

Iranian Journal of Mathematical Sciences and Informatics  
 Vol. 12, No. 2 (2017), pp 101-109.  
 DOI: 10.7508/ijmsi.2017.2.006

## Inequalities for the Derivatives of a Polynomial

Abdullah Mir<sup>a,\*</sup>, Q. M. Dawood<sup>a</sup> and Bilal Dar<sup>b</sup>

<sup>a</sup>Department of Mathematics, University of Kashmir,  
 Srinagar, 190006, India.

<sup>b</sup>Govt. Degree College (Boys), Pulwama-192301.

E-mail: mabdullah\_mir@yahoo.co.in

E-mail: qdawood@gmail.com

E-mail: darbilal85@ymail.com

ABSTRACT. The paper presents an  $L^r$ - analogue of an inequality regarding the  $s^{th}$  derivative of a polynomial having zeros outside a circle of arbitrary radius but greater or equal to one. Our result provides improvements and generalizations of some well-known polynomial inequalities.

**Keywords:** Polynomial, Zeros,  $s^{th}$  Derivative.

**2010 Mathematics subject classification:** 30A10, 30C10, 30C15.

### 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $P(z)$  be a polynomial of degree at most  $n$  and  $P'(z)$  be its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

---

\*Corresponding Author

and for every  $r \geq 1$ ,

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}. \quad (1.2)$$

Inequality (1.1) is a classical result of Bernstein[6] whereas inequality (1.2) is due to Zygmund[15] who proved it for all trigonometric polynomials of degree  $n$  and not only for those which are of the form  $P(e^{i\theta})$ . Arestov[1] proved that (1.2) remains true for  $0 < r < 1$  as well. If  $r \rightarrow \infty$  in inequality (1.2), we get (1.1).

If we restrict ourselves to the class of polynomials having no zeros in  $|z| < 1$ , then both the inequalities (1.1) and (1.2) can be sharpened. In fact, If  $P(z) \neq 0$  in  $|z| < 1$ , then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \quad (1.3)$$

and

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n A_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \quad (1.4)$$

$$\text{where } A_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}.$$

Inequality (1.3) was conjectured by Erdős and later verified by Lax[11], whereas inequality (1.4) was proved by De-Bruijn[7] for  $r \geq 1$ . Rahman and Schemeisser[13] later proved that (1.4) holds for  $0 < r < 1$  also. If  $r \rightarrow \infty$  in (1.4), we get (1.3).

As a generalization of (1.3) Malik[12] proved that if  $P(z) \neq 0$  in  $|z| < k$ ,  $k \geq 1$ , then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|, \quad (1.5)$$

whereas under the same hypothesis, Govil and Rahman[9] extended inequality (1.4) by showing that

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n E_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \quad (1.6)$$

$$\text{where } E_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |k + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}, \quad r \geq 1.$$

In the same paper, Govil and Rahman[9, Theorem 4] extended inequality (1.5) to the  $s^{th}$  derivative of a polynomial and proved under the same hypothesis

for  $1 \leq s < n$  that

$$\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1+k^s} \max_{|z|=1} |P(z)|. \tag{1.7}$$

Inequality (1.7) was refined by Aziz and Rather [3, Corollary 1] by involving the binomial coefficients  $C(n, s)$ ,  $1 \leq s < n$  and coefficients of the polynomial  $P(z)$ . In fact they proved that if  $P(z) = \sum_{j=0}^n a_j z^j$  does not vanish in  $|z| < k$ ,  $k \geq 1$ , then for  $1 \leq s < n$ ,

$$\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1+\psi_{k,s}} \max_{|z|=1} |P(z)|, \tag{1.8}$$

where

$$\psi_{k,s} = k^{s+1} \left( \frac{1 + \frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^{s-1}}{1 + \frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^{s+1}} \right). \tag{1.9}$$

In the literature there exist various results regarding the estimates for polynomials and for general analytic functions and also the approximations of polynomials and their derivatives (for example see[8],[14]). In this paper, we prove the following result which refines the inequality (1.8).

**Theorem 1.1.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , and  $m = \min_{|z|=k} |P(z)|$  then for  $1 \leq s < n$ ,*

$$\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1+\psi_{k,s}} \left( \max_{|z|=1} |P(z)| - \frac{m\psi_{k,s}}{k^n} \right), \tag{1.10}$$

where  $\psi_{k,s}$  is defined by (1.9).

The result is best possible for  $k = 1$  and equality holds for  $P(z) = z^n + 1$ .

*Remark 1.2.* For  $s = 1$  and  $m = 0$ , Theorem 1.1 reduces to a result of Govil et. al.[10, Theorem 1] and for  $k = s = 1$ , inequality (1.10) reduces to a result of Aziz and Dawood[2, Theorem A].

*Remark 1.3.* Note by inequality (2.2) of Lemma 2.1 (stated in section 2) that  $\frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1$ , which can easily be shown to be equivalent to  $\psi_{k,s} \geq k^s$ ,  $1 \leq s < n$ . Using this fact in inequality (1.10), we get the following improvement of inequality (1.7).

**Corollary 1.4.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , and  $m = \min_{|z|=k} |P(z)|$  then for  $1 \leq s < n$ ,*

$$\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1+k^s} \left( \max_{|z|=1} |P(z)| - \frac{m}{k^{n-s}} \right). \tag{1.11}$$

In order to prove the Theorem 1.1, we prove the following more general result which extends Theorem 1.1 to its corresponding  $L^r$ - analogue.

**Theorem 1.5.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , and  $m = \min_{|z|=k} |P(z)|$ , then for every complex number  $\beta$  with  $|\beta| \leq 1$  and  $1 \leq s < n$ , we have*

$$\left\{ \int_0^{2\pi} \left| P^{(s)}(e^{i\theta}) + \frac{\beta m n(n-1) \cdots (n-s+1) \psi_{k,s}}{k^n (1 + \psi_{k,s})} \right|^r d\theta \right\}^{\frac{1}{r}} \\ \leq n(n-1) \cdots (n-s+1) C_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \quad (1.12)$$

where  $C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}$ ,  $r > 0$  and  $\psi_{k,s}$  is defined by (1.9).

*Remark 1.6.* Using the fact that  $\psi_{k,s} \geq k^s$  and take  $\beta = 0$  in inequality (1.12), we obtain a result of Aziz and Shah[5].

## 2. LEMMAS

We need the following lemmas for the proofs of Theorems. Here, throughout this paper we write  $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ .

**Lemma 2.1.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  which does not vanish in  $|z| < k$ ,  $k \geq 1$ , then for  $1 \leq s < n$  and  $|z| = 1$ ,*

$$|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)|, \quad (2.1)$$

and

$$\frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1, \quad (2.2)$$

where  $\psi_{k,s}$  is defined by (1.9).

The above lemma is due to Aziz and Rather[3].

**Lemma 2.2.** *If  $P(z)$  is a polynomial of degree  $n$ , then for each  $\alpha$ ,  $0 \leq \alpha < 2\pi$  and  $r > 0$ , we have*

$$\int_0^{2\pi} \int_0^{2\pi} \left| Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta}) \right|^r d\theta d\alpha \leq 2\pi n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \quad (2.3)$$

The above lemma is due to Aziz and Shah[4].

**Lemma 2.3.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  which does not vanish in  $|z| < k, k \geq 1$ , then for  $1 \leq s < n$  and  $|z| = 1$ ,*

$$|Q^{(s)}(z)| \geq \psi_{k,s} |P^{(s)}(z)| + \frac{mn(n-1) \cdots (n-s+1)}{k^n} \psi_{k,s}, \tag{2.4}$$

where  $m = \min_{|z|=k} |P(z)|$ .

*Proof.* Since  $m \leq |P(z)|$  for  $|z| = k$ , we have for every  $\beta$  with  $|\beta| < 1$ ,

$$\left| \frac{m\beta z^n}{k^n} \right| < |P(z)| \text{ for } |z| = k.$$

Therefore by Rouché's theorem  $P(z) + \frac{m\beta z^n}{k^n}$  has no zero in  $|z| < k, k \geq 1$ . Applying Lemma 2.1 to the polynomial  $P(z) + \frac{m\beta z^n}{k^n}$ , we get for  $1 \leq s < n$  and  $|z| = 1$ ,

$$|Q^{(s)}(z)| \geq \psi_{k,s} \left| P^{(s)}(z) + \frac{mn(n-1) \cdots (n-s+1)\beta}{k^n} \right|. \tag{2.5}$$

Choose the argument of  $\beta$  so that

$$\left| P^{(s)}(z) + \frac{mn(n-1) \cdots (n-s+1)\beta z^{n-s}}{k^n} \right| = |P^{(s)}(z)| + \frac{mn(n-1) \cdots (n-s+1)|\beta z^{n-s}|}{k^n},$$

it follows from (2.5) that for  $|z| = 1$ ,

$$|Q^{(s)}(z)| \geq \psi_{k,s} \left| P^{(s)}(z) \right| + \frac{mn(n-1) \cdots (n-s+1)|\beta z^{n-s}|}{k^n} \psi_{k,s}. \tag{2.6}$$

Letting  $|\beta| \rightarrow 1$  in inequality (2.6), we get

$$|Q^{(s)}(z)| \geq \psi_{k,s} \left| P^{(s)}(z) \right| + \frac{mn(n-1) \cdots (n-s+1)}{k^n} \psi_{k,s}.$$

This completes the proof of Lemma 2.3. □

**Lemma 2.4.** *If  $A, B, C$  are non-negative real numbers such that  $B + C \leq A$ . Then for every real  $\alpha$ ,*

$$|(A - C) + e^{i\alpha}(B + C)| \leq |A + e^{i\alpha}B|. \tag{2.7}$$

The above lemma is due to Aziz and Shah[4].

### 3. PROOFS OF THEOREMS

**Proof of the Theorem 1.5.** Since  $P(z)$  is a polynomial of degree  $n$ ,  $P(z) \neq 0$  in  $|z| < k, k \geq 1$ , and  $Q(z) = z^n P(\frac{1}{z})$ . Therefore, for each  $\alpha, 0 \leq \alpha < 2\pi, F(z) = Q(z) + e^{i\alpha}P(z)$  is a polynomial of degree  $n$  and we have

$$F^{(s)}(z) = Q^{(s)}(z) + e^{i\alpha}P^{(s)}(z),$$

which is clearly a polynomial of degree  $n - s, 1 \leq s < n$ . By the repeated application of inequality (1.2), we have for each  $r > 0$ ,

$$\begin{aligned}
 & \int_0^{2\pi} |Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta})|^r d\theta \\
 & \leq (n - s + 1)^r \int_0^{2\pi} |Q^{(s-1)}(e^{i\theta}) + e^{i\alpha} P^{(s-1)}(e^{i\theta})|^r d\theta \\
 & \leq (n - s + 1)^r (n - s + 2)^r \int_0^{2\pi} |Q^{(s-2)}(e^{i\theta}) + e^{i\alpha} P^{(s-2)}(e^{i\theta})|^r d\theta \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \leq (n - s + 1)^r (n - s + 2)^r \dots (n - 1)^r \int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta})|^r d\theta.
 \end{aligned} \tag{3.1}$$

Integrating inequality (3.1) with respect to  $\alpha$  over  $[0, 2\pi]$  and using inequality (2.3) of Lemma 2.2, we get

$$\begin{aligned}
 & \int_0^{2\pi} \int_0^{2\pi} |Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta})|^r d\theta d\alpha \\
 & \leq 2\pi (n - s + 1)^r (n - s + 2)^r \dots (n - 1)^r n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta.
 \end{aligned} \tag{3.2}$$

Now, from inequality (2.4) of Lemma 2.3, it easily follows that

$$\begin{aligned}
 & \psi_{k,s} \left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \\
 & \leq |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.
 \end{aligned} \tag{3.3}$$

Taking  $A = |Q^{(s)}(e^{i\theta})|, B = |P^{(s)}(e^{i\theta})|, C = \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}$  and noting that  $\psi_{k,s} \geq k^s \geq 1, 1 \leq s < n$ , so that by (3.3),

$$B + C \leq \psi_{k,s}(B + C) \leq A - C \leq A,$$

we get from Lemma 2.4 that

$$\begin{aligned}
 & \left| \left\{ |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \right. \\
 & \quad \left. + e^{i\alpha} \left\{ |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\} \right| \\
 & \leq \left| |Q^{(s)}(e^{i\theta})| + e^{i\alpha} |P^{(s)}(e^{i\theta})| \right|.
 \end{aligned}$$

This implies for each  $r > 0$ ,

$$\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r d\alpha \leq \int_0^{2\pi} \left| |Q^{(s)}(e^{i\theta})| + e^{i\alpha} |P^{(s)}(e^{i\theta})| \right|^r d\alpha, \quad (3.4)$$

where

$$F(\theta) = |Q^{(s)}(e^{i\theta})| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}$$

and

$$G(\theta) = |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.$$

Integrating inequality (3.4) with respect to  $\theta$  on  $[0, 2\pi]$  and using inequality (3.2), we obtain

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r d\alpha d\theta \\ & \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| |Q^{(s)}(e^{i\theta})| + e^{i\alpha} |P^{(s)}(e^{i\theta})| \right|^r d\alpha d\theta \\ & = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left| Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta}) \right|^r d\alpha d\theta \\ & \leq (n-s+1)^r (n-s+2)^r \dots (n-1)^r n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \end{aligned} \quad (3.5)$$

Now for every real number  $\alpha$  and  $t_1 \geq t_2 \geq 1$ , we have

$$|t_1 + e^{i\alpha}| \geq |t_2 + e^{i\alpha}|,$$

which implies for every  $r > 0$ ,

$$\int_0^{2\pi} |t_1 + e^{i\alpha}|^r d\alpha \geq \int_0^{2\pi} |t_2 + e^{i\alpha}|^r d\alpha.$$

If  $G(\theta) \neq 0$ , we take  $t_1 = \left| \frac{F(\theta)}{G(\theta)} \right|$  and  $t_2 = \psi_{k,s}$ , then from (3.3) and noting that  $\psi_{k,s} \geq 1$ , we have  $t_1 \geq t_2 \geq 1$ , hence

$$\begin{aligned} \int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r d\alpha &= |G(\theta)|^r \int_0^{2\pi} \left| \frac{F(\theta)}{G(\theta)} + e^{i\alpha} \right|^r d\alpha \\ &= |G(\theta)|^r \int_0^{2\pi} \left| \left| \frac{F(\theta)}{G(\theta)} \right| + e^{i\alpha} \right|^r d\alpha \\ &\geq |G(\theta)|^r \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha \\ &= \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r \\ &\quad \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha. \end{aligned} \quad (3.6)$$

For  $G(\theta) = 0$ , this inequality is trivially true. Using this in (3.5), it follows for each  $r > 0$ ,

$$\begin{aligned} \int_0^{2\pi} \left\{ \left| P^{(s)}(e^{i\theta}) \right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right\}^r d\theta \\ \leq \frac{(n-s+1)^r(n-s+2)^r\dots(n-1)^r n^r}{\frac{1}{2\pi} \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha} \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \end{aligned} \quad (3.7)$$

Now using the fact that for every  $\beta$  with  $|\beta| \leq 1$ ,

$$\left| P^{(s)}(e^{i\theta}) + \frac{\beta mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right| \leq |P^{(s)}(e^{i\theta})| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})},$$

the desired result follows from (3.7).

**Proof of the Theorem 1.1** Making  $r \rightarrow \infty$  and choosing the argument of  $\beta$  suitably with  $|\beta| = 1$  in (1.12), Theorem 1.1 follows.

#### ACKNOWLEDGEMENT

The authors are highly thankful to the referee for his valuable suggestions regarding the paper.

#### REFERENCES

1. V. V. Arestov, On Integral Inequalities for Trigonometric Polynomials and Their Derivatives, *Izv. Akad. Nauk. SSSR. Ser. Mat.*, **45**, (1981), 3-22.
2. A. Aziz, Q. M. Dawood, Inequalities for a Polynomial and its Derivative, *J. Approx. Theory*, **54**, (1988), 306-313.
3. A. Aziz, N. A. Rather, Some Zygmund Type  $L^p$  Inequalities for Polynomials, *J. Math. Anal. Appl.*, **289**, (2004), 14-29.
4. A. Aziz, W. M. Shah,  $L^p$  Inequalities for Polynomials with Restricted Zeros, *Glas. Mate.*, **32**, (1997), 247-258.



5. A. Aziz, W. M. Shah,  $L^p$  Inequalities for Polynomials with Restricted Zeros, *Proc. Indian Acad. Sci. (Math. Sci.)*, **108**, (1998), 63-68.
6. S. Bernstein, *Lecons Sur Les Proprietes Extremals et la Meilleure Approximation des Fonctions Analytiques d'une Fonctions Reelle*, Gauthier-villars Paris, 1926.
7. N. G. De-Bruijn, Inequalities Concerning Polynomials in the Complex Domain, *Nederl. Akad. Wetnesch Proc.*, **50**, (1947), 1265-127.
8. M. R. Eslahchi, Sonaz Amani, The Best Uniform Polynomial Approximation of Two Classes of Rational Functions, *Iranian Journal of Mathematical Sciences and Informatics*, **7**, (2012), 93-102.
9. N. K. Govil, , Q. I. Rahman, Functions of Exponential Type not Vanishing in a Half-Plane and Related Polynomials, *Trans. Amer. Math. Soc.*, **137**, (1969), 501-517.
10. N. K. Govil, Q. I. Rahman, G. Schemesser, On the Derivative of a Polynomial, *Illinois J. Math.*, **23**, (1979), 319-330.
11. P. D. Lax, Proof of a Conjecture of P. Erdős on the Derivative of a Polynomial, *Bull. Amer. Math. Soc.*, **50**, (1944), 509-513.
12. M. A. Malik, On the Derivative of a Polynomial, *J. London Math. Soc.*, **1**, (1969), 57-60.
13. Q. I. Rahman, G. Schemesser,  $L^p$  Inequalities for Polynomials, *J. Approx. Theory*, **53**, (1988), 26-32.
14. Rasoul Aghalary, Applications of the Norm Estimates for Univalence of Analytic Functions, *Iranian Journal of Mathematical Sciences and Informatics*, **9**, (2014), 101-108.
15. A. Zygmund, A Remark on Conjugate Series, *Proc. London Math. Soc.*, **34**, (1932), 392-400.