Inequalities for the Derivatives of a Polynomial

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Abstract. The paper presents an $L^r$– analogue of an inequality regarding the $s^{th}$ derivative of a polynomial having zeros outside a circle of arbitrary radius but greater or equal to one. Our result provides improvements and generalizations of some well-known polynomial inequalities.

Keywords: Polynomial, Zeros, $s^{th}$ Derivative.

1. Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree at most $n$ and $P'(z)$ be its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

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and for every $r \geq 1$,
\[
\left\{ \frac{2\pi}{0} \left| P'(e^{i\theta}) \right|^r d\theta \right\}^\frac{1}{r} \leq n \left\{ \frac{2\pi}{0} \left| P(e^{i\theta}) \right|^r d\theta \right\}^\frac{1}{r}.
\] (1.2)

Inequality (1.1) is a classical result of Bernstein[6] whereas inequality (1.2) is due to Zygmund[15] who proved it for all trigonometric polynomials of degree $n$ and not only for those which are of the form $P(e^{i\theta})$. Arestov[1] proved that (1.2) remains true for $0 < r < 1$ as well. If $r \to \infty$ in inequality (1.2), we get (1.1).

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then both the inequalities (1.1) and (1.2) can be sharpened. In fact, If $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|
\] (1.3)
and
\[
\left\{ \frac{2\pi}{0} \left| P'(e^{i\theta}) \right|^r d\theta \right\}^\frac{1}{r} \leq nA_r \left\{ \frac{2\pi}{0} \left| P(e^{i\theta}) \right|^r d\theta \right\}^\frac{1}{r},
\] (1.4)
where
\[
A_r = \left\{ \frac{2\pi}{0} \left| 1 + e^{i\alpha} \right|^r d\alpha \right\}^{-\frac{1}{r}}.
\]

Inequality (1.3) was conjectured by Erdős and later verified by Lax[11], whereas inequality (1.4) was proved by De-Bruijn[7] for $r \geq 1$. Rahman and Schemeisser[13] later proved that (1.4) holds for $0 < r < 1$ also. If $r \to \infty$ in (1.4), we get (1.3).

As a generalization of (1.3) Malik[12] proved that if $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|,
\] (1.5)
whereas under the same hypothesis, Govil and Rahman[9] extended inequality (1.4) by showing that
\[
\left\{ \frac{2\pi}{0} \left| P'(e^{i\theta}) \right|^r d\theta \right\}^\frac{1}{r} \leq nE_r \left\{ \frac{2\pi}{0} \left| P(e^{i\theta}) \right|^r d\theta \right\}^\frac{1}{r},
\] (1.6)
where
\[
E_r = \left\{ \frac{2\pi}{0} \left| k + e^{i\alpha} \right|^r d\alpha \right\}^{-\frac{1}{r}}, \quad r \geq 1.
\]

In the same paper, Govil and Rahman[9, Theorem 4] extended inequality (1.5) to the $s^{th}$ derivative of a polynomial and proved under the same hypothesis
for \(1 \leq s < n\) that

\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + k^s} \max_{|z|=1} |P(z)|.
\]  

(1.7)

Inequality (1.7) was refined by Aziz and Rather [3, Corollary 1] by involving the binomial coefficients \(C(n,s)\), \(1 \leq s < n\) and coefficients of the polynomial \(P(z)\). In fact they proved that if \(P(z) = \sum_{j=0}^{n} a_j z^j\) does not vanish in \(|z| < k\), \(k \geq 1\), then for \(1 \leq s < n\),

\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + \psi_{k,s}} \max_{|z|=1} |P(z)|,
\]  

(1.8)

where

\[
\psi_{k,s} = k^{s+1} \left( \frac{1 + \frac{1}{C(n,s)} \frac{a_s}{a_0} k^{s-1}}{1 + \frac{1}{C(n,s)} \frac{a_s}{a_0} k^{s+1}} \right).
\]  

(1.9)

In the literature there exist various results regarding the estimates for polynomials and for general analytic functions and also the approximations of polynomials and their derivatives (for example see\([8],[14]\)). In this paper, we prove the following result which refines the inequality (1.8).

**Theorem 1.1.** If \(P(z) = \sum_{j=0}^{n} a_j z^j\) is a polynomial of degree \(n\) having no zeros in \(|z| < k\), \(k \geq 1\), and \(m = \min_{|z|=k} |P(z)|\) then for \(1 \leq s < n\),

\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + \psi_{k,s}} \left( \max_{|z|=1} |P(z)| - \frac{m \psi_{k,s}}{k^n} \right),
\]  

(1.10)

where \(\psi_{k,s}\) is defined by (1.9).

The result is best possible for \(k = 1\) and equality holds for \(P(z) = z^n + 1\).

**Remark 1.2.** For \(s = 1\) and \(m = 0\), Theorem 1.1 reduces to a result of Govil et al.[10, Theorem 1] and for \(k = s = 1\), inequality (1.10) reduces to a result of Aziz and Dawood[2, Theorem A].

**Remark 1.3.** Note by inequality (2.2) of Lemma 2.1 (stated in section 2) that

\[
\frac{1}{C(n,s)} \sum_{j=0}^{n} |a_j| k^s \leq 1,
\]

which can easily be shown to be equivalent to \(\psi_{k,s} \geq k^s\). \(1 \leq s < n\). Using this fact in inequality (1.10), we get the following improvement of inequality (1.7).

**Corollary 1.4.** If \(P(z) = \sum_{j=0}^{n} a_j z^j\) is a polynomial of degree \(n\) having no zeros in \(|z| < k\), \(k \geq 1\), and \(m = \min_{|z|=k} |P(z)|\) then for \(1 \leq s < n\),

\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + k^s} \left( \max_{|z|=1} |P(z)| - \frac{m}{k^{n-s}} \right).
\]  

(1.11)
In order to prove the Theorem 1.1, we prove the following more general result which extends Theorem 1.1 to its corresponding $L^r$–analogue.

**Theorem 1.5.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |P(z)|$, then for every complex number $\beta$ with $|\beta| \leq 1$ and $1 \leq s < n$, we have

$$\left\{ \frac{2\pi}{\psi_{k,s}} \right\}^\frac{1}{r} \leq n(n-1) \cdots (n-s) \frac{1}{C_r} \left\{ \frac{2\pi}{|P|} \right\}^\frac{1}{r}, \quad (1.12)$$

where $C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha \right\}^{-\frac{1}{r}}$, $r > 0$ and $\psi_{k,s}$ is defined by (1.9).

**Remark 1.6.** Using the fact that $\psi_{k,s} \geq k^s$ and take $\beta = 0$ in inequality (1.12), we obtain a result of Aziz and Shah.[5]

2. **Lemmata**

We need the following lemmas for the proofs of Theorems. Here, throughout this paper we write $Q(z) = z^n P\left(\frac{1}{z}\right)$.

**Lemma 2.1.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ which does not vanish in $|z| < k$, $k \geq 1$, then for $1 \leq s < n$ and $|z| = 1$,

$$|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)|, \quad (2.1)$$

and

$$\frac{1}{C(n,s)} \frac{a_s}{a_0} k^s \leq 1, \quad (2.2)$$

where $\psi_{k,s}$ is defined by (1.9).

The above lemma is due to Aziz and Rather[3].

**Lemma 2.2.** If $P(z)$ is a polynomial of degree $n$, then for each $\alpha$, $0 \leq \alpha < 2\pi$ and $r > 0$, we have

$$\int_0^{2\pi} \int_0^{2\pi} |Q(e^{i\theta}) + e^{i\alpha} P(e^{i\theta})|^r d\theta d\alpha \leq 2\pi n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \quad (2.3)$$
The above lemma is due to Aziz and Shah[4].

**Lemma 2.3.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k, k \geq 1 \), then for \( 1 \leq s < n \) and \( |z| = 1 \),

\[
|Q^{(s)}(z)| \geq \psi_{k,s}|P^{(s)}(z)| + \frac{mn(n-1) \cdots (n-s+1)\beta}{k^n} \psi_{k,s}, \tag{2.4}
\]

where \( m = \min_{|z|=k} |P(z)| \).

**Proof.** Since \( |Q(z)| = |P(z)| \) for \( |z| = k \), we have for every \( \beta \) with \( |\beta| < 1 \),

\[
|\frac{m\beta z^n}{k^n}| < |P(z)| \text{ for } |z| = k.
\]

Therefore by Rouche’s theorem \( P(z) + \frac{m\beta z^n}{k^n} \) has no zero in \( |z| < k, k \geq 1 \). Applying Lemma 2.1 to the polynomial \( P(z) + \frac{m\beta z^n}{k^n} \), we get for \( 1 \leq s < n \) and \( |z| = 1 \),

\[
|Q^{(s)}(z)| \geq \psi_{k,s}|P^{(s)}(z)| + \frac{mn(n-1) \cdots (n-s+1)\beta}{k^n} \psi_{k,s}. \tag{2.5}
\]

Choose the argument of \( \beta \) so that

\[
|P^{(s)}(z) + \frac{mn(n-1) \cdots (n-s+1)\beta z^{n-s}}{k^n}| = |P^{(s)}(z)| + \frac{mn(n-1) \cdots (n-s+1)\beta z^{n-s}}{k^n},
\]

it follows from (2.5) that for \( |z| = 1 \),

\[
|Q^{(s)}(z)| \geq \psi_{k,s}|P^{(s)}(z)| + \frac{mn(n-1) \cdots (n-s+1)\beta z^{n-s}}{k^n} \psi_{k,s}. \tag{2.6}
\]

Letting \( |\beta| \to 1 \) in inequality (2.6), we get

\[
|Q^{(s)}(z)| \geq \psi_{k,s}|P^{(s)}(z)| + \frac{mn(n-1) \cdots (n-s+1)\beta z^{n-s}}{k^n} \psi_{k,s}.
\]

This completes the proof of Lemma 2.3.

**Lemma 2.4.** If \( A, B, C \) are non-negative real numbers such that \( B + C \leq A \). Then for every real \( \alpha \),

\[
|(A - C) + e^{i\alpha}(B + C)| \leq |A + e^{i\alpha}B|. \tag{2.7}
\]

The above lemma is due to Aziz and Shah[4].

3. PROOFS OF THEOREMS

**Proof of the Theorem 1.5.** Since \( P(z) \) is a polynomial of degree \( n \), \( P(z) \neq 0 \) in \( |z| < k, k \geq 1 \), and \( Q(z) = z^n P\left(\frac{1}{z}\right) \). Therefore, for each \( \alpha, 0 \leq \alpha < 2\pi, F(z) = Q(z) + e^{i\alpha}P(z) \) is a polynomial of degree \( n \) and we have

\[
F^{(s)}(z) = Q^{(s)}(z) + e^{i\alpha}P^{(s)}(z),
\]
which is clearly a polynomial of degree \( n-s, 1 \leq s < n \). By the repeated application of inequality (1.2), we have for each \( r > 0 \),

\[
\int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\theta \\
\leq (n-s+1)^r \int_0^{2\pi} \left| Q^{(s-1)}(e^{i\theta}) + e^{i\alpha} P^{(s-1)}(e^{i\theta}) \right|^r d\theta \\
\leq (n-s+1)^r (n-s+2)^r \int_0^{2\pi} \left| Q^{(s-2)}(e^{i\theta}) + e^{i\alpha} P^{(s-2)}(e^{i\theta}) \right|^r d\theta \\
\ldots \\
\leq (n-s+1)^r (n-s+2)^r \ldots (n-1)^r \int_0^{2\pi} \left| Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta}) \right|^r d\theta.
\tag{3.1}
\]

Integrating inequality (3.1) with respect to \( \alpha \) over \([0, 2\pi]\) and using inequality (2.3) of Lemma 2.2, we get

\[
\int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\theta d\alpha \\
\leq 2\pi (n-s+1)^r (n-s+2)^r \ldots (n-1)^r n^r \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta. \tag{3.2}
\]

Now, from inequality (2.4) of Lemma 2.3, it easily follows that

\[
\psi_{k,s} \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
\leq \left| Q^{(s)}(e^{i\theta}) \right| - \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}. \tag{3.3}
\]

Taking \( A = \left| Q^{(s)}(e^{i\theta}) \right|, B = \left| P^{(s)}(e^{i\theta}) \right|, C = \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \) and noting that \( \psi_{k,s} \geq k^s \geq 1, 1 \leq s < n \), so that by (3.3),

\[
B + C \leq \psi_{k,s}(B + C) \leq A - C \leq A,
\]

we get from Lemma 2.4 that

\[
\left\{ \left| Q^{(s)}(e^{i\theta}) \right| - \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
+ e^{i\alpha} \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
\leq \left| Q^{(s)}(e^{i\theta}) \right| + e^{i\alpha} \left| P^{(s)}(e^{i\theta}) \right|.
\]
This implies for each \( r > 0 \),
\[
\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^r d\alpha \leq \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\alpha, \tag{3.4}
\]
where
\[
F(\theta) = \left| Q^{(s)}(e^{i\theta}) \right| - \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}
\]
and
\[
G(\theta) = \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.
\]

Integrating inequality (3.4) with respect to \( \theta \) on \([0, 2\pi]\) and using inequality (3.2), we obtain
\[
\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^r d\alpha d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\alpha d\theta
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\alpha d\theta
\]
\[
\leq (n-s+1)^r(n-s+2)^r\ldots(n-1)^r n^r \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta.
\tag{3.5}
\]

Now for every real number \( \alpha \) and \( t_1 \geq t_2 \geq 1 \), we have
\[
|t_1 + e^{i\alpha}| \geq |t_2 + e^{i\alpha}|
\]
which implies for every \( r > 0 \),
\[
\int_0^{2\pi} |t_1 + e^{i\alpha}|^r d\alpha \geq \int_0^{2\pi} |t_2 + e^{i\alpha}|^r d\alpha.
\]
If $G(\theta) \neq 0$, we take $t_1 = \left| \frac{F(\theta)}{G(\theta)} \right|$ and $t_2 = \psi_{k,s}$, then from (3.3) and noting that
\[\psi_{k,s} \geq 1, \quad \text{hence} \]
\[
\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^r \, d\alpha = \left| G(\theta) \right|^r \int_0^{2\pi} \left| \frac{F(\theta)}{G(\theta)} + e^{i\alpha} \right|^r \, d\alpha
\]
\[
\geq \left| G(\theta) \right|^r \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha
\]
\[
= \left\{ \left| P(s)(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r
\int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha. \quad (3.6)
\]

For $G(\theta) = 0$, this inequality is trivially true. Using this in (3.5), it follows for each $r > 0$,
\[
\int_0^{2\pi} \left\{ \left| P(s)(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r d\theta
\]
\[
\leq \frac{(n-s+1)^r(n-s+2)^r \ldots (n-1)^r n^r}{12\pi} \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha \int_0^{2\pi} \left| P(s)(e^{i\theta}) \right|^r d\theta. \quad (3.7)
\]

Now using the fact that for every $\beta$ with $|\beta| \leq 1$,
\[
\left| P(s)(e^{i\theta}) + \frac{\beta mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right| \leq \left| P(s)(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})},
\]
the desired result follows from (3.7).

**Proof of the Theorem 1.1** Making $r \to \infty$ and choosing the argument of $\beta$ suitably with $|\beta| = 1$ in (1.12), Theorem 1.1 follows.

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**References**