Iranian Journal of Mathematical Sciences and Informatics
Vol. 12, No. 1 (2017), pp 35-46
DOI: 10.7508/ijmsi.2017.01.004

Graph Convergence for $H(\cdot,\cdot)$-co-Accretive Mapping with over-relaxed Proximal Point Method for Solving a Generalized Variational Inclusion Problem

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Abstract. In this paper, we use the concepts of graph convergence of $H(\cdot,\cdot)$-co-accretive mapping introduced by [R. Ahmad, M. Akram, M. Dilshad, Graph convergence for the $H(\cdot,\cdot)$-co-accretive mapping with an application, Bull. Malays. Math. Sci. Soc., doi: 10.1007/s40840-014-0103-z, 2014] and define an over-relaxed proximal point method to obtain the solution of a generalized variational inclusion problem in Banach spaces. Our results can be viewed as an extension of some previously known results in this direction.

Keywords: Graph convergence, Proximal point method, Accretive mapping, Variational inclusion, Convergence.

2000 Mathematics subject classification: 47J22, 49J40, 65K15.

1. Introduction

Variational inequalities were extended and generalized in various ways using different concepts and obtained application oriented shapes. They are widely applied in mechanics, physics, optimization, economics, engineering sciences

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Received 29 December 2014; Accepted 28 September 2015

and general sciences etc. Variational inclusions are generalized forms of variational inequalities, which is mainly due to *Hassouni* and *Moudafi* [10]. The one of the most efficient and effective technique for solving variational inclusions is the resolvent operator technique, see e.g. [2, 3, 5, 6, 7, 9, 12, 13].

Over-relaxed factors are significant parameters affecting the convergence of a numerical scheme. They represent the fraction of the solution being carried forward from one iteration to the next for the various equations being solved during the simulation.

Verma [26] introduced a general framework for the over-relaxed A-proximal point algorithm based on A-maximal monotonicity and stated that it is application oriented. *Pan et al.* [23] solved a general nonlinear mixed set-valued variational inclusions by constructing an over-relaxed A-proximal point algorithm based on (A, η)-accretive mappings. For related work, see [17, 27, 20].

Li and *Huang* [18] introduced the concepts of graph convergence for $H(\cdot, \cdot)$-accretive mappings and applied it to solve a variational inclusion problem. After that, *Ahmad et al.* [4] introduced the concept of graph convergence for $H(\cdot, \cdot)$-co-accretive mappings for solving a generalized variational inclusion problem. Very recently, *Lan* [19] designed the graph convergence analysis of over-relaxed (A, η, m)-proximal point iterative methods for solving general nonlinear operator equations. A quite reasonable work is done in this direction to solve some classes of variational inclusion problems. For more details of the related work, we refer to [1, 8, 22, 14, 15, 16, 21, 24, 25] and references therein.

In this communication, we design an over-relaxed proximal point algorithm for solving a generalized variational inclusion problem by using the concept of $H(\cdot, \cdot)$-co-accretive mapping due to *Ahmad et al.* [4]. We prove an existence result for generalized variational inclusion problem and show that the sequences generated by our algorithm converge to a solution of generalized variational inclusion problem.

2. Preliminaries

Let X be a real Banach space with its norm $\|\cdot\|$, X^* be the topological dual of X and d be the metric induced by the norm $\|\cdot\|$. Let $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^*, $CB(X)$ (respectively, 2^X) be the family of all nonempty closed and bounded subsets (respectively, all non-empty subsets) of X and $\mathcal{D}$ be the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\},$$

where $A, B \in CB(X)$, $d(x, B) = \inf_{y \in B} d(x, y)$ and $d(A, y) = \inf_{x \in A} d(x, y)$.

The generalized duality mapping $J_q : X \longrightarrow 2^{X^*}$ is defined by

$$J_q(x) = \left\{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\right\}, \ \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that $J_q(x) = \|x\|^{q-1}J_2(x)$, for all $x(\neq 0) \in X$. If X is a Hilbert space, then J_2 becomes the identity mapping on X.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space X is said to be uniformly smooth if,

$$\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0.$$

Also, X is called q-uniformly smooth if, there exists a constant $C > 0$ such that

$$\rho_X(t) \leq Ct^q, \ q > 1.$$

Note that J_q is single-valued, if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, *Xu* [28] proved the following lemma.

Lemma 2.1. *Let $q > 1$ be a real number and X be a real smooth Banach space. Then X is q-uniformly smooth if and only if, there exists a constant $C_q > 0$ such that for every $x, y \in X$*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

Throughout the paper unless otherwise specified, we take X to be q-uniformly smooth Banach space. Now, we recall some definitions and results which will be used in subsequent section.

Definition 2.2. A mapping $A : X \longrightarrow X$ is said to be

(i) accretive if,

$$\langle Ax - Ay, J_q(x-y)\rangle \geq 0, \ \forall x, y \in X;$$

(ii) strongly accretive if,

$$\langle Ax - Ay, J_q(x-y)\rangle > 0, \ \forall x, y \in X,$$

and the equality holds if and only if $x = y$;

(iii) δ-strongly accretive if, there exists a constant $\delta > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq \delta\|x-y\|^q, \ \forall x, y \in X;$$

(iv) β-relaxed accretive if, there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq (-\beta)\|x-y\|^q, \ \forall x, y \in X;$$

(*v*) μ-cocoercive if, there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq \mu\|Ax - Ay\|^q,\ \forall x, y \in X;$$

(*vi*) γ-relaxed cocoercive if, there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, J_q(x-y)\rangle \geq (-\gamma)\|Ax - Ay\|^q,\ \forall x, y \in X;$$

(*vii*) σ-Lipschitz continuous if, there exists a constant $\sigma > 0$ such that

$$\|Ax - Ay\| \leq \sigma\|x - y\|,\ \forall x, y \in X;$$

(*viii*) η-expansive if, there exists a constant $\eta > 0$ such that

$$\|Ax - Ay\| \geq \eta\|x - y\|,\ \forall x, y \in X,$$

if $\eta = 1$, then it is expansive.

Definition 2.3. A set-valued mapping $T : X \longrightarrow CB(X)$ is said to be $\mathcal{D}$-Lipschitz continuous if, there exists a constant $\lambda_{\mathcal{D}_T} > 0$ such that

$$\mathcal{D}(T(x), T(y)) \leq \lambda_{\mathcal{D}_T}\|x - y\|,\ \forall x, y \in X.$$

Definition 2.4. Let $H : X \times X \longrightarrow X$ and $A, B : X \longrightarrow X$ be single-valued mappings. Then

(*i*) $H(A, \cdot)$ is said to be μ_1-cocoercive with respect to A if for a fixed $u \in X$, there exists a constant $\mu_1 > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x-y)\rangle \geq \mu_1\|Ax - Ay\|^q,\ \forall x, y \in X;$$

(*ii*) $H(\cdot, B)$ is said to be γ_1-relaxed cocoercive with respect to B if for a fixed $u \in X$, there exists a constant $\gamma_1 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x-y)\rangle \geq (-\gamma_1)\|Bx - By\|^q,\ \forall x, y \in X;$$

(*iii*) $H(A, B)$ is said to be symmetric cocoercive with respect to A and B if, $H(A, \cdot)$ is cocoercive with respect to A and $H(\cdot, B)$ is relaxed cocoercive with respect to B;

(*iv*) $H(A, \cdot)$ is said to be α_1-strongly accretive with respect to A if for a fixed $u \in X$, there exists a constant $\alpha_1 > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x-y)\rangle \geq \alpha_1\|x - y\|^q,\ \forall x, y \in X;$$

(*v*) $H(\cdot, B)$ is said to be β_1-relaxed accretive with respect to B if for a fixed $u \in X$, there exists a constant $\beta_1 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x-y)\rangle \geq (-\beta_1)\|x - y\|^q,\ \forall x, y \in X;$$

(*vi*) $H(A, B)$ is said to be symmetric accretive with respect to A and B if, $H(A, \cdot)$ is strongly accretive with respect to A and $H(\cdot, B)$ is relaxed accretive with respect to B;

(vii) $H(A,\cdot)$ is said to be ξ_1-Lipschitz continuous with respect to A if for a fixed $u \in X$, there exists a constant $\xi_1 > 0$ such that

$$\|H(Ax,u) - H(Ay,u)\| \le \xi_1\|x-y\|,\ \forall x,y \in X;$$

$(viii)$ $H(\cdot,B)$ is said to be ξ_2-Lipschitz continuous with respect to B if for a fixed $u \in X$, there exists a constant $\xi_2 > 0$ such that

$$\|H(u,Bx) - H(u,By)\| \le \xi_2\|x-y\|,\ \forall x,y \in X.$$

Definition 2.5. Let $f,g : X \longrightarrow X$ be single-valued mappings and $M : X \times X \longrightarrow 2^X$ be a set-valued mapping. Then

(i) $M(f,\cdot)$ is said to be α-strongly accretive with respect to f if, there exists a constant $\alpha > 0$ such that

$$\langle u-v, J_q(x-y)\rangle \ge \alpha\|x-y\|^q,\ \forall x,y,w \in X,$$

and for all $u \in M(f(x),w), v \in M(f(y),w)$;

(ii) $M(\cdot,g)$ is said to be β-relaxed accretive with respect to g if, there exists a constant $\beta > 0$ such that

$$\langle u-v, J_q(x-y)\rangle \ge (-\beta)\|x-y\|^q,\ \forall x,y,w \in X,$$

and for all $u \in M(w,g(x)), v \in M(w,g(y))$;

(iii) $M(f,g)$ is said to be symmetric accretive with respect to f and g if, $M(f,\cdot)$ is strongly accretive with respect to f and $M(\cdot,g)$ is relaxed accretive with respect to g.

Definition 2.6. A sequence $\{x_i\}$ is said to converge linearly to x^* if, there exists a constant $0 < c < 1$ such that

$$\|x_{i+1} - x^*\| \le c\|x_i - x^*\|,$$

for all $i > m$, for some natural number $m > 0$.

Definition 2.7 ([4]). Let $A,B,f,g : X \longrightarrow X$ and $H : X \times X \longrightarrow X$ be single-valued mappings. Let $M : X \times X \longrightarrow 2^X$ be the set-valued mapping. The mapping M is said to be $H(\cdot,\cdot)$-co-accretive with respect to A,B,f and g if, $H(A,B)$ is symmetric cocoercive with respect to A and B, $M(f,g)$ is symmetric accretive with respect to f and g, and

$$\left[H(A,B) + \lambda M(f,g)\right](X) = X,\ \forall \lambda > 0.$$

Theorem 2.8 ([4]). *Let $A,B,f,g : X \longrightarrow X$ and $H : X \times X \longrightarrow X$ be single-valued mappings. Let $M : X \times X \longrightarrow 2^X$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A,B,f and g. Let A be η-expansive and B be σ-Lipschitz continuous and $\alpha > \beta$, $\mu > \gamma$ and $\eta \ge \sigma$. Then the mapping $[H(A,B) + \lambda M(f,g)]^{-1}$ is single-valued, for every $\lambda > 0$.*

Definition 2.9 ([4])**.** Let $A, B, f, g : X \longrightarrow X$ and $H : X \times X \longrightarrow X$ be single-valued mappings. Let $M : X \times X \longrightarrow 2^X$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. Then the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : X \longrightarrow X$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) = [H(A,B) + \lambda M(f,g)]^{-1}(u), \ \forall u \in X, \lambda > 0.$$

Theorem 2.10 ([4])**.** *Let $A, B, f, g : X \longrightarrow X$ and $H : X \times X \longrightarrow X$ be single-valued mappings. Let $M : X \times X \longrightarrow 2^X$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous and $\alpha > \beta$, $\mu > \gamma$ and $\eta \geq \sigma$. Then the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ is Lipschitz continuous with constant θ, i.e.,*

$$\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\| \leq \theta\|u - v\|, \ \forall u, v \in X, \lambda > 0,$$

where $\theta = \dfrac{1}{\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)}$.

Definition 2.11. Let $M : X \times X \longrightarrow 2^X$ be a set-valued mapping. The graph of M is denoted by $\mathcal{G}(M)$ and defined by

$$\mathcal{G}(M) = \{((x,y),z) : z \in M(x,y)\}, \ \forall x, y \in X.$$

Definition 2.12. Let $A, B, f, g : X \longrightarrow X$ and $H : X \times X \longrightarrow X$ be single-valued mappings. Let $M_n, M : X \times X \longrightarrow 2^X$ be $H(\cdot,\cdot)$-co-accretive mappings, for $n = 0, 1, 2, \cdots$. The sequence M_n is said to be graph convergent to M, denoted by $M_n \xrightarrow{\mathcal{G}} M$ if, for every $((f(x), g(x)), z) \in \mathcal{G}(M)$, there exists a sequence $((f(x_n), g(x_n)), z_n) \in \mathcal{G}(M_n)$ such that

$$f(x_n) \longrightarrow f(x), \ g(x_n) \longrightarrow g(x) \ \textit{and} \ z_n \longrightarrow z, \ \textit{as} \ n \to \infty.$$

Theorem 2.13 ([4])**.** *Let $M_n, M : X \times X \longrightarrow 2^X$ be $H(\cdot,\cdot)$-co-accretive mappings with respect to A, B, f and g. Let $H : X \times X \longrightarrow X$ be a single-valued mapping such that $H(A,B)$ is ξ_1-Lipschitz continuous with respect to A and ξ_2-Lipschitz continuous with respect to B. Suppose that f is τ-expansive mapping. Then, $M_n \xrightarrow{\mathcal{G}} M$ if and only if*

$$R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(u) \longrightarrow R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u), \ \forall u \in X, \lambda > 0.$$

3. Formulation of the problem and Algorithm framework

Let $T : X \longrightarrow CB(X)$ and $M : X \times X \longrightarrow 2^X$ be set-valued mappings and $f, g : X \longrightarrow X$ be single-valued mappings. We consider the following problem of finding $x \in X$, $w \in T(x)$ such that

$$0 \in w + M(f(x), g(x)). \tag{3.1}$$

Problem (3.1) is called generalized variational inclusion problem.

Special Cases:

(i) If $M(f(x),\cdot) = M(f(x))$ and $g \equiv 0$, then problem (3.1) is equivalent to the problem of finding $x \in X$ such that

$$0 \in w + M\left(f(x)\right). \tag{3.2}$$

Problem (3.2) was introduced and studied by *Huang* [11] in the setting of Banach spaces.

(ii) If T is single-valued, $f \equiv I$, the identity mapping, then problem (3.2) is equivalent to the problem

$$0 \in T(x) + M(x). \tag{3.3}$$

Problem (3.3) is studied by *Li* and *Huang* [18].

We remark that problem (3.1) includes many variational inequalities (inclusions) and complementarity problems as special cases.

Lemma 3.1. *The elements $x \in X$, $w \in T(x)$ are the solutions of generalized variational inclusion problem* (3.1) *if and only if, they satisfy the following equation:*

$$x = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}\left[H(Ax,Bx) - \lambda w\right], \tag{3.4}$$

where $\lambda > 0$ and $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) = \left[H(A,B) + \lambda M(f,g)\right]^{-1}(x)$, $\forall x \in X$.

Algorithm 3.2. *Step 1. Choose an arbitrary initial point $x_0 \in X$ and $w_0 \in T(x_0)$.*

Step 2. Compute the sequence $\{x_n\}$ and $\{w_n\}$ by the following iterative procedure:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n,\ \ n \geq 0, \tag{3.5}$$

where for some $P_n \in T(y_n)$, y_n satisfies

$$\left\|y_n - R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax_n,Bx_n) - \lambda w_n]\right\| \leq \sigma_n\left(\|y_n - x_n\| + \lambda\|P_n - w_n\|\right); \tag{3.6}$$

and

$$\|w_n - w_{n-1}\| \leq \mathcal{D}(T(x_n),T(x_{n-1})), \tag{3.7}$$

where $\{\alpha_n\} \subseteq [0,\infty)$ is a sequence of over-relaxed factors, $\{\sigma_n\}$ is a scalar sequence, $n \geq 0$, $\lambda > 0$, $\sum\limits_{n=0}^{\infty}\sigma_n < \infty$, $\sigma_n \to 0$ and $\alpha = \limsup\limits_{n\to\infty}\alpha_n < 1$.

Step 3. If $\{x_n\}$ and $\{y_n\}$ satisfy (3.5), (3.6) *and $\{w_n\}$ satisfies* (3.7) *to an amount of accuracy, Stop. Otherwise, set $n = n+1$ and repeat the Step 2.*

Theorem 3.3. *Let X be a q-uniformly smooth Banach space. Let $A, B : X \longrightarrow X$ and $H : X \times X \longrightarrow X$ be mappings such that H is symmetric cocoercive with respect to A and B with constants μ and γ, respectively; r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B; A is η-expansive and B is σ-Lipschitz continuous. Let $T : X \longrightarrow CB(X)$ be $\mathcal{D}$-Lipschitz continuous with constant δ_T and the mappings $M_n, M : X \times X \longrightarrow 2^X$*

be $H(\cdot,\cdot)$-co-accretive mappings such that $M_n \xrightarrow{\mathcal{G}} M$. In addition, if for some $\lambda > 0$, the following condition holds:

$$\theta(r_1 + r_2) + \lambda\theta\delta_T < 1, \tag{3.8}$$

where $\theta = \dfrac{1}{\lambda(\alpha-\beta) + (\mu\eta^q - \gamma\sigma^q)}$, $\mu > \gamma$, $\eta \geq \sigma$ and $\alpha > \beta$. Then, the generalized variational inclusion problem (3.1) *admits a solution (x^*, w^*), $x^* \in X$, $w^* \in T(x^*)$, and the sequences $\{x_n\}$ and $\{w_n\}$ defined in Algorithm 3.2 converge linearly to x^* and w^*, respectively.*

Proof. For any $\lambda > 0$, we define a mapping $G : X \longrightarrow X$ by

$$G(x) = R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}\left[H(Ax, Bx) - \lambda w_1\right], \ \forall x \in X, w_1 \in T(x).$$

Since the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}$ is θ-Lipschitz continuous, H is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B, T is δ_T-Lipschitz continuous, hence, for any $x, y \in X$, $w_1 \in T(x)$, $w_2 \in T(y)$, we estimate

$$\begin{aligned}
\|G(x) - G(y)\| &= \left\| R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}[H(Ax, Bx) - \lambda w_1] - R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}[H(Ay, By) - \lambda w_2] \right\| \\
&\leq \theta \left\| H(Ax, Bx) - H(Ay, By) - \lambda(w_1 - w_2) \right\| \\
&\leq \theta \left\| H(Ax, Bx) - H(Ay, By) \right\| + \lambda\theta \|(w_1 - w_2)\| \\
&\leq \theta(r_1 + r_2)\|x - y\| + \lambda\theta\mathcal{D}(T(x), T(y)) \\
&\leq \theta(r_1 + r_2)\|x - y\| + \lambda\theta\delta_T\|x - y\| \\
&= \left(\theta(r_1 + r_2) + \lambda\theta\delta_T\right)\|x - y\|,
\end{aligned}$$

which implies that

$$\|G(x) - G(y)\| \leq P(\theta_1)\|x - y\|, \tag{3.9}$$

where $P(\theta_1) = \theta(r_1 + r_2) + \lambda\theta\delta_T$ and $\theta = \dfrac{1}{\lambda(\alpha-\beta) + (\mu\eta^q - \gamma\sigma^q)}$. It follows from condition (3.8) that $0 \leq P(\theta_1) < 1$, and so G is a contraction mapping i.e., G has a unique fixed point in X.

Next, we prove that (x^*, w^*), $x^* \in X$, $w^* \in T(x^*)$ is a solution of the problem (3.1). It follows from Lemma 3.1 that

$$x^* = (1 - \alpha_n)x^* + \alpha_n R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}\left[H(Ax^*, Bx^*) - \lambda w^*\right]. \tag{3.10}$$

Let

$$z_{n+1} = (1 - \alpha_n)x_n + \alpha_n R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}\left[H(Ax_n, Bx_n) - \lambda w_n\right]. \tag{3.11}$$

Using the Lipschitz continuity of the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$, we evaluate

$$\begin{aligned}
& \|z_{n+1}-x^*\| \\
= \; & \big\|(1-\alpha_n)(x_n-x^*)+\alpha_n\big\{R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax_n,Bx_n)-\lambda w_n]- \\
& R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]\big\}\big\| \\
\leq \; & (1-\alpha_n)\|x_n-x^*\|+\alpha_n\big\|R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax_n,Bx_n)-\lambda w_n]- \\
& R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]\big\|+\alpha_n\big\|R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]- \\
& R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]\big\| \\
\leq \; & (1-\alpha_n)\|x_n-x^*\|+\alpha_n\theta\|H(Ax_n,Bx_n)-H(Ax^*,Bx^*)-\lambda(w_n-w^*)\|+ \\
& \alpha_n\big\|R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]\big\|.
\end{aligned} \tag{3.12}$$

By using Lemma 3.1 and as $H(A,B)$ is μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B, we have

$$\begin{aligned}
& \|H(Ax_n,Bx_n)-H(Ax^*,Bx^*)-\lambda(w_n-w^*)\|^q \\
\leq \; & \lambda^q\|w_n-w^*\|^q+C_q\|H(Ax_n,Bx_n)-H(Ax^*,Bx^*)\|^q- \\
& 2q\lambda\langle H(Ax_n,Bx_n)-H(Ax^*,Bx^*),J_q(w_n-w^*)\rangle \\
\leq \; & \lambda^q\|w_n-w^*\|^q+C_q(r_1+r_2)^q\|x_n-x^*\|^q- \\
& 2q\lambda(\mu\|Ax_n-Ax^*\|^q-\gamma\|Bx_n-Bx^*\|^q) \\
\leq \; & \lambda^q\delta_T^q\|x_n-x^*\|^q+C_q(r_1+r_2)^q\|x_n-x^*\|^q-2q\lambda(\mu\eta^q-\gamma\sigma^q)\|x_n-x^*\|^q \\
= \; & \left[\lambda^q\delta_T^q+C_q(r_1+r_2)^q--2q\lambda(\mu\eta^q-\gamma\sigma^q)\right]\|x_n-x^*\|^q,
\end{aligned}$$

which implies that

$$\begin{aligned}
& \|H(Ax_n,Bx_n)-H(Ax^*,Bx^*)-\lambda(w_n-w^*)\| \\
\leq \; & \sqrt[q]{\lambda^q\delta_T^q+C_q(r_1+r_2)^q-2q\lambda(\mu\eta^q-\gamma\sigma^q)}\|x_n-x^*\|.
\end{aligned} \tag{3.13}$$

By Theorem 2.13, we have

$$R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]\longrightarrow R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*].$$

Let

$$b_n=R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*]-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}[H(Ax^*,Bx^*)-\lambda w^*], \tag{3.14}$$

then, $b_n\to 0$ as $n\to\infty$.

By using of (3.13) and (3.14), (3.12) becomes

$$\|z_{n+1}-x^*\|\leq\{(1-\alpha_n)+\alpha_n\theta L_1\}\|x_n-x^*\|+\alpha_n\|b_n\|, \tag{3.15}$$

where $L_1=\sqrt[q]{\lambda^q\delta_T^q+C_q(r_1+r_2)^q-2q\lambda(\mu\eta^q-\gamma\sigma^q)}$.

Since $x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n$, $x_{n+1} - x_n = \alpha_n(y_n - x_n)$, it follows that

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\| &= \big\|(1-\alpha_n)x_n + \alpha_n y_n - [(1-\alpha_n)x_n + \\
&\quad \alpha_n R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}\{H(Ax_n, Bx_n) - \lambda w_n\}\big\| \\
&= \Big\|\alpha_n \Big[y_n - R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}\{H(Ax_n, Bx_n) - \lambda w_n\}\Big]\Big\| \\
&\leq \alpha_n \sigma_n \big(\|y_n - x_n\| + \lambda\|P_n - w_n\|\big) \\
&\leq \alpha_n \sigma_n \|y_n - x_n\| + \alpha_n \sigma_n \lambda D(T(y_n), T(x_n)) \\
&\leq \alpha_n \sigma_n \|y_n - x_n\| + \alpha_n \sigma_n \lambda \delta_T \|y_n - x_n\| \\
&= \alpha_n \sigma_n (1 + \lambda\delta_T)\|y_n - x_n\| \\
&= \sigma_n (1 + \lambda\delta_T)\|\alpha_n(y_n - x_n)\| \\
&= \sigma_n (1 + \lambda\delta_T)\|x_{n+1} - x_n\|. \qquad (3.16)
\end{aligned}$$

Using the above discussed arguments, we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|x_{n+1} - z_{n+1}\| + \|z_{n+1} - x^*\| \\
&\leq \sigma_n(1 + \lambda\delta_T)\|x_{n+1} - x_n\| + \{(1-\alpha_n) + \alpha_n \theta L_1\}\|x_n - x^*\| + \alpha_n\|b_n\| \\
&= \sigma_n(1 + \lambda\delta_T)\|x_{n+1} - x^* + x^* - x_n\| + \{(1-\alpha_n) + \alpha_n \theta L_1\}\|x_n - x^*\| \\
&\quad + \alpha_n\|b_n\| \\
&\leq \sigma_n(1 + \lambda\delta_T)\|x_{n+1} - x^*\| + \sigma_n(1 + \lambda\delta_T)\|x_n - x^*\| + \\
&\quad \{(1-\alpha_n) + \alpha_n \theta L_1\}\|x_n - x^*\| + \alpha_n\|b_n\|
\end{aligned}$$

which implies that

$$\|x_{n+1} - x^*\| \leq \frac{\sigma_n(1 + \lambda\delta_T) + (1-\alpha_n) + \alpha_n\theta L_1}{1 - \sigma_n(1 + \lambda\delta_T)}\|x_n - x^*\| + \frac{\alpha_n}{1 - \sigma_n(1 + \lambda\delta_T)}\|b_n\|. \qquad (3.17)$$

From (3.14) and (3.17), it follows that x_n converges to x^* linearly. Also from Algorithm 3.2 and $\mathcal{D}$-Lipschitz continuity T, we have

$$\begin{aligned}
\|w_n - w_{n-1}\| &\leq \mathcal{D}(T(x_n), T(x_{n-1})) \\
&\leq \delta_T \|x_n - x_{n-1}\|. \qquad (3.18)
\end{aligned}$$

Since x_n converges to x^* linearly, it follows from (3.18) that w_n converges to w linearly. This completes the proof. □

Acknowledgments

The authors are grateful to the referees and editors for their valuable suggestions which improved this paper.

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