

On the Vector Variational-like Inequalities with Relaxed $\eta - \alpha$ Pseudomonotone Mappings

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ABSTRACT. In this paper we introduce some new conditions of the solutions existence for variational-like inequalities with relaxed $\eta - \alpha$ pseudomonotone mappings in Banach spaces. The advantage of these new conditions is that they are easier to be verified than those that appear in some of the previous corresponding articles.

Key words: Variational-like Inequalities, Relaxed $\eta - \alpha$ Pseudomonotone, Relaxed $\eta - \alpha$ Demipseudomonotone, η -Hemicontinuous Mapping.

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2000 Mathematics subject classification: Primary 49J40, Secondary 65K10, 90C33, 47J20.

1. INTRODUCTION

Variational inequalities problems play a critical role in many fields and just because of this ability, they have been generalized in various directions by several authors. Monotonicity as a powerful weapon in almost all of these researches project has a notable position and during the recent years many generalization of monotonicity have been introduced to study various classes of variational inequalities. Briefly, a historical order of these researches are:

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Chen [2] who introduced the concept of semimonotonicity and applied it in the semimonotone scalar variational inequalities in Banach space, Fang and Huang [4] who introduced a new concept of relaxed $\eta - \alpha$ monotonicity and obtained some existence theorems of solutions for variational-like inequalities with relaxed $\eta - \alpha$ monotone mappings in reflexive Banach spaces, Bai, Zhou and Ni [1] who introduced a new concept of relaxed $\eta - \alpha$ pseudomonotone and obtained some existence of the solutions for variational-like inequalities with relaxed $\eta - \alpha$ pseudomonotone mappings in reflexive Banach spaces and very recently the works of Wu and Huang [6] who introduced the new concepts of relaxed $\eta - \alpha$ pseudomonotone and demipseudomonotone mappings and obtained some existence results for solutions of vector variational-like inequalities with relaxed $\eta - \alpha$ pseudomonotone and demipseudomonotone mappings by means of KKM technique and Glicksberg fixed point theorem in reflexive Banach spaces. In this paper we try to replace some conditions of the works of Wu and Huang [6] with some new conditions. It is claimed that these conditions are checked somewhat easier in practice than those that appear in the former works.

Consider a Banach space X and a pointed convex closed cone P with $\text{int}P \neq \emptyset$, where $\text{int}P$ is the interior of P . We now define

$$\begin{aligned} x \geq y &\iff x - y \in P \\ x \not\geq y &\iff x - y \notin P \\ x > y &\iff x - y \in \text{int}P \\ x \not> y &\iff x - y \notin \text{int}P. \end{aligned}$$

Throughout this section, unless otherwise specified, suppose that K is a nonempty closed convex subset of X . Let D be a Banach space induced by the convex closed cone P such that (D, \leq) is an ordered Banach space. Denote the space of all bounded linear operators from X to D by $L(X, D)$.

Definition 1.1. See Ref.[6] A mapping $T : K \rightarrow L(X, D)$ is said to be relaxed $\eta - \alpha$ pseudomonotone if there exist the mappings $\eta : K \times K \rightarrow X$ and $\alpha : X \rightarrow D$ with $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in X$ such that

$$(1) \quad \langle Ty, \eta(x, y) \rangle \not\leq 0 \implies \langle Tx, \eta(x, y) \rangle \geq \alpha(x - y)$$

where $p > 1$ is a constant.

Definition 1.2. See Ref.[6] Let $T : K \rightarrow L(X, D)$ and $\eta : K \times K \rightarrow X$ be two mappings. T is said to be η -hemicontinuous if, for any $x, y \in K$ the mapping

$$t \mapsto \langle T(x + t(y - x)), \eta(y, x) \rangle$$

is continuous at 0^+ .

Definition 1.3. See Ref.[6] A mapping $T : K \rightarrow D$ is said to be completely continuous if, for any net $\{x_\lambda\} \in K$, $x_\lambda \rightharpoonup x_0$ (weakly convergence), then $Tx_\lambda \rightarrow Tx_0$ in norm.

Definition 1.4. See [6] A mapping $G : X \rightarrow 2^X$ is said to be a KKM mapping if, for any finite set $B \subset X$, $\text{co}B \subset \bigcup_{x \in B} G(x)$, where 2^X denotes the family of the all nonempty subsets of X and $\text{co}B$ is the convex hull B .

Lemma 1.1. ([5]) *Let B be a nonempty subset of a topological vector space X and $G : B \rightarrow 2^X$ be a KKM mapping. If $G(x)$ is closed in X for every $x \in B$ and compact for some $x \in B$, then $\bigcap_{x \in B} G(x) \neq \emptyset$.*

Lemma 1.2. ([3]) *Let (D, \leq) be an ordered Banach space induced by the pointed closed convex cone P with $\text{int}P \neq \emptyset$. For any $a, b, c \in D$, the following hold:*

- (i) $c \not\prec a \geq b$ implies $b \not\prec c$;
- (ii) $c \not\prec a \leq b$ implies $b \not\prec c$.

Lemma 1.3. ([6]) *Let $T : K \rightarrow L(X, D)$ be η -hemicontinuous and $\eta - \alpha$ pseudomonotone. Suppose that*

- (i) $\eta(x, x) = 0$, for all $x \in K$;
- (ii) for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y) \rangle$ is convex.

Then the following problems are equivalent:

- (a) $x \in K, \langle Tx, \eta(y, x) \rangle \not\prec 0, \forall y \in K$;
- (b) $x \in K, \langle Ty, \eta(y, x) \rangle \geq \alpha(y - x), \forall y \in K$.

2. VECTOR VARIATIONAL-LIKE INEQUALITIES WITH RELAXED $\eta - \alpha$ PSEUDOMONOTONE MAPPINGS

In this section, we suppose that K is a nonempty closed convex subset of a real reflexive Banach space X and (D, \leq) is an ordered Banach space induced by the pointed closed convex cone P with $\text{int}P \neq \emptyset$. Denote by $L(X, D)$ the space of all the continuous linear mappings from X to D . We will discuss the existence of solutions for the vector variational-like inequality with relaxed $\eta - \alpha$ pseudomonotone mappings.

Theorem 2.1. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X . Let $T : K \rightarrow L(X, D)$ be η -hemicontinuous and relaxed $\eta - \alpha$ pseudomonotone. Suppose that*

- (i) $\eta(x, x) = 0, \forall x \in K$;
- (ii) for any given points $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y) \rangle$ be convex and the mapping $x \mapsto \langle Ty, \eta(y, x) \rangle - \alpha(y - x)$ be continuous and concave;
- (iii) $\alpha : X \rightarrow D$ is continuous.

Then the following problem is solvable: find $x \in K$ such that

$$(2) \quad \langle Tx, \eta(y, x) \rangle \not\leq 0 \quad \forall y \in K.$$

Proof. Let $F, G : K \rightarrow 2^X$ be two set-valued mappings defined by:

$$F(y) = \{x \in K : \langle Tx, \eta(y, x) \rangle \not\leq 0\}, \forall y \in K;$$

and

$$G(y) = \{x \in K : \langle Ty, \eta(y, x) \rangle \geq \alpha(y - x)\}, \forall y \in K,$$

respectively. We claim that F is a KKM mapping. Indeed, if F is not a KKM mapping, then there exists $\{y_1, \dots, y_n\} \subset K$ and $t_i > 0$, $i = 1, \dots, n$ with $\sum_{i=1}^n t_i = 1$ such that

$$y = \sum_{i=1}^n t_i y_i \notin \bigcup_{i=1}^n F(y_i).$$

By definition of F , we must have

$$(3) \quad \langle Ty, \eta(y_i, y) \rangle < 0, \quad i = 1, \dots, n.$$

Equation (3) together with condition (ii) yield

$$\begin{aligned} 0 &= \langle Ty, \eta(y, y) \rangle \\ &= \langle Ty, \eta\left(\sum_{i=1}^n t_i y_i, y\right) \rangle \\ &\leq \sum_{i=1}^n t_i \langle Ty, \eta(y_i, y) \rangle \\ &< 0, \end{aligned}$$

which is a clear contradiction. This guarantees that F is a KKM mapping. Easily we can deduce that $F(y) \subset G(y)$, for all $y \in K$. So $G(\cdot)$ is a KKM mapping as well. By the reflexivity of the space and the assumptions that K is bounded, closed and convex we deduce that K is weakly compact. Now let for any given points $y, z \in K$, $\xi_{y,z} : X \rightarrow D$ be a mapping defined by

$$\xi_{y,z}(x) = \langle Tz, \eta(y, x) \rangle - \alpha(y - x).$$

Based on hypothesis we may deduce that $\xi_{y,y}(\cdot)$ is a continuous mapping, for each $y \in K$. So for each $y \in K$, $\xi_{y,y}^{-1}(P)$ is a closed subset of X . On the other hand we have $G(y) = \xi_{y,y}^{-1}(P)$, $\forall y \in K$ which proves that $G(y)$ is closed for each $y \in K$. Let x, z be two points in $G(y)$ for an arbitrary $y \in K$ and $0 \leq t \leq 1$ be a real number. It follows that $\xi_{y,y}(x)$ and $\xi_{y,y}(z)$ lie both in P and hence $t\xi_{y,y}(x) + (1-t)\xi_{y,y}(z)$ lies in P . By condition (ii) it follows that the mapping $\xi_{y,y}$ is concave. Hence $\xi_{y,y}(tx + (1-t)z) \in P$. In other words $tx + (1-t)z \in \xi_{y,y}^{-1}(P)$. This implies that $G(y)$ is convex. So $G(y)$ is weakly

closed for each $y \in K$. Easily we can deduce that $G(y)$ is bounded and hence is weakly compact, for each $y \in K$. It follows from Lemmas (1.1) and (1.3) that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$

This implies that there exists $x \in K$ such that

$$\langle Tx, \eta(y, x) \rangle \not\leq 0, \quad \forall y \in K.$$

This completes the proof. \square

Remark 2.1. Comparing the above theorem with Theorem (2.1) in Ref. [6], we see that the strong condition completely continuity of the mappings α and $x \mapsto \langle Tz, \eta(y, x) \rangle$ has been removed.

We now have the following theorem in the case that K is not necessary a bounded subset of X .

Theorem 2.2. *Let $T : K \rightarrow L(X, D)$ be η -hemicontinuous and relaxed $\eta - \alpha$ pseudomonotone. Assume that K is a unbounded subset of X . Suppose that*

(i) *there exist a constant $r > 0$ and $y_0 \in K$ with $\|y_0\| = r$ such that*

$$\langle Tz, \eta(x, y_0) \rangle > 0, \quad \forall z \in K \text{ with } \|z\| = r;$$

(ii) $\eta(x, y) + \eta(y, x) = 0, \quad \forall x, y \in K;$

(iii) *for any given point $y, z \in K$, the mapping $x \mapsto \langle Ty, \eta(x, y) \rangle$ is continuous as well as the mapping $x \mapsto \langle Tz, \eta(y, x) \rangle - \alpha(y - x)$ is concave;*

(iv) $\alpha : X \rightarrow D$ *is concave and continuous.*

Then the problem (2) is solvable.

Proof. First, we claim that for any given $y, z \in K$ the mapping $x \mapsto \langle Tz, \eta(x, y) \rangle$ is convex. In fact by condition (iii), the mapping $x \mapsto \langle Tz, \eta(y, x) \rangle - \alpha(y - x)$ is concave, for any $z, y \in K$. Thus for every $x, u \in K$ and $0 \leq t \leq 1$:

$$\begin{aligned} & \langle Tz, \eta(y, tx + (1-t)u) \rangle - \alpha(y - (tx + (1-t)u)) \geq \\ & t[\langle Tz, \eta(y, x) \rangle - \alpha(y - x)] + (1-t)[\langle Tz, \eta(y, u) \rangle - \alpha(y - u)]. \end{aligned}$$

Letting $y = ty + (1-t)y$ and applying it in the above formula gives

$$\begin{aligned} & \langle Tz, \eta(y, tx + (1-t)u) \rangle - \alpha(t(y - x) + (1-t)(y - u)) \geq \\ & t\langle Tz, \eta(y, x) \rangle - t\alpha(y - x) + (1-t)\langle Tz, \eta(y, u) \rangle - (1-t)\alpha(y - u). \end{aligned}$$

Thus

$$\begin{aligned} & \langle Tz, \eta(y, tx + (1-t)u) \rangle - \{t\langle Tz, \eta(y, x) \rangle + (1-t)\langle Tz, \eta(y, u) \rangle\} \geq \\ & \alpha(t(y - x) + (1-t)(y - u)) - \{t\alpha(y - x) + (1-t)\alpha(y - u)\} \end{aligned}$$

Based on condition (iv), the right-hand side of the inequality above is greater than the zero and so the left-hand side must be positive which proves our assertion above.

Let $K_r = \{z \in K : \|z\| \leq r\}$. It follows from Theorem (2.1) that there exists $x \in K_r$ such that

$$(4) \quad \langle Tx, \eta(y, x) \rangle \not\leq 0, \quad \forall y \in K_r.$$

Letting $y = y_0$ in (4), it follows that

$$\langle Tx, \eta(y_0, x) \rangle \not\leq 0.$$

So, by condition (ii)

$$(5) \quad \langle Tx, \eta(x, y_0) \rangle \not\geq 0.$$

Combining (5) and condition (i), it follows that $\|x\| < r$. Let $y \in K$ be any given. We can choose $t > 0$ small enough such that $x + t(y - x) \in K_r$. It follows from (4) and condition (iii) that

$$(6) \quad (1 - t)\langle Tx, \eta(x, x) \rangle + t\langle Tx, \eta(y, x) \rangle \geq \langle Tx, \eta(x + t(y - x), x) \rangle \not\leq 0.$$

The equation (6) together with Lemma (1.2) yield the result. \square

Acknowledgement. We are grateful to the referees for their valuable suggestions, which have improved this paper.

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