega p(t) \ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega p(t) \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t) \\ &= D_{KL}(p,q). \end{aligned}$$

We have $f'(t) = -\frac{1}{t}$ and $f''(t) = \frac{1}{t^2}$ and then we can choose $l = \frac{1}{R^2}$ and $L = \frac{1}{r^2}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(p,q) - \frac{R^2+r^2}{4R^2r^2} D_{\chi^2}(p,q) \right| \leq \frac{R^2-r^2}{4R^2r^2} D_{\chi^2}(p,q). \tag{5.17}$$

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References

1. S. M. Ali, S. D. Silvey, A general class of coefficients of divergence of one distribution from another, *J. Roy. Statist. Soc. Sec B,* **28**, (1966), 131-142.
2. G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135**(3), (2002), 175–189.
3. A. Bhattacharyya, On a measure of divergence between two statistical populations defined by their probability distributions, *Bull. Calcutta Math. Soc.,* **35**, (1943), 99-109.
4. M. Beth Bassat, f-entropies, probability of error and feature selection, *Inform. Control*, **39**, (1978), 227-242.
5. I. Burbea, C. R. Rao, On the convexity of some divergence measures based on entropy function, *IEEE Trans. Inf. Th.,* **28**(3), (1982), 489-495.
6. P. Cerone, S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, (2000), 135-200.
7. P. Cerone, S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, *Advances in Statistics Combinatorics and Related Areas*, (2002), 53-62.
8. P. Cerone, S. S. Dragomir, C. E. M. Pearce, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.,* **24**(2), (2000), 147-163.
9. P. Cerone, S. S. Dragomir, J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2), (1999), 697—712.
10. C. H. Chen, L. F. Pau, P. S. P. Wang *Statistical Pattern Recognition,*Rocelle Park, New York, Hoyderc Book Co., 1973.
11. C. K. Chow, C .N. Lin, Approximating discrete probability distributions with dependence trees, *IEEE Trans. Inf. Th.,* **14**(3), (1968), 462-467.
12. I. Csiszár, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica,* **2**, (1967), 299-318.

13. I. I. Csiszár, On topological properties of $f-$divergences, *Studia Math. Hungarica,* **2**, (1967), 329-339.
14. I. I. Csiszár, J. Körner, *Information Theory: Coding Theorem for Discrete Memoryless Systems,* Academic Press, New York, 1981.
15. S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1), (1999), 127-135.
16. S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38**, (1999), 33–37.
17. S. S. Dragomir, A converse result for Jensen's discrete inequality via Gruss' inequality and applications in information theory. *An. Univ. Oradea Fasc. Mat.*, **7**, (1999), 178–189.
18. S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.,* **60**(1), (1999), 495-508.
19. S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications, *Kragujevac J. Math.,* **22**, (2000), 13-18.
20. S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7**, (2000), 477-485.
21. S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1), (2001), 33-40.
22. S. S. Dragomir, On a reverse of Jessen's inequality for isotonic linear functionals, *J. Ineq. Pure & Appl. Math.*, **2**(3), (2001), Article 36.
23. S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1), (2001), 35-45.
24. S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5), (2002), Art. 68.
25. S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.*, **3**(2), (2002), Article 31, 8 pp.
26. S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications, *Libertas Math.*, **22**, (2002), 49–63.
27. S. S. Dragomir, Some companions of Ostrowski's inequality for absolutely continuous functions and applications, *Bull. Korean Math. Soc.*, **42**(2), (2005), 213–230.
28. S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2), (2003), 373-382.
29. S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications, *Demonstratio Math.*, **36**(3), (2003), 551–562.
30. S. S. Dragomir, Bounds for the deviation of a function from the chord generated by its extremities, *Bull. Aust. Math. Soc.*, **78**(2), (2008), 225–248.
31. S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.*, Springer Briefs in Mathematics, Springer, New York, 2012.
32. S. S. Dragomir, Perturbed companions of Ostrowski's inequality for absolutely continuous functions (I), Preprint *RGMIA Res. Rep. Coll.*, **17** Art 7 , (2014), 15 pp.
33. S. S. Dragomir, Reverses of the Jensen inequality in terms of first derivative and applications, *Acta Math. Vietnam*, **38**(3), (2013), 429-446.
34. S. S. Dragomir, Jensen and Ostrowski type inequalities for general Lebesgue integral with applications (I), *RGMIA Res. Rep. Coll.*, **17**, (2014).
35. S. S. Dragomir, P. Cerone, J. Roumeliotis, S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4), (1999), 301-314.

36. S. S. Dragomir, N. M. Ionescu, Some converse of Jensen's inequality and applications, *Rev. Anal. Numér. Théor. Approx.* , **23**(1), (1994), 71–78.
37. S. S. Dragomir, Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.
38. S. S. Dragomir, S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** , (1997), 239–244.
39. S. S. Dragomir, S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** , (1998), 105–109.
40. S. S. Dragomir, S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3),(1998), 245–304.
41. A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.,* **42**(117), (1992), 298-310.
42. D. V. Gokhale, S. Kullback, *Information in Contingency Tables*, New York, Marcel Decker, 1978.
43. A. Guessab and G. Schmeisser, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Th.,* **115** , (2002), 260-288.
44. J. H. Havrda, F. Charvat, Quantification method classification process: concept of structural α-entropy, *Kybernetika*, **3**, (1967), 30-35.
45. E. Hellinger, Neue Bergrüirdung du Theorie quadratisher Formerus von uneudlichvieleu Veränderlicher, *J. für reine and Augeur. Math.,* **36**, (1909), 210-271.
46. A. Sheikh Hossein, A generalized singular value inequality for Heinz means, *Iranian Journal of Mathematical Sciences and Informatics*, **10**(2), (2015), 23–27.
47. H. Jeffreys, An invariant form for the prior probability in estimating problems, *Proc. Roy. Soc. London*, **186**A, (1946), 453–461.
48. T. T. Kadota, L. A. Shepp, On the best finite set of linear observables for discriminating two Gaussian signals, *IEEE Trans. Inf. Th.,* **13**, (1967), 288-294.
49. T. Kailath, The divergence and Bhattacharyya distance measures in signal selection, *IEEE Trans. Comm. Technology.,* **COM-15**, (1967), 52-60.
50. J. N. Kapur, A comparative assessment of various measures of directed divergence, *Advances in Management Studies*, **3**, (1984), 1-16.
51. D. Kazakos, T. Cotsidas, A decision theory approach to the approximation of discrete probability densities, *IEEE Trans. Perform. Anal. Machine Intell.,* **1**, (1980), 61-67.
52. S. Kullback, R. A. Leibler, On information and sufficiency, *Annals Math. Statist.,* **22**, (1951), 79-86.
53. J. Lin, Divergence measures based on the Shannon entropy, *IEEE Trans. Inf. Th.,* **37** (1) (1991), 145-151.
54. M. Mei, The theory of genetic distance and evaluation of human races, *Japan J. Human Genetics*, **23** (1978), 341-369.
55. A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.
56. E. C. Pielou, *Ecological Diversity,* Wiley, New York, 1975.
57. C. R. Rao, Diversity and dissimilarity coefficients: a unified approach, *Theoretic Population Biology*, **21** , (1982), 24–43.
58. A. Rényi, On measures of entropy and information, *Proc. Fourth Berkeley Symp. Math. Stat. and Prob., University of California Press*, **1**, (1961), 547–561.
59. A. W. Roberts, D. E. Varberg, *Convex Functions,* Academic Press, 1973.
60. A. Sen, *On Economic Inequality,* Oxford University Press, London, 1973.

61. M. Z. Sarikaya, A. Saglamb and H. Yildirim, On generalization of Čebyšev type inequalities, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(1), (2010), 41–48.
62. B. D. Sharma, D. P. Mittal, New non-additive measures of relative information, *Journ. Comb. Inf. Sys. Sci.,* **2**(4), (1977), 122–132.
63. H. Shioya, T. Da-Te, A generalisation of Lin divergence and the derivative of a new information divergence, *Elec. and Comm. in Japan*, **78** (7), (1995), 37–40.
64. I. J. Taneja, *Generalised Information Measures and their Applications*, (`http://www.mtm.ufsc.br/~taneja/bhtml/bhtml.html`).
65. F. Topsoe, Some inequalities for information divergence and related measures of discrimination, *Res. Rep. Coll., RGMIA,* **2** (1), (1999), 85-98.
66. H. Theil, *Economics and Information Theory*, North-Holland, Amsterdam, 1967.
67. H. Theil, *Statistical Decomposition Analysis*, North-Holland, Amsterdam, 1972.
68. I. Vajda, *Theory of Statistical Inference and Information,* Dordrecht-Boston, Kluwer Academic Publishers, 1989.