Inequalities for the Derivatives of a Polynomial

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\textbf{Abstract}. The paper presents an $L^{-r}$ analogue of an inequality regarding the $s^{th}$ derivative of a polynomial having zeros outside a circle of arbitrary radius but greater or equal to one. Our result provides improvements and generalizations of some well-known polynomial inequalities.

\textbf{Keywords}: Polynomial, Zeros, $s^{th}$ Derivative.


1. Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree at most $n$ and $P'(z)$ be its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

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Received 26 February 2015; Accepted 05 December 2015
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and for every $r \geq 1$,
\[
\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}. \tag{1.2}
\]

Inequality (1.1) is a classical result of Bernstein[6] whereas inequality (1.2) is due to Zygmund[15] who proved it for all trigonometric polynomials of degree $n$ and not only for those which are of the form $P(e^{i\theta})$. Arestov[1] proved that (1.2) remains true for $0 < r < 1$ as well. If $r \to \infty$ in inequality (1.2), we get (1.1).

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then both the inequalities (1.1) and (1.2) can be sharpened. In fact, If $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}
\]
and
\[
\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n A_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.4}
\]
where $A_r = \left\{ \int_0^{2\pi} \frac{1}{|1 + e^{i\alpha}|^r} d\alpha \right\}^{\frac{1}{r}}$.

Inequality (1.3) was conjectured by Erdős and later verified by Lax[11], whereas inequality (1.4) was proved by De-Bruijn[7] for $r \geq 1$. Rahman and Schemeisser[13] later proved that (1.4) holds for $0 < r < 1$ also. If $r \to \infty$ in (1.4), we get (1.3).

As a generalization of (1.3) Malik[12] proved that if $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |P(z)|, \tag{1.5}
\]
whereas under the same hypothesis, Govil and Rahman[9] extended inequality (1.4) by showing that
\[
\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n E_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.6}
\]
where $E_r = \left\{ \int_0^{2\pi} \frac{1}{|k + e^{i\alpha}|^r} d\alpha \right\}^{\frac{1}{r}}$, $r \geq 1$.

In the same paper, Govil and Rahman[9, Theorem 4] extended inequality (1.5) to the $s^{th}$ derivative of a polynomial and proved under the same hypothesis
for \(1 \leq s < n\) that
\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \max_{|z|=1} |P(z)|. \tag{1.7}
\]

Inequality (1.7) was refined by Aziz and Rather [3, Corollary 1] by involving the binomial coefficients \(C(n,s)\), \(1 \leq s < n\) and coefficients of the polynomial \(P(z)\). In fact they proved that if \(P(z) = \sum_{j=0}^{n} a_j z^j\) does not vanish in \(|z| < k\), \(k \geq 1\), then for \(1 \leq s < n\),
\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1)\cdots(n-s+1)}{1+\psi_{k,s}} \max_{|z|=1} |P(z)|, \tag{1.8}
\]
where
\[
\psi_{k,s} = k^{s+1} \left( \frac{1 + \frac{1}{C(n,s)} \frac{a_s}{a_n} k^{s-1}}{1 + \frac{1}{C(n,s)} \frac{a_s}{a_n} k^{s+1}} \right). \tag{1.9}
\]

In the literature there exist various results regarding the estimates for polynomials and for general analytic functions and also the approximations of polynomials and their derivatives (for example see [8],[14]). In this paper, we prove the following result which refines the inequality (1.8).

**Theorem 1.1.** If \(P(z) = \sum_{j=0}^{n} a_j z^j\) is a polynomial of degree \(n\) having no zeros in \(|z| < k\), \(k \geq 1\), and \(m = \min_{|z|=k} |P(z)|\) then for \(1 \leq s < n\),
\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1)\cdots(n-s+1)}{1 + \psi_{k,s}} \left( \max_{|z|=1} |P(z)| - \frac{m \psi_{k,s}}{k^n} \right), \tag{1.10}
\]
where \(\psi_{k,s}\) is defined by (1.9).

The result is best possible for \(k = 1\) and equality holds for \(P(z) = z^n + 1\).

**Remark 1.2.** For \(s = 1\) and \(m = 0\), Theorem 1.1 reduces to a result of Govil et. al.[10, Theorem 1] and for \(k = s = 1\), inequality (1.10) reduces to a result of Aziz and Dawood[2, Theorem A].

**Remark 1.3.** Note by inequality (2.2) of Lemma 2.1 (stated in section 2) that \(\frac{1}{C(n,s)} \frac{a_s}{a_n} k^s \leq 1\), which can easily be shown to be equivalent to \(\psi_{k,s} \geq k^s\), \(1 \leq s < n\). Using this fact in inequality (1.10), we get the following improvement of inequality (1.7).

**Corollary 1.4.** If \(P(z) = \sum_{j=0}^{n} a_j z^j\) is a polynomial of degree \(n\) having no zeros in \(|z| < k\), \(k \geq 1\), and \(m = \min_{|z|=k} |P(z)|\) then for \(1 \leq s < n\),
\[
\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \left( \max_{|z|=1} |P(z)| - \frac{m}{k^n} \right). \tag{1.11}
\]
In order to prove the Theorem 1.1, we prove the following more general result which extends Theorem 1.1 to its corresponding $L^r$-analogue.

**Theorem 1.5.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |P(z)|$, then for every complex number $\beta$ with $|\beta| \leq 1$ and $1 \leq s < n$, we have

$$\left\{ \int_0^{2\pi} |p(s)(e^{i\theta}) + \beta mn(n-1)\cdots(n-s+1)\psi_{k,s}\cdots(n-s+1)\psi_{k,s} + 1 \right\} \leq n(n-1)\cdots(n-s+1)C_{r}\left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \quad (1.12)$$

where $C_r = \left\{ \frac{2\pi}{2\pi} \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha \right\}^{-\frac{1}{r}}$, $r > 0$ and $\psi_{k,s}$ is defined by (1.9).

**Remark 1.6.** Using the fact that $\psi_{k,s} \geq k^s$ and take $\beta = 0$ in inequality (1.12), we obtain a result of Aziz and Shah[5].

2. **Lemmas**

We need the following lemmas for the proofs of Theorems. Here, throughout this paper we write $Q(z) = z^n P(\frac{z}{z})$.

**Lemma 2.1.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ which does not vanish in $|z| < k, k \geq 1$, then for $1 \leq s < n$ and $|z| = 1$,

$$|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)|,$$

and

$$\frac{1}{C(n,s)} \frac{a_s}{a_0} k^s \leq 1,$$

where $\psi_{k,s}$ is defined by (1.9).

The above lemma is due to Aziz and Rather[3].

**Lemma 2.2.** If $P(z)$ is a polynomial of degree $n$, then for each $\alpha, \ 0 \leq \alpha < 2\pi$ and $r > 0$, we have

$$\int_0^{2\pi} \int_0^{2\pi} |Q^{(e^{i\theta})} + e^{i\alpha} P^{(e^{i\theta})})|^r d\theta d\alpha \leq 2\pi n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \quad (2.3)$$
The above lemma is due to Aziz and Shah[4].

**Lemma 2.3.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) which does not vanish in \(|z| < k, k \geq 1\), then for \( 1 \leq s < n \) and \(|z| = 1\),

\[
|Q^{(s)}(z)| \geq \psi_{k,s}(|P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)}{k^n} \psi_{k,s}), \tag{2.4}
\]

where \( m = \min_{|z|=k} |P(z)| \).

**Proof.** Since \( m \leq |P(z)| \) for \(|z| = k\), we have for every \( \beta \) with \(|\beta| < 1\),

\[
|\frac{m\beta z^n}{k^n}| < |P(z)| \text{ for } |z| = k.
\]

Therefore by Rouche’s theorem \( P(z) + \frac{m\beta z^n}{k^n} \) has no zero in \(|z| < k, k \geq 1\).

Applying Lemma 2.1 to the polynomial \( P(z) + \frac{m\beta z^n}{k^n} \), we get for \( 1 \leq s < n \) and \(|z| = 1\),

\[
|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)\beta}{k^n} \psi_{k,s}. \tag{2.5}
\]

Choose the argument of \( \beta \) so that

\[
|P^{(s)}(z) + \frac{mn(n-1)\cdots(n-s+1)\beta z^{n-s}}{k^n}| = |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)|\beta z^{n-s}|}{k^n},
\]

it follows from (2.5) that for \(|z| = 1\),

\[
|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)|\beta z^{n-s}|}{k^n} \psi_{k,s}. \tag{2.6}
\]

Letting \(|\beta| \to 1\) in inequality (2.6), we get

\[
|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1)\cdots(n-s+1)}{k^n} \psi_{k,s}.
\]

This completes the proof of Lemma 2.3. \(\Box\)

**Lemma 2.4.** If \( A, B, C \) are non-negative real numbers such that \( B + C \leq A \). Then for every real \( \alpha \),

\[
|A - C + e^{i\alpha}(B + C)| \leq |A + e^{i\alpha}B|. \tag{2.7}
\]

The above lemma is due to Aziz and Shah[4].

3. PROOFS OF THEOREMS

**Proof of the Theorem 1.5.** Since \( P(z) \) is a polynomial of degree \( n \), \( P(z) \neq 0 \) in \(|z| < k, k \geq 1\), and \( Q(z) = z^n P(\frac{1}{z}) \). Therefore, for each \( \alpha, 0 \leq \alpha < 2\pi, F(z) = Q(z) + e^{i\alpha}P(z) \) is a polynomial of degree \( n \) and we have

\[
F^{(s)}(z) = Q^{(s)}(z) + e^{i\alpha}P^{(s)}(z),
\]
which is clearly a polynomial of degree \( n - s, 1 \leq s < n \). By the repeated application of inequality (1.2), we have for each \( r > 0 \),

\[
\int_0^{2\pi} |Q^{(s)}(e^{i\theta}) + e^{i\alpha}P^{(s)}(e^{i\theta})|^r d\theta
\]

\[
\leq (n-s+1)^r \int_0^{2\pi} |Q^{(s-1)}(e^{i\theta}) + e^{i\alpha}P^{(s-1)}(e^{i\theta})|^r d\theta
\]

\[
\leq (n-s+1)^r(n-s+2)^r \int_0^{2\pi} |Q^{(s-2)}(e^{i\theta}) + e^{i\alpha}P^{(s-2)}(e^{i\theta})|^r d\theta
\]

\[
\vdots
\]

\[
\leq (n-s+1)^r(n-s+2)^r \ldots (n-1)^r \int_0^{2\pi} |Q^{(0)}(e^{i\theta}) + e^{i\alpha}P^{(0)}(e^{i\theta})|^r d\theta.
\]

Integrating inequality (3.1) with respect to \( \alpha \) over \([0, 2\pi]\) and using inequality (2.3) of Lemma 2.2, we get

\[
\int_0^{2\pi} \int_0^{2\pi} |Q^{(s)}(e^{i\theta}) + e^{i\alpha}P^{(s)}(e^{i\theta})|^r d\theta d\alpha
\]

\[
\leq 2\pi(n-s+1)^r(n-s+2)^r \ldots (n-1)^r n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \tag{3.2}
\]

Now, from inequality (2.4) of Lemma 2.3, it easily follows that

\[
\psi_{k,s} \left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}
\]

\[
\leq |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}. \tag{3.3}
\]

Taking \( A = |Q^{(s)}(e^{i\theta})|, B = |P^{(s)}(e^{i\theta})|, C = \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \)
and noting that \( \psi_{k,s} \geq k^s \geq 1, 1 \leq s < n \), so that by (3.3),

\[
B + C \leq \psi_{k,s}(B + C) \leq A - C \leq A,
\]

we get from Lemma 2.4 that

\[
\left\{ |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}
\]

\[
+ e^{i\alpha} \left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}
\]

\[
\leq |Q^{(s)}(e^{i\theta})| + e^{i\alpha} |P^{(s)}(e^{i\theta})|.
\]
This implies for each \( r > 0 \),
\[
\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^r d\alpha \leq \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\alpha, \tag{3.4}
\]

where
\[
F(\theta) = \left| Q^{(s)}(e^{i\theta}) \right| - \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1 + \psi_{k,s})}
\]

and
\[
G(\theta) = \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1 + \psi_{k,s})}.
\]

Integrating inequality (3.4) with respect to \( \theta \) on \([0, 2\pi]\) and using inequality (3.2), we obtain
\[
\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^r d\alpha d\theta
\leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\alpha d\theta
\leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\alpha d\theta
\leq (n-s+1)^r(n-s+2)^r \ldots (n-1)^r n^r \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta. \tag{3.5}
\]

Now for every real number \( \alpha \) and \( t_1 \geq t_2 \geq 1 \), we have
\[
|t_1 + e^{i\alpha}| \geq |t_2 + e^{i\alpha}|,
\]
which implies for every \( r > 0 \),
\[
\int_0^{2\pi} |t_1 + e^{i\alpha}|^r d\alpha \geq \int_0^{2\pi} |t_2 + e^{i\alpha}|^r d\alpha.
\]
If $G(\theta) \neq 0$, we take $t_1 = |\frac{F(\theta)}{G(\theta)}|$ and $t_2 = \psi_{k,s}$, then from (3.3) and noting that $\psi_{k,s} \geq 1$, we have $t_1 \geq t_2 \geq 1$, hence

$$
\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r \, d\alpha = |G(\theta)|^r \int_0^{2\pi} \left| \frac{F(\theta)}{G(\theta)} + e^{i\alpha} \right|^r \, d\alpha
$$

$$
= |G(\theta)|^r \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha
$$

$$
\geq |G(\theta)|^r \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha
$$

$$
= \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha.
$$

(3.6)

For $G(\theta) = 0$, this inequality is trivially true. Using this in (3.5), it follows for each $r > 0$,

$$
\int_0^{2\pi} \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r \, d\theta
$$

$$
\leq \frac{(n-s+1)^r(n-s+2)^r \ldots (n-1)^r n^r}{\frac{1}{2\pi} \int_0^{2\pi} \left| \psi_{k,s} + e^{i\alpha} \right|^r \, d\alpha} \int_0^{2\pi} \left| P^{(s)}(e^{i\theta}) \right|^r \, d\theta.
$$

(3.7)

Now using the fact that for every $\beta$ with $|\beta| \leq 1$,

$$
\left| P^{(s)}(e^{i\beta}) + \frac{\beta mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right| \leq \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1) \ldots (n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})},
$$

the desired result follows from (3.7).

**Proof of the Theorem 1.1** Making $r \to \infty$ and choosing the argument of $\beta$ suitably with $|\beta| = 1$ in (1.12), Theorem 1.1 follows.

**Acknowledgement**

The authors are highly thankful to the referee for his valuable suggestions regarding the paper.

**References**


