Inequalities for the Derivatives of a Polynomial

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Abstract. The paper presents an $L^r$-analogue of an inequality regarding the $s^{th}$ derivative of a polynomial having zeros outside a circle of arbitrary radius but greater or equal to one. Our result provides improvements and generalizations of some well-known polynomial inequalities.

Keywords: Polynomial, Zeros, $s^{th}$ Derivative.


1. Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree at most $n$ and $P'(z)$ be its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

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and for every \( r \geq 1 \),
\[
\left\{ \frac{2\pi}{0} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \frac{2\pi}{0} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}.
\tag{1.2}
\]

Inequality (1.1) is a classical result of Bernstein\[6\] whereas inequality (1.2) is due to Zygmund\[15\] who proved it for all trigonometric polynomials of degree \( n \) and not only for those which are of the form \( P(e^{i\theta}) \). Arestov\[1\] proved that (1.2) remains true for \( 0 < r < 1 \) as well. If \( r \to \infty \) in inequality (1.2), we get (1.1).

If we restrict ourselves to the class of polynomials having no zeros in \( |z| < 1 \), then both the inequalities (1.1) and (1.2) can be sharpened. In fact, if \( P(z) \neq 0 \) in \( |z| < 1 \), then (1.1) and (1.2) can be respectively replaced by
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|
\tag{1.3}
\]
and
\[
\left\{ \frac{2\pi}{0} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq nA_r \left\{ \frac{2\pi}{0} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}},
\tag{1.4}
\]
where
\[
A_r = \left\{ \frac{2\pi}{0} \frac{1}{1 + |e^{i\alpha}|^r} d\alpha \right\}^{\frac{1}{r}}.
\]

Inequality (1.3) was conjectured by Erdős and later verified by Lax\[11\], whereas inequality (1.4) was proved by De-Bruijn\[7\] for \( r \geq 1 \). Rahman and Schemeisser\[13\] later proved that (1.4) holds for \( 0 < r < 1 \) also. If \( r \to \infty \) in (1.4), we get (1.3).

As a generalization of (1.3) Malik\[12\] proved that if \( P(z) \neq 0 \) in \( |z| < k, \ k \geq 1 \), then
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |P(z)|,
\tag{1.5}
\]
whereas under the same hypothesis, Govil and Rahman\[9\] extended inequality (1.4) by showing that
\[
\left\{ \frac{2\pi}{0} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq nE_r \left\{ \frac{2\pi}{0} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}},
\tag{1.6}
\]
where
\[
E_r = \left\{ \frac{2\pi}{0} \frac{1}{1 + |e^{i\alpha}|^r} d\alpha \right\}^{\frac{1}{r}}, \ r \geq 1.
\]

In the same paper, Govil and Rahman\[9, Theorem 4\] extended inequality (1.5) to the \( s^{th} \) derivative of a polynomial and proved under the same hypothesis
for $1 \leq s < n$ that
\[\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + k^s} \max_{|z|=1} |P(z)|.\] (1.7)

Inequality (1.7) was refined by Aziz and Rather [3, Corollary 1] by involving the binomial coefficients $C(n, s)$, $1 \leq s < n$ and coefficients of the polynomial $P(z)$. In fact they proved that if $P(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| < k$, $k \geq 1$, then for $1 \leq s < n$,
\[\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + \psi_{k, s}} \max_{|z|=1} |P(z)|,\] (1.8)

where
\[\psi_{k, s} = k^{s+1} \left(1 + \frac{1}{C(n, s)} \frac{a_n}{a_0} k^s \right)^{s+1}.\] (1.9)

In the literature there exist various results regarding the estimates for polynomials and for general analytic functions and also the approximations of polynomials and their derivatives (for example see[8],[14]). In this paper, we prove the following result which refines the inequality (1.8).

**Theorem 1.1.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |P(z)|$ then for $1 \leq s < n$,
\[\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + \psi_{k, s}} \left(\max_{|z|=1} |P(z)| - \frac{m \psi_{k, s}}{k^n}\right),\] (1.10)

where $\psi_{k, s}$ is defined by (1.9).

The result is best possible for $k = 1$ and equality holds for $P(z) = z^n + 1$.

**Remark 1.2.** For $s = 1$ and $m = 0$, Theorem 1.1 reduces to a result of Govil et. al.[10, Theorem 1] and for $k = s = 1$, inequality (1.10) reduces to a result of Aziz and Dawood[2, Theorem A].

**Remark 1.3.** Note by inequality (2.2) of Lemma 2.1 (stated in section 2) that $\frac{1}{C(n, s)} \frac{a_n}{a_0} k^s \leq 1$, which can easily be shown to be equivalent to $\psi_{k, s} \geq k^s$, $1 \leq s < n$. Using this fact in inequality (1.10), we get the following improvement of inequality (1.7).

**Corollary 1.4.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |P(z)|$ then for $1 \leq s < n$,
\[\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + k^s} \left(\max_{|z|=1} |P(z)| - \frac{m}{k^n}\right).\] (1.11)
In order to prove the Theorem 1.1, we prove the following more general result which extends Theorem 1.1 to its corresponding $L^r$ - analogue.

**Theorem 1.5.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |P(z)|$, then for every complex number $\beta$ with $|\beta| \leq 1$ and $1 \leq s < n$, we have

$$\left\{ \int_{0}^{2\pi} |P(s)(e^{i\theta}) + e^{i\alpha}P(e^{i\theta})|^r \, d\theta \right\}^{\frac{1}{r}} \leq \frac{\beta mn(n-1)\cdots(n-s+1)\psi_{k,s}^r}{k^n(1+\psi_{k,s})} \leq n(n-1)\cdots(n-s+1)C_r \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^r \, d\theta \right\}^{\frac{1}{r}},$$

(1.12)

where $C_r = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r \, d\alpha \right\}^{-\frac{1}{r}}$, $r > 0$ and $\psi_{k,s}$ is defined by (1.9).

**Remark 1.6.** Using the fact that $\psi_{k,s} \geq k^s$ and take $\beta = 0$ in inequality (1.12), we obtain a result of Aziz and Shah[5].

2. **Lemmas**

We need the following lemmas for the proofs of Theorems. Here, throughout this paper we write $Q(z) = z^nP(\frac{1}{z})$.

**Lemma 2.1.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ which does not vanish in $|z| < k, k \geq 1$, then for $1 \leq s < n$ and $|z| = 1$,

$$|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)|,$$

(2.1)

and

$$\frac{1}{C(n,s)} \frac{a_s}{a_0} |k^s| \leq 1,$$

(2.2)

where $\psi_{k,s}$ is defined by (1.9).

The above lemma is due to Aziz and Rather[3].

**Lemma 2.2.** If $P(z)$ is a polynomial of degree $n$, then for each $\alpha$, $0 \leq \alpha < 2\pi$ and $r > 0$, we have

$$\int_{0}^{2\pi} \int_{0}^{2\pi} |Q'(e^{i\theta}) + e^{i\alpha}P'(e^{i\theta})|^r \, d\theta d\alpha \leq 2\pi n^r \int_{0}^{2\pi} |P(e^{i\theta})|^r \, d\theta.$$

(2.3)
The above lemma is due to Aziz and Shah[4].

**Lemma 2.3.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k, k \geq 1 \), then for \( 1 \leq s < n \) and \( |z| = 1 \),
\[
|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)}{k^n} \psi_{k,s},
\]
where \( m = \min_{|z|=k} |P(z)| \).

**Proof.** Since \( |P(z)| \leq 1 \), we have for every \( \beta \) with \( |\beta| < 1 \),
\[
|m\beta z^n| < |P(z)| \text{ for } |z| = k.
\]
Therefore by Rouche’s theorem \( P(z) + \frac{m\beta z^n}{k^n} \) has no zero in \( |z| < k, k \geq 1 \). Applying Lemma 2.1 to the polynomial \( P(z) + \frac{m\beta z^n}{k^n} \), we get for \( 1 \leq s < n \) and \( |z| = 1 \),
\[
|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)\beta}{k^n} |Q^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)\beta z^n}{k^n} |P^{(s)}(z)|.
\]
Choose the argument of \( \beta \) so that
\[
|\beta| \rightarrow 1 \text{ in inequality (2.6), we get}
\]
\[
|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)|\beta z^n|}{k^n} \psi_{k,s}.
\]
This completes the proof of Lemma 2.3. \( \square \)

**Lemma 2.4.** If \( A, B, C \) are non-negative real numbers such that \( B + C \leq A \). Then for every real \( \alpha \),
\[
|(A - C) + e^{i\alpha}(B + C)| \leq |A + e^{i\alpha}B|.
\]

The above lemma is due to Aziz and Shah[4].

### 3. Proofs of Theorems

**Proof of the Theorem 1.5.** Since \( P(z) \) is a polynomial of degree \( n \), \( P(z) \neq 0 \) in \( |z| < k, k \geq 1 \), and \( Q(z) = z^n P(z) \). Therefore, for each \( \alpha, 0 \leq \alpha < 2\pi, F(z) = Q(z) + e^{i\alpha}P(z) \) is a polynomial of degree \( n \) and we have
\[
F^{(s)}(z) = Q^{(s)}(z) + e^{i\alpha}P^{(s)}(z),
\]
which is clearly a polynomial of degree \( n - s, 1 \leq s < n \). By the repeated application of inequality (1.2), we have for each \( r > 0 \),
\[
\int_0^{2\pi} |Q^{(s)}(e^{i\theta}) + e^{i\alpha}P^{(s)}(e^{i\theta})|^r d\theta \\
\leq (n - s + 1)^r \int_0^{2\pi} |Q^{(s-1)}(e^{i\theta}) + e^{i\alpha}P^{(s-1)}(e^{i\theta})|^r d\theta \\
\leq (n - s + 1)^r (n - s + 2)^r \int_0^{2\pi} |Q^{(s-2)}(e^{i\theta}) + e^{i\alpha}P^{(s-2)}(e^{i\theta})|^r d\theta \\
\ldots
\leq (n - s + 1)^r (n - s + 2)^r \ldots (n - 1)^r \int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\alpha}P'(e^{i\theta})|^r d\theta. \tag{3.1}
\]
Integrating inequality (3.1) with respect to \( \alpha \) over \([0, 2\pi]\) and using inequality (2.3) of Lemma 2.2, we get
\[
\int_0^{2\pi} \int_0^{2\pi} |Q^{(s)}(e^{i\theta}) + e^{i\alpha}P^{(s)}(e^{i\theta})|^r d\theta d\alpha \\
\leq 2\pi(n - s + 1)^r (n - s + 2)^r \ldots (n - 1)^r n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \tag{3.2}
\]
Now, from inequality (2.4) of Lemma 2.3, it easily follows that
\[
\psi_{k,s}\left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
\leq |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}. \tag{3.3}
\]
Taking \( A = |Q^{(s)}(e^{i\theta})|, B = |P^{(s)}(e^{i\theta})|, C = \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \) and noting that \( \psi_{k,s} \geq k^s \geq 1, 1 \leq s < n \), so that by (3.3),
\[
B + C \leq \psi_{k,s}(B + C) \leq A - C \leq A,
\]
we get from Lemma 2.4 that
\[
\left\{ |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} + e^{i\alpha}\left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
\leq |Q^{(s)}(e^{i\theta})| + e^{i\alpha}|P^{(s)}(e^{i\theta})|. \]
This implies for each \( r > 0 \),
\[
\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r \, d\alpha \leq \int_0^{2\pi} \left| \left| Q^{(s)}(e^{i\theta}) \right| + e^{i\alpha} \left| P^{(s)}(e^{i\theta}) \right| \right|^r \, d\alpha, \tag{3.4}
\]
where
\[
F(\theta) = \left| Q^{(s)}(e^{i\theta}) \right| - \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}
\]
and
\[
G(\theta) = \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.
\]
Integrating inequality (3.4) with respect to \( \theta \) on \([0, 2\pi]\) and using inequality (3.2), we obtain
\[
\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r \, d\alpha \, d\theta
\leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) \right| + e^{i\alpha} \left| P^{(s)}(e^{i\theta}) \right| \right|^r \, d\alpha \, d\theta
= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r \, d\alpha \, d\theta
\leq (n-s+1)^r (n-s+2)^r \ldots (n-1)^r \int_0^{2\pi} |P(e^{i\theta})|^r \, d\theta.
\tag{3.5}
\]
Now for every real number \( \alpha \) and \( t_1 \geq t_2 \geq 1 \), we have
\[
|t_1 + e^{i\alpha}| \geq |t_2 + e^{i\alpha}|,
\]
which implies for every \( r > 0 \),
\[
\int_0^{2\pi} |t_1 + e^{i\alpha}|^r \, d\alpha \geq \int_0^{2\pi} |t_2 + e^{i\alpha}|^r \, d\alpha.
\]
If \( G(\theta) \neq 0 \), we take \( t_1 = \left| \frac{F(\theta)}{G(\theta)} \right| \) and \( t_2 = \psi_{k,s} \), then from (3.3) and noting that \( \psi_{k,s} \geq 1 \), we have \( t_1 \geq t_2 \geq 1 \), hence

\[
\int_0^{2\pi} \left| F(\theta) + e^{i\alpha}G(\theta) \right|^r \, d\alpha = |G(\theta)|^r \int_0^{2\pi} \left| \frac{F(\theta)}{G(\theta)} + e^{i\alpha} \right|^r \, d\alpha
\]
\[
\geq |G(\theta)|^r \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha
\]
\[
= \left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha. \tag{3.6}
\]

For \( G(\theta) = 0 \), this inequality is trivially true. Using this in (3.5), it follows for each \( r > 0 \),

\[
\int_0^{2\pi} \left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r \, d\theta
\]
\[
\leq \frac{(n-s+1)^r(n-s+2)^r\ldots(n-1)^rn^r}{2\pi} \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r \, d\alpha \int_0^{2\pi} \left| P^{(s)}(e^{i\theta}) \right|^r \, d\theta. \tag{3.7}
\]

Now using the fact that for every \( \beta \) with \( |\beta| \leq 1 \),

\[
\left| P^{(s)}(e^{i\theta}) + \frac{\beta mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right| \leq \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\ldots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})},
\]

the desired result follows from (3.7).

**Proof of the Theorem 1.1** Making \( r \to \infty \) and choosing the argument of \( \beta \) suitably with \( |\beta| = 1 \) in (1.12), Theorem 1.1 follows.

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**References**