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Common Zero Points of Two Finite Families of Maximal Monotone Operators Via Iteration Methods

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Abstract. In this work, we have presented some iterative schemes for achieving to common points of the solutions set of the system of generalized mixed equilibrium problems, solutions set of the variational inequality for an inverse-strongly monotone operator, common fixed points set of two infinite sequences of relatively nonexpansive mappings and common zero points set of two finite sequences of maximal monotone operators.

Keywords: Maximal monotone operator, Equilibrium problem, Variational inequality.

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1. Introduction

Let E be a Banach space with dual E^*. The pairing between E and E^* is denoted by $\langle .,. \rangle$ and norm both in E and E^* is designed by $\|.\|$. We denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integers, respectively. The normalized duality mapping $J : E \rightrightarrows E^*$ is defined by

$$J(x) = \{x^* \in E^* | \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

If E is a Hilbert space, then the duality mapping is identity. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The duality mapping J is said to be weakly sequentially continuous if $\{Jx_n\}$ converges to Jx in the weak* topology of E^* whenever $\{x_n\}$ is a sequence of E such that $x_n \rightharpoonup x$. A Banach space E is strictly convex if and only if $x \neq y$ and $\|x\| = \|y\| = 1$ together imply that $\|x + y\| < 2$. A Banach space E is uniformly convex if for any $\varepsilon > 0$ there exists some $\delta > 0$ so that for any two vectors with $\|x\| = 1$ and $\|y\| = 1$, the condition $\|x - y\| \geq \varepsilon$ implies that $\|\frac{x+y}{2}\| \leq 1 - \delta$. Also a Banach space E is uniformly smooth if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$, where,

$$\rho_E(t) = \sup\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\},$$

is modulus of smoothness of E. Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. Here, some notes in [7, 18] is recalled:

(1) If E is an arbitrary Banach space, then J is monotone and bounded.
(2) If E is smooth, then J is single-valued and semi-continuous.
(3) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E and E is smooth and reflexive.
(4) If E is strictly convex, then J is one-to-one and strictly monotone.
(5) If E is reflexive, then J is surjective.
(6) If E is reflexive, smooth and strictly convex, then J is single valued, one-to-one and onto.
(7) If E is uniformly convex, then it is reflexive.
(8) If E is uniformly smooth, then E is smooth and reflexive.
(9) E is uniformly smooth if and only if E^* is uniformly convex.

A set-valued operator $T : E \rightrightarrows E^*$ is

- monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0 \;\forall\; (x, x^*),\; (y, y^*) \in G(T),$$

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- strictly monotone if

$$\langle x-y, x^*-y^*\rangle = 0 \ (x,x^*),\ (y,y^*) \in G(T) \Rightarrow \ x=y.$$

Also a single-valued operator $T : E \to E^*$ is α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx-Ty, x-y\rangle \geq \alpha\|Tx-Ty\|^2,\ \forall\ (x,Tx),\ (y,Ty) \in G(T),$$

where $G(T) := \{(x,y) \in E \times E^*;\ y \in Tx\}$. The domain of T is $D(T) := \{x \in E;\ Tx \neq \emptyset\}$, range of T is denoted by $R(T) := T(E)$ and $F(T) := \{x \in E; Tx = x\}$ is the set of fixed points of T. A point p in closed and convex subset C of E is called an asymptotic fixed point of T [14] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. A monotone operator T is called maximal monotone, if its graph has no monotone extension graph in the sense of inclusion. Minty surjectivity theorem indicates $R(T+J) = E^*$ for maximal monotone operator T in reflexive Banach space E. Resolvent operator of T on Banach space E denotes by ${J_r}^T = (J+rT)^{-1}J$ for all $r > 0$. For any $r > 0$, the Yosida approximation of T is defined by $T_r = \frac{(J-JJ_r^T)}{r}$. Notice that $T_r(x) \in T({J_r}^T(x))$ for any $x \in E$. The graph of maximal monotone operator $T : E \rightrightarrows E^*$ is demiclosed, i.e., if $\{x_k\} \subset E$ converges weakly to x_0, $\{u_k \in Tx_k\}$ converges strongly to u_0, then $u_0 \in Tx_0$.

Assume E is smooth and consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2 < x, Jy > +\|y\|^2 \ for\ x,\ y \in E,$$

the above function was studied in [1, 9, 14]. It is comprehensible from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \qquad \forall x,\ y \in E. \tag{1.1}$$

Let C be a nonempty, closed and convex subset of E. A mapping $T : C \to C$ is called:

(D_1) quasi-ϕ-nonexpansive [13, 23] if $F(T) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p,x),\ \forall x \in C\ p \in F(T);$$

(D_2) relatively nonexpansive [2, 4] if $\widehat{F}(T) = F(T)$ and

$$\phi(p, Tx) \leq \phi(p,x),\ \forall\ x \in C,\ p \in F(T).$$

The mapping $\Pi_C : E \to C$ that assigns to an arbitrary point $x \in E$ the

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minimum point of the functional $\phi(x,y)$, i.e.,

$$\phi(\overline{x},x) = inf_{y\in C}\phi(y,x),$$

is called *generalized projection*.
From the properties of the functional $\phi(y,x)$ and the strict monotonicity of the mapping J we conclude the existence and uniqueness of the operator Π_C [18, 7, 9]. If E be a Hilbert space, then Π_C reduce into metric projection.

Let $f_i : C\times C \to \mathbb{R}$ be a bifunction, $\phi_i : C \to \mathbb{R}$ be a real valued function and $A_i : C \to E^*$ be a nonlinear mapping for $i = 1,2,...,m$. The system of generalized mixed equilibrium problems is as follows:
find $x\in C$ such that for all $y \in C$,

$$\begin{cases} f_1(x,y) + \langle y-x, A_1x\rangle + \phi_1(y) - \phi_1(x) \geq 0, \\ f_2(x,y) + \langle y-x, A_2x\rangle + \phi_2(y) - \phi_2(x) \geq 0, \\ \vdots \\ f_m(x,y) + \langle y-x, A_mx\rangle + \phi_m(y) - \phi_m(x) \geq 0. \end{cases} \tag{1.2}$$

We denote by $\Omega(f_i, A_i, \phi_i)$ the solutions set of (1.2).
If $f_i = f$, $A_i = A$ and $\phi_i = \phi$ for all $i = 1,2,...,m$ the problem (1.2) reduce into generalized mixed equilibrium problem, i.e., finding $x \in C$ such that

$$f(x,y) + \langle y-x, Ax\rangle + \phi(y) - \phi(x) \geq 0,\ \forall y \in C. \tag{1.3}$$

The solutions set of the problem (1.3) is denoted by Ω. If $f = 0$ and $\phi = 0$ the problem (1.3) reduce into variational inequality problem, denoted by $VI(C,A)$, i.e., finding $x\in C$ such that

$$\langle y-x, Ax\rangle \geq 0,\ \forall y \in C.$$

If $A_i = 0$ for all $i = 1,2,...,m$ the problem (1.2) reduce into the system of mixed equilibrium problem for f, i.e., finding $x \in C$ such that

$$\begin{cases} f_1(x,y) + \phi_1(y) - \phi_1(x) \geq 0, \\ f_2(x,y) + \phi_2(y) - \phi_2(x) \geq 0, \\ \vdots \\ f_m(x,y) + \phi_m(y) - \phi_m(x) \geq 0. \end{cases}$$

If $f_i = 0$ for all $i = 1,2,...,m$ the problem (1.2) reduce into the system of mixed variational inequality of Browder type, i.e., finding $x\in C$ such that

$$\begin{cases} \langle y-x, A_1x\rangle + \phi_1(y) - \phi_1(x) \geq 0, \\ \langle y-x, A_2x\rangle + \phi_2(y) - \phi_2(x) \geq 0, \\ \vdots \\ \langle y-x, A_mx\rangle + \phi_m(y) - \phi_m(x) \geq 0. \end{cases}$$

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If $\phi_i = 0$ for all $i = 1, 2, ..., m$ the problem (1.2) reduce into the system of generalized equilibrium problems denoted by GEP(f_i, A_i), i.e., finding $x \in C$ such that

$$\begin{cases} f_1(x,y) + \langle y - x, A_1 x\rangle \geq 0, \\ f_2(x,y) + \langle y - x, A_2 x\rangle \geq 0, \\ \vdots \\ f_m(x,y) + \langle y - x, A_m x\rangle \geq 0. \end{cases}$$

If $A_i = 0$ and $\phi_i = 0$ for all $i = 1, 2, ..., m$ the problem (1.2) reduce into the system of equilibrium problem for f_i, i.e.,

$$\begin{cases} f_1(x,y) \geq 0, \\ f_2(x,y) \geq 0, \\ \vdots \\ f_m(x,y) \geq 0. \end{cases}$$

One of the main problems in the theory of monotone operators is

$$find\ x \in E\ such\ that\ 0 \in Tx,$$

where $T : E \rightrightarrows E^*$ is a maximal monotone operator.
For this problem, first Martinet [11] presented proximal point algorithm and then Rockafellar [15] generalized it in a framework of maximal monotone operators in a Hilbert space.
In 2012 Wattanawitoon and Kumam [19] considered the the following iterative method for a maximal monotone operator T on a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$\begin{aligned} u_n &= K_{r_n} x_n, \\ z_n &= \Pi_C J^{-1}(J - \lambda_n A)u_n, \\ \gamma_n &= J^{-1}(\beta_n J x_n + (1 - \beta_n) JSJ_{r_n} z_n), \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n J x + (1 - \alpha_n) J\gamma_n), \end{aligned}$$

for all $n \in N$, where Π_C is the generalized projection from E onto C, S is a relatively nonexpansive mapping from C into itself, $B : C \to E^*$ is a continuous and monotone operators, $\phi : C \to \mathbb{R}$ is convex and lower semicontinuous, $f : C \times C \to \mathbb{R}$ is a bifunction and

$$K_r(x) = \{u \in C : f(x,y) + \langle Bu, y - u\rangle + \phi(y) - \phi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \forall y \in C\}$$

for all $x \in E$. They proved the sequence $\{x_n\}$ converges strongly to $\Pi_F x$, where $F = F(K_r) \cap VI(C, A) \cap T^{-1}0 \cap F(S)$.

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In 2013 Saewan and et al. [16] introduced the following algorithm: $x_1 \in C$ and

$$\begin{aligned} z_n &= J^{B_l}_{\lambda_l,n} \circ J^{B_{l-1}}_{\lambda_{l-1},n} \circ ... \circ J^{B_1}_{\lambda_1,n} x_n, \\ u_n &= K^{\Phi_m}_{r_m,n} \circ K^{\Phi_{m-1}}_{r_{m-1},n} \circ ... \circ K^{\Phi_1}_{r_1,n} z_n, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_1, \ \forall n \ge 1, \end{aligned}$$

where $f_i : C \times C \to \mathbb{R}$ for any $i = 1, 2, 3, ..., m$ is a bifunction, $\{A_i\}$ is a finite family of continuous and monotone mappings from C to E^*, $B_j \subset E \times E^*$ for all $j \in \{1, 2, ..., l\}$ is maximal monotone operator and

$$K^{\Phi_i}_{r_i}(x) = \{z \in C : \Phi_i(z, y) + \frac{1}{r_i}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in C\} \ \forall \ x \in C,$$

where $\Phi_i(z, y) = f_i(z, y) + \langle y - z, A_i z\rangle$. They showed the sequence $\{x_n\}$ converges strongly to a point $p \in F = (\bigcap_{i=1}^m GEP(f_i, A_i) \cap (\bigcap_{j=1}^l B_j^{-1}0)$, where $p = \Pi_F x_1$.
Also Cai and Bu [3] in 2013 investigated the sequence $\{x_n\}$ generated from the following scheme: $u_1 \in C$ and

$$\begin{aligned} &x_n \in C \ such \ that \ f(x_n, y) + \langle Bx_n, y - x_n\rangle + \frac{1}{r_n}\langle y - x_n, Jx_n, Ju_n\rangle \ge 0, \ \forall y \in C, \\ &z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \qquad (1.4) \\ &u_{n+1} = J^{-1}(\alpha_n Jx_n + \beta_n JT_n z_n + \gamma_n JS_n z_n), \ \forall n \ge 1, \end{aligned}$$

where $f : C \times C \to \mathbb{R}$ is a bifunction, $B : C \to E^*$ is a β-inverse strongly monotone operator, $A : C \to E^*$ is an α-inverse strongly monotone operator, $\{T_n\}$ and $\{S_n\}$ are two sequences of relatively nonexpansive mappings from C into C. They proved $\{x_n\}$ converges weakly to $z \in F = \bigcap_{i=1}^\infty F(T_i) \cap \bigcap_{i=1}^\infty F(S_i) \cap GEP(f, A) \cap VI(C, A)$, where $z = \lim_{n\to\infty} \Pi_F x_n$.

There is the fact that projection method and its different kinds including the Wiener-Hopf equations can not be expanded and improved to solve mixed equilibrium-variational inequalities due to the nature of the problem. This truth motivated us to study the auxiliary flexible methods. With this motivation, we present two iteration methods which converges weakly to common point of the solutions set of the system of generalized mixed equilibrium problems, solutions set of the variational inequality for an inverse-strongly monotone operator, common fixed points set of two infinite sequences of relatively nonexpansive mappings and common zero points set of two finite sequences of maximal monotone operators.

2. Preliminaries

Following lemmas and notations are necessary for proof of our main results.

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Lemma 2.1. [17] *If $\{a_n\}$ and $\{b_n\}$ are two real nonnegative sequences satisfying*

$$a_{n+1} \leq a_n + b_n \ \forall n \geq 1,$$
$$\sum_{n=1}^{\infty} a_n < \infty,$$

then $\{a_n\}$ is convergent.

Lemma 2.2. [20] *If E be a 2-uniformly convex Banach space, then for all $x, y \in E$ we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $\frac{1}{c}$ $(0 < c \leq 1)$ is the 2-uniformly convex constant of E.

Lemma 2.3. [6] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu\gamma g(\|x - y\|),$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

We see important consequences that are proved by Kamimura and Takahashi.

Lemma 2.4. [9] *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \varphi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5. [9] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.*

Lemma 2.6. [9] *Let E be a strictly convex, smooth and reflexive Banach space and $T : E \rightrightarrows E^*$ be a maximal monotone operator. If $T^{-1}(0) \neq \emptyset$, then for any $r > 0$,*

$$\phi(u, J_r^T x) + \phi(J_r^T x, x) \leq \phi(u, x),$$

where $u \in T^{-1}(0)$ and $x \in E$.

We now record famous consequences of Alber:

Lemma 2.7. [1] *Let E be a smooth Banach space, let C be a nonempty closed convex subset of E, and let $x \in E$ and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if $\langle y - x_0, Jx - Jx_0\rangle \leq 0$ for all $y \in C$.*

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Lemma 2.8. [1] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \ \forall y \in C.$$

Also another concept used in this paper which was studied by Alber [1] is mapping $V : E \times E^* \to \mathbb{R}$:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2,$$

for all $x \in E$ and $x^* \in E^*$. It is easy to see that $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for each $x \in E$ and $x^* \in E^*$. Further, it is noteworthy that V is continuous and convex respect to second component. More mention on functional V is in order.

Lemma 2.9. [10] *Let E be a strictly convex, smooth and reflexive Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

To solve the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, suppose that f satisfies the following conditions:
$(C1)$ $f(x, x) = 0$ for all $x \in C$;
$(C2)$ f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
$(C3)$ for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

$(C4)$ for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

The following results for solving the different types of equilibrium problems are noteworthy and important.

Lemma 2.10. [15] *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone operator from C to E^* such that $D(A) = C$ and $t \to A(tx + (1 - t)y)$ for all $x, y \in C$, is continuous with respect to the weak* topology of E^*. Let $B \subset E \times E^*$ be an operator defined as follows:*

$$Bv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \neq C, \end{cases}$$

where $N_C(v) := \{x^* \in E^* : \langle v, y - x \rangle \geq 0, \ \forall y \in C\}$ is the normal cone for C at a point $v \in C$. Then B is maximal monotone and $B^{-1}(0) = VI(C, A)$.

Lemma 2.11. [21]*Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\phi : C \to \mathbb{R}$ is convex and lower semi-continuous and f be a bifunction*

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from $C \times C$ to $\mathbb{R}$ satisfying $(C1)$-$(C4)$. For $r > 0$ and $x \in E$, then there exists $u \in C$ such that

$$f(u, y) + \langle Au, y - u\rangle + \phi(y) - \phi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C.$$

Define a mapping $K_r^{\Theta} : C \to C$ as follows:

$$K_r^{\Theta}(x) = \{u \in C : \Theta(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\}\ \forall x \in C, \quad (2.1)$$

where $\Theta(u, y) = f(u, y)+ < Au, y - u > +\phi(y) - \phi(u)$ for all $u, y \in C$. Then the followings hold:
(a) K_r^{Θ} is single-valued;
(b) K_r^{Θ} is firmly nonexpansive, i.e., for all $x, y \in E$,

$$\langle K_r^{\Theta}x - K_r^{\Theta}y, JK_r^{\Theta}x - JK_r^{\Theta}y\rangle \leq \langle K_r^{\Theta}x - K_r^{\Theta}y, Jx - Jy\rangle;$$

(c) $F(K_r^{\Theta}) = \Omega$;
(d) Ω is closed and convex;
(e) $\phi(p, K_r^{\Theta}z) + \phi(K_r^{\Theta}z, z) \leq \phi(p, z)$, $\forall p \in F(K_r^{\Theta})$, $z \in E$.

Hereinafter, we define $K_{r_i}^{\Theta_i}(x) : C \to C$ by

$$K_{r_i}^{\Theta_i}(x) = \{u \in C : \Theta_i(u, y) + \frac{1}{r_i}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\}\ \forall x \in C,$$

where $\Theta_i(u, y) = f_i(u, y) + \langle A_i u, y - u\rangle + \phi_i(y) - \phi_i(u)$.

Remark 2.12. [21] It follows from Lemma 2.11 that the mapping $K_r^{\Theta} : C \to C$ defined by (2.1) is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.

During the last three decades, many researchers have been achieved the number of results on common fixed point of different types of mappings, see [5, 8, 12, 13]. A few consequences follow.

Lemma 2.13. [12] *Let E be a strictly convex Banach space whose norm is Frechét differentiable, let C be a nonempty closed convex subsets of E, and let $\{T_n\}$ be a countable family of relatively nonexpansive mappings from C into E such that $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Then $F(T)$ is closed and convex.*

Lemma 2.14. [12] *Let C be a nonempty subset of a Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E. Suppose that for any bounded subset B of C there exists a continuous increasing function h_B from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_B(0) = 0$ and $\lim_{k,l\to\infty} \rho_l^k = 0$, where $\rho_l^k := \sup\{h_B(\|T_k z - T_l z\|) : z \in B\} < \infty$ for all $k, l \in N$. Then*

$$\lim_{n\to\infty} \sup\{h_B(\|Tz - T_n z\|) : z \in B\} = 0.$$

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Lemma 2.15. [12] *If* $\sum_{n=1}^{\infty}\sup\{h_B(\|T_{n+1}z - T_n z\|) : z \in B\} < \infty$ *and* $h_B : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous, increasing function such that* $h_B(0) = 0$*, then*

$$\limsup_{n\to\infty}\{h_B(\|T_k z - T_l z\|) : z \in B\} = 0.$$

Lemma 2.16. [13] *Let* E *be a strictly convex and smooth Banach space,* C *be a nonempty closed convex subset of* E *and* $T : C \to C$ *be a relatively quasi-nonexpansive mapping. Then* $F(T)$ *is a closed convex subset of* C*.*

Lemma 2.17. [22] *Let* C *be a nonempty closed and convex subset of a real uniformly convex Banach space* E*. Let* $T_j : C \to E$*,* $j = 1, 2, ...$ *be closed relatively quasi-nonexpansive mappings such that* $\bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset$*. Then the mapping* $T := J^{-1}(\sum_{j=0}^{\infty}\xi_j JT_j) : C \to E$ *is closed relatively quasi-nonexpansive and* $F(T) = \bigcap_{j=1}^{\infty} F(T_j)$*, where* $\sum_{j=0}^{\infty}\xi_j = 1$ *and* $T_0 = I$*.*

3. Main Results

In this section we will present present our main results.

Theorem 3.1. *Let* E *be a 2-uniformly convex and smooth Banach space, let* C *be a nonempty closed convex subset of* E*. For any* $i = 1, 2, 3, ..., m$*, let* f_i *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(C1) - (C4)$ *and let* $\{A_i\}$ *be a finite family of continuous and monotone mappings from* C *to* E^* *and* $\{\phi_i\}$ *be a finite family of proper lower semicontinuous and convex functions from* C *to* $\mathbb{R}$*. Let* $A : C \to E^*$ *be an* α*-inverse strongly monotone operator. Let* $T_j, S_k \subset E \times E^*$ *be maximal monotone operators satisfying* $D(T_j), D(S_k) \subset C$ *and* $J_{\rho_{j,n}}^{T_j} = (J + \rho_{j,n}T_j)^{-1}J$ *for all* $\rho_{j,n} > 0$ *and* $j = 1, 2, ..., l$*, also* $J_{\rho'_{k,n}}^{S_k} = (J + \rho'_{k,n}S_k)^{-1}J$ *for all* $\rho'_{k,n} > 0$ *and* $k = 1, 2, ..., l'$*. Let* $\{T'_n\}$ *and* $\{S'_n\}$ *be two sequences of relatively nonexpansive mappings from* C *into itself such that* $F := \bigcap_{i=1}^{\infty} F(T'_i) \cap \bigcap_{i=1}^{\infty} F(S'_i) \cap \bigcap_{i=1}^{l} T_i^{-1}0 \cap \bigcap_{i=1}^{l'} S_i^{-1}0 \cap \bigcap_{i=1}^{m}\Omega(f_i, A_i, \phi_i) \cap VI(C, A) \neq \emptyset$ *and* $\|Ax\| \leq \|Ax - Av\|$ *for all* $x \in C$ *and* $v \in F$*.*
Assume that for any bounded subset B *of* C *there exists an increasing, continuous and convex function* h_B *from* $\mathbb{R}^+$ *into* $\mathbb{R}_+$ *such that* $h_B(0) = 0$ *and*

$$\limsup_{k,l\to\infty}\{h_B(\|T'_k z - T'_l z\|) : z \in B)\} = 0, \limsup_{k,l\to\infty}\{h_B(\|S'_k z - S'_l z\|) : z \in B)\} = 0. \quad (3.1)$$

Let $T', S' : C \to E$ *such that* $T'x = \lim_{n\to\infty} T'_n x$*,* $S'x = \lim_{n\to\infty} S'_n x$ *for all* $x \in C$ *and suppose that* $F(T') = \bigcap_{i=1}^{\infty} F(T'_i) = \bigcap_{i=1}^{\infty}\widehat{F}(T'_i) = \widehat{F}(T')$ *and* $F(S') = \bigcap_{i=1}^{\infty} F(S'_i) = \bigcap_{i=1}^{\infty}\widehat{F}(S'_i) = \widehat{F}(S')$*.*
Assume that $u \in E$ *is given,* $\{\alpha_n\}, \{\mu_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\} \subseteq [0, 1]$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *such that*
i. $\alpha_n + \beta_n + \gamma_n + \eta_n = 1$*,* *ii.* $\sum_{n=1}^{\infty}\mu_n\alpha_n < \infty$*,* *iii.* $\sum_{n=1}^{\infty}\mu_n\beta_n < \infty$*,*
iv. $\liminf_{n\to\infty}\eta_n\mu_n(1 - \mu_n) > 0$*,* *v.* $\liminf_{n\to\infty}\mu_n\gamma_n(1 - \mu_n) > 0$*,*
vi. $\liminf_{n\to\infty}\rho_{j,n} > 0$ *for each* $j \in \{1, 2, ..., l\}$*,*

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vii. $\liminf_{n\to\infty} \rho'_{j,n} > 0$ *for each* $j \in \{1, 2, ..., l'\}$,
viii. $\{r_{i,n}\} \subset [a, \infty)$ *for some* $a > 0$ *and for any* $i \in \{1, 2, ..., m\}$,
ix. $\{\lambda_n\} \subset [d, e]$ *for some* d, e *with* $0 < d < e < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity constant of* E,
x. the sequence $\{e_n\} \subset E$ *is bounded.*
For arbitrary $u \in E$ *generate a sequence* $\{x_n\}$ *by following method*

$$\begin{aligned} x_n &= K^{\Theta_m}_{r_m,n} \circ K^{\Theta_{m-1}}_{r_{m-1},n} \circ ... K^{\Theta_1}_{r_1,n} u_n, \\ z_n &= \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_{n+1} &= J^{T_l}_{\rho_l,n} \circ J^{T_{l-1}}_{\rho_{l-1},n} \circ ... J^{T_1}_{\rho_1,n} J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n \\ &\quad + \gamma_n JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n + \eta_n JT'_n z_n) + (1-\mu_n)Jx_n]. \end{aligned}$$

If J *is weakly sequentially continuous, then the sequence* $\{x_n\}$ *converges weakly to a point* $x^* \in F$.

Proof. The proof will be accomplished in 10 steps.
Step 1. First, we will prove $\{x_n\}$ is bounded. From Lemmas 2.8 and 2.9 and for $u^* \in F$,

$$\begin{aligned} \phi(u^*, z_n) &= \phi(u^*, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &\le \phi(u^*, J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &= V(u^*, Jx_n - \lambda_n Ax_n) \\ &\le V(u^*, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - u^*, \lambda_n Ax_n\rangle \\ &= V(u^*, Jx_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - u^*, \lambda_n Ax_n\rangle \qquad (3.2) \\ &= \phi(u^*, x_n) - 2\lambda_n\langle x_n - u^*, Ax_n\rangle + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle. \end{aligned}$$

Since $u \in VI(C, A)$ and A is α-strongly monotone,

$$\begin{aligned} -2\lambda_n\langle x_n - u^*, Ax_n\rangle &= -2\lambda_n\langle x_n - u^*, Ax_n - Au^*\rangle - 2\lambda_n\langle x_n - u^*, Au^*\rangle \\ &\le -2\alpha\lambda_n\|Ax_n - Au^*\|^2. \qquad (3.3) \end{aligned}$$

By Lemma 2.2 and condition $\|Ay\| \le \|Ay - Au^*\|$ for all $y \in C$ and $u^* \in F$,

$$\begin{aligned} 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) \\ &\quad - J^{-1}Jx_n, -\lambda_n Ax_n\rangle \le 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n\|\|\lambda_n Ax_n\| \\ &\le \frac{4}{c^2}\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}Jx_n\|\|\lambda_n Ax_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\ &\le \frac{4}{c^2}\lambda_n^2\|Ax_n - Au^*\|^2. \qquad (3.4) \end{aligned}$$

By (3.2), (3.3) and (3.4) and assumption (ix),

$$\phi(u^*, z_n) \le \phi(u^*, x_n) + 2\lambda_n(\frac{2\lambda_n}{c^2} - \alpha)\|Ax_n - Au^*\|^2 \le \phi(u^*, x_n). \qquad (3.5)$$

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For $u^* \in F$, by (3.5), condition (x), Lemmas 2.11 and 2.1 and relatively nonexpansivity of mappings T'_n and S'_n for any $n \in \mathbb{N}$,

$$\begin{aligned}
\phi(u^*, x_{n+1}) &= \phi(u^*, K^{\Theta_m}_{r_m,n+1} \circ K^{\Theta_{m-1}}_{r_{m-1},n+1} \circ ... K^{\Theta_1}_{r_1,n+1} u_{n+1}) \leq \phi(u^*, u_{n+1}) \\
&= \phi(u^*, J^{T_l}_{\rho_l,n} \circ J^{T_{l-1}}_{\rho_{l-1},n} \circ ... J^{T_1}_{\rho_1,n} J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n \\
&\quad + \gamma_n JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n + \eta_n JT'_n z_n) + (1-\mu_n) Jx_n]) \\
&\leq \phi(u^*, J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n \\
&\quad + \eta_n JT'_n z_n) + (1-\mu_n) Jx_n]) \\
&= V(u^*, \mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n \\
&\quad + \eta_n JT'_n z_n) + (1-\mu_n) Jx_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, S'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n) \\
&\quad + \mu_n \eta_n \phi(u^*, T'_n z_n) + (1-\mu_n) \phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n) \\
&\quad + \mu_n \eta_n \phi(u^*, z_n) + (1-\mu_n) \phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + (\mu_n \gamma_n + \mu_n \eta_n) \phi(u^*, z_n) + (1-\mu_n) \phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + (1 - \mu_n(1-\gamma_n-\eta_n)) \phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + \phi(u^*, x_n), \qquad (3.6)
\end{aligned}$$

where $\phi(u^*, e_n) \leq L$. From (ii), (iii) and Lemma 2.1, we obtain that $\lim_{n\to\infty} \phi(u^*, x_n)$ exists. Without any loss of generality we assume that $\lim_{n\to\infty} \phi(u^*, x_n) = p$. Then (1.1) will let us conclude that $\{x_n\}$ is bounded and there exist a subsequence $\{x_{n_k}\}$ and $x^* \in E$ such that $x_{n_k} \rightharpoonup x^*$ as $n_k \to \infty$. By (3.5), we have $\{z_n\}$ is also bounded.

Step 2. We prove that $\|x_n - z_n\| \to 0$ as $n \to \infty$, so, $z_{n_k} \rightharpoonup x^*$ as $n_k \to \infty$. Using (3.5) and (3.6),

$$\begin{aligned}
\phi(u^*, x_{n+1}) &\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + (1-\mu_n) \phi(u^*, x_n) \\
&\quad + (\mu_n \gamma_n + \mu_n \eta_n) \phi(u^*, z_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + (1-\mu_n) \phi(u^*, x_n) \\
&\quad + (\mu_n \gamma_n + \mu_n \eta_n)[\phi(u^*, x_n) + 2\lambda_n(\frac{2\lambda_n}{c^2} - \alpha)\|Ax_n - Au^*\|^2] \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + \phi(u^*, x_n) + 2\lambda_n(\frac{2\lambda_n}{c^2} - \alpha)\|Ax_n - Au^*\|^2.
\end{aligned}$$

This gives us

$$2\lambda_n(\alpha - \frac{2\lambda_n}{c^2})\|Ax_n - Au^*\|^2 \leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + \phi(u^*, x_n) - \phi(u^*, x_{n+1}).$$

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By (ii), (iii), (ix) and since $\{\phi(u^*, x_n)\}$ is convergent, we have

$$\lim_{n\to\infty} \|Ax_n - Au^*\| = 0.$$

Applying Lemmas 2.2, 2.8 and 2.9 and condition $\|Ax\| \leq \|Ax - Av\|$ for all $x \in C$ and $v \in F$, we get

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq V(x_n, Jx_n - \lambda_n Ax_n) \\
&\leq V(x_n, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\
&= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle \\
&= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n, -\lambda_n Ax_n\rangle \\
&\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n\|\|\lambda_n Ax_n\| \\
&\leq \frac{4}{c^2}\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}Jx_n\|\|\lambda_n Ax_n\| \\
&\leq \frac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\
&\leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Au^*\|^2 \\
&\to 0 \ as \ n \to \infty.
\end{aligned}$$

From Lemma 2.4, we see that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.7}$$

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Step 3. We show that $\|T'_n z_n - z_n\| \to 0$ as $n \to \infty$. Since

$$\begin{aligned}
\phi(u^*, x_{n+1}) &\le \phi(u^*, u_{n+1}) \\
&\le \phi(u^*, J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n \\
&\quad + \eta_n JT'_n z_n) + (1-\mu_n)Jx_n]) \\
&= \|u^*\|^2 - 2\mu_n\alpha_n\langle u^*, Ju\rangle - 2\mu_n\beta_n\langle u^*, Je_n\rangle \\
&\quad - 2\mu_n\gamma_n\langle u^*, JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n\rangle \\
&\quad - 2\mu_n\eta_n\langle u^*, JT'_n z_n\rangle - 2(1-\mu_n)\langle u^*, Jx_n\rangle \\
&\quad + \|\mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n + \eta_n JT'_n z_n) \\
&\quad + (1-\mu_n)Jx_n\|^2 - \eta_n\mu_n(1-\mu_n)g_1(\|JT'_n z_n - Jx_n\|) \\
&\le \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n\phi(u^*, e_n) + \mu_n\gamma_n\phi(u^*, S'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n) \\
&\quad + \mu_n\eta_n\phi(u^*, T'_n z_n) + (1-\mu_n)\phi(u^*, x_n) - \eta_n\mu_n(1-\mu_n)g_1(\|JT'_n z_n - Jx_n\|) \\
&\le \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n\phi(u^*, e_n) + (\mu_n\gamma_n + \mu_n\eta_n)\phi(u^*, z_n) \\
&\quad + (1-\mu_n)\phi(u^*, x_n) - \eta_n\mu_n(1-\mu_n)g_1(\|JT'_n z_n - Jx_n\|) \\
&\le \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n L + (1-\mu_n(1-\gamma_n-\eta_n))\phi(u^*, x_n) \\
&\quad - \eta_n\mu_n(1-\mu_n)g_1(\|JT'_n z_n - Jx_n\|) \\
&\le \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n L + \phi(u^*, x_n) - \eta_n\mu_n(1-\mu_n)g_1(\|JT'_n z_n - Jx_n\|).
\end{aligned}$$

It is easy to see,

$$\eta_n\mu_n(1-\mu_n)g_1(\|JT'_n z_n - Jx_n\|) \le \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n L + \phi(u^*, x_n) - \phi(u^*, x_{n+1}).$$

By (ii), (iii) and (iv), it is obtained that $\lim_{n\to\infty} g_1(\|JT'_n z_n - Jx_n\|) = 0$. From the property of g_1

$$\lim_{n\to\infty} \|JT'_n z_n - Jx_n\| = 0, \tag{3.8}$$

and since J^{-1} is uniformly norm to norm continuous on bounded sets, we get

$$\lim_{n\to\infty} \|T'_n z_n - x_n\| = 0.$$

From this, by (3.7) and

$$\|T'_n z_n - z_n\| \le \|T'_n z_n - x_n\| + \|x_n - z_n\|,$$

it is obtained that

$$\lim_{n\to\infty} \|T'_n z_n - z_n\| = 0. \tag{3.9}$$

Step 4. We demonstrate that $x^* \in \bigcap_{j=1}^{l'} S^{-1}0$. Set $\Delta'^{j}_n := J^{S_j}_{\rho'_j,n} \circ J^{S_{j-1}}_{\rho'_{j-1},n} \circ ... J^{S_1}_{\rho'_1,n}$. By Lemma 2.6 and for $j \in \{1, 2, ..., l'\}$

$$\phi(u^*, \Delta'^{j}_n z_n) \le \phi(u^*, \Delta'^{j-1}_n z_n) - \phi(\Delta'^{j}_n z_n, \Delta'^{j-1}_n z_n), \tag{3.10}$$

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and

$$\phi(u^*, \Delta_n^{'j} z_n) \leq \phi(u^*, z_n) - \phi(\Delta_n^{'j} z_n, z_n). \tag{3.11}$$

From (3.6) and (3.10) for all $j \in \{1, 2, ..., l'\}$,

$$\begin{aligned}
\phi(u^*, x_{n+1}) &\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, J_{\rho'_{l'},n}^{S_{l'}} \circ J_{\rho'_{l'-1},n}^{S_{l'-1}} \circ ...J_{\rho'_1,n}^{S_1} z_n) \\
&+ \mu_n \eta_n \phi(u^*, z_n) + (1 - \mu_n)\phi(u^*, x_n) \\
&= \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, \Delta_n^{'j} z_n) \\
&+ \mu_n \eta_n \phi(u^*, z_n) + (1 - \mu_n)\phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n (\phi(u^*, \Delta_n^{'j-1} z_n) - \phi(\Delta_n^{'j} z_n, \Delta_n^{'j-1} z_n)) \\
&+ \mu_n \eta_n \phi(u^*, z_n) + (1 - \mu_n)\phi(u^*, x_n),
\end{aligned}$$

which implies,

$$\begin{aligned}
\mu_n \gamma_n \phi(\Delta_n^{'j} z_n, \Delta_n^{'j-1} z_n) &\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, \Delta_n^{'j-1} z_n) \\
&+ \mu_n \eta_n \phi(u^*, z_n) + (1 - \mu_n)\phi(u^*, x_n) - \phi(u^*, x_{n+1}) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, \Delta_n^{'j-2} z_n) \\
&+ \mu_n \eta_n \phi(u^*, z_n) + (1 - \mu_n)\phi(u^*, x_n) - \phi(u^*, x_{n+1}) \\
&\leq ... \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + (1 - \mu_n + \mu_n \gamma_n + \mu_n \eta_n)\phi(u^*, x_n)) \\
&- \phi(u^*, x_{n+1}) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + \phi(u^*, x_n)) - \phi(u^*, x_{n+1}).
\end{aligned}$$

Conditions (ii), (iii) and (v) and convergence of $\{\phi(u^*, x_n)\}$ follow that

$$\lim_{n \to \infty} \phi(\Delta_n^{'j} z_n, \Delta_n^{'j-1} z_n) = 0.$$

From this and Lemma 2.4, it is easy to see

$$\lim_{n \to \infty} \|\Delta_n^{'j} z_n - \Delta_n^{'j-1} z_n\| = 0. \tag{3.12}$$

Also from (3.6) and (3.11) for all $j \in \{1, 2, ..., l'\}$ and by similar path, we have

$$\lim_{n \to \infty} \|\Delta_n^{'j} z_n - z_n\| = 0. \tag{3.13}$$

This term and (3.7) affirm that

$$\lim_{n \to \infty} \|\Delta_n^{'j} z_n - x_n\| = 0 \ and \ \Delta_{n_k}^{'j} z_{n_k} \rightharpoonup x^* \ as \ n_k \to \infty. \tag{3.14}$$

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Since J is uniformly norm to norm continuous on bounded subsets of E, $\Delta_n^{'j} z_n = J^{S_j}_{\rho'_{j,n}} \Delta_n^{'j-1} z_n$ and by (3.12) and (vi), it is obtained that

$$\lim_{n\to\infty} \|S_{\rho'_{j,n}} \Delta_n^{'j-1} z_n\| = \lim_{n\to\infty} \frac{\|J\Delta_n^{'j-1} z_n - JJ^{S_j}_{\rho'_{j,n}} \Delta_n^{'j-1} z_n\|}{\rho'_{j,n}}$$
$$= \lim_{n\to\infty} \frac{\|J\Delta_n^{'j-1} z_n - J\Delta_n^{'j} z_n\|}{\rho'_{j,n}} = 0. \tag{3.15}$$

For any $(v, v^*) \in G(S_j)$, $(\Delta_{n_k}^{'j} z_{n_k}, S_{\rho'_{j,n_k}} \Delta_{n_k}^{'j-1} z_n) \in G(S_j)$ for each $j \in \{1, 2, ..., l'\}$ and by monotonicity of $G(S_j)$

$$\langle v - \Delta_{n_k}^{'j} z_{n_k}, v^* - S_{\rho'_{j,n_k}} \Delta_{n_k}^{'j-1} z_{n_k}\rangle \geq 0,$$

for any $j \in \{1, 2, ..., l'\}$. Tending $n_k \to \infty$ at the above inequality, from (3.14), (3.15) and demiclosedness of $G(S_j)$ for all $j \in \{1, 2, ..., l'\}$, we get

$$\langle v - x^*, v^*\rangle \geq 0.$$

Maximality of $G(S_j)$ for all $j \in \{1, 2, ..., l'\}$ indicates that $x^* \in \bigcap_{j=1}^{l'} S_j^{-1}0$.
Step 5. We assert that $x^* \in \bigcap_{j=1}^{l} T_j^{-1}0$. By (3.6)

$$\begin{aligned}\phi(u^*, x_{n+1}) &\leq \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n\phi(u^*, e_n) + \mu_n\gamma_n\phi(u^*, S'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n)\\
&+ \mu_n\eta_n\phi(u^*, T'_n z_n) + (1-\mu_n)\phi(u^*, x_n)\\
&- \gamma_n\mu_n(1-\mu_n)g_2(\|JS'_n J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ... J^{S_1}_{\rho'_1,n} z_n - Jx_n\|)\\
\\
&= \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n\phi(u^*, e_n) + \mu_n\gamma_n\phi(u^*, S'_n\Delta_n^{'l'} z_n)\\
&+ \mu_n\eta_n\phi(u^*, T'_n z_n) + (1-\mu_n)\phi(u^*, x_n)\\
&- \gamma_n\mu_n(1-\mu_n)g_2(\|JS'_n\Delta_n^{'l'} z_n - Jx_n\|),\end{aligned}$$

so

$$\begin{aligned}\gamma_n\mu_n(1-\mu_n)g_2(\|JS'_n\Delta_n^{'l'} z_n - Jx_n\|) &\leq \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n\phi(u^*, e_n)\\
&+ \mu_n\gamma_n\phi(u^*, S'_n\Delta_n^{'l'} z_n) + \mu_n\eta_n\phi(u^*, T'_n z_n)\\
&+ (1-\mu_n)\phi(u^*, x_n) - \phi(u^*, x_{n+1})\\
&\leq \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n L - \phi(u^*, x_{n+1})\\
&+ (1-\mu_n+\mu_n\gamma_n+\mu_n\eta_n)\phi(u^*, x_n)\\
&\leq \mu_n\alpha_n\phi(u^*, u) + \mu_n\beta_n L\\
&+ \phi(u^*, x_n) - \phi(u^*, x_{n+1}).\end{aligned}$$

By (ii), (iii), (v), convergence of $\{\phi(u^*, x_n)\}$ and property of g_2, we have

$$\lim_{n\to\infty} \|JS'_n\Delta_n^{'l'} z_n - Jx_n\| = 0. \tag{3.16}$$

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Set for any $j \in \{1, 2, ..., l\}$

$$\Delta_n^j := J_{\rho_j,n}^{T_j} \circ J_{\rho_{j-1},n}^{T_{j-1}} \circ ... J_{\rho_1,n}^{T_1} J^{-1},$$

$$t_n := J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS'_n \Delta_n^{'l'} z_n + \eta_n JT'_n z_n) + (1 - \mu_n)Jx_n].$$

Then $u_{n+1} = \Delta_n^l t_n$. By Lemma 2.2, nonexpansivity of resolvent operator and triangular inequality,

$$\begin{aligned}
\|\Delta_n^l t_n - \Delta_n^l x_n\| &\leq \|J_{\rho_{l,n}}^{T_l} \Delta_n^{l-1} t_n - J_{\rho_{l,n}}^{T_l} \Delta_n^{l-1} x_n\| \\
&\leq ... \leq \|t_n - x_n\| \leq \frac{2}{c^2}\|Jt_n - Jx_n\| \\
&\leq \frac{2}{c^2}\|J[J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS'_n \Delta_n^{'l'} z_n + \eta_n JT'_n z_n) \\
&+ (1 - \mu_n)Jx_n] - JJ^{-1}Jx_n\| \\
&\leq \frac{2}{c^2}[\mu_n\alpha_n\|Ju - Jx_n\| + \mu_n\beta_n\|Je_n - Jx_n\| \\
&+ \mu_n\gamma_n\|JS'_n \Delta_n^{'l'} z_n - Jx_n\| + \mu_n\eta_n\|JT'_n z_n - Jx_n\|].
\end{aligned}$$

From (ii), (iii), (3.8), (3.16) and boundedness of $\{x_n\}$ and $\{e_n\}$, we will find that

$$\lim_{n\to\infty} \Delta_n^l t_n = \lim_{n\to\infty} \Delta_n^l x_n. \tag{3.17}$$

By (3.6), conditions (ii) and (iii) and convergence of $\{\phi(u^*, x_n)\}$, we can find

$$\lim_{n\to\infty} \phi(u^*, u_{n+1}) = p. \tag{3.18}$$

From (3.17) and (3.18) and for all $j \in \{1, 2, ..., l\}$

$$\begin{aligned}
p = \lim_{n\to\infty} \phi(u^*, u_{n+1}) &= \phi(u^*, \lim_{n\to\infty} u_{n+1}) = \phi(u^*, \lim_{n\to\infty} \Delta_n^l t_n) \\
&= \phi(u^*, \lim_{n\to\infty} \Delta_n^l x_n) = \lim_{n\to\infty} \phi(u^*, \Delta_n^l x_n) \\
&\leq ... \leq \lim_{n\to\infty} \phi(u^*, \Delta_n^j x_n) \leq ... \\
&\leq \lim_{n\to\infty} \phi(u^*, \Delta_n^1 x_n) \leq \lim_{n\to\infty} \phi(u^*, x_n) = p.
\end{aligned}$$

Hence for all $j \in \{1, 2, ..., l\}$

$$\lim_{n\to\infty} \phi(u^*, \Delta_n^j x_n) = p. \tag{3.19}$$

Applying Lemma 2.6 for all $j \in \{1, 2, ..., l\}$

$$\phi(\Delta_n^j x_n, x_n) \leq \phi(u^*, x_n) - \phi(u^*, \Delta_n^j x_n),$$

so by (3.19), we get

$$\lim_{n\to\infty} \phi(\Delta_n^j x_n, x_n) = 0.$$

From Lemma 2.4, we derive

$$\lim_{n\to\infty} \|\Delta_n^j x_n - x_n\| = 0. \tag{3.20}$$

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It follows that

$$\Delta_{n_k}^{j} x_{n_k} \rightharpoonup x^* \ as\ n_k \to \infty. \tag{3.21}$$

Nonexpansivity of resolvent operator and (3.20) allows us to conclude that

$$\begin{aligned}\|\Delta_n^j x_n - \Delta_n^{j-1} x_n\| &\le \|\Delta_n^{j-1} x_n - \Delta_n^{j-2} x_n\| \\ &\le ... \le \|\Delta_n^1 x_n - x_n\| \to 0,\ as\ n \to \infty,\end{aligned} \tag{3.22}$$

for all $j \in \{1, 2, ..., l\}$. Since J is uniformly norm to norm continuous on bounded subset of E and (3.22) we achieve to

$$\lim_{n\to\infty} \|J\Delta_n^j x_n - J\Delta_n^{j-1} x_n\| = 0. \tag{3.23}$$

From condition (vi) and (3.23), we are able to deduce that

$$\lim_{n\to\infty} \|T_{\rho_{j,n}} \Delta_n^{j-1} x_n\| = \lim_{n\to\infty} \frac{\|J\Delta_n^j x_n - J\Delta_n^{j-1} x_n\|}{\rho_{j,n}} = 0. \tag{3.24}$$

For any $(w, w^*) \in G(T_j)$, $(\Delta_{n_k}^j x_{n_k}, T_{\rho_{j,n_k}} \Delta_{n_k}^{j-1} x_{n_k}) \in G(T_j)$ for each $j \in \{1, 2, ..., l\}$ and by monotonicity of $G(T_j)$

$$\langle v - \Delta_{n_k}^j x_{n_k}, v^* - T_{\rho_{j,n_k}} \Delta_{n_k}^{j-1} x_{n_k}\rangle \ge 0,$$

for any $j \in \{1, 2, ..., l\}$. When $n_k \to \infty$ at the above inequality, from (3.21), (3.24) and demiclosedness of $G(T_j)$ for all $j \in \{1, 2, ..., l\}$, we get

$$\langle w - x^*, w^*\rangle \ge 0.$$

Maximality of $G(T_j)$ for all $j \in \{1, 2, ..., l\}$ indicate that $x^* \in \bigcap_{j=1}^{l} T_j^{-1} 0$.
Step 6. We indicate $\lim_{n\to\infty} \|S'_n \Delta_n^{'l'} z_n - \Delta_n^{'l'} z_n\| = 0$. By (3.16) and since J^{-1} is uniformly norm to norm continuous on bounded subset of E^*, we see

$$\lim_{n\to\infty} \|S'_n \Delta_n^{'l'} z_n - x_n\| = 0.$$

From this, (3.14) and

$$\|S'_n \Delta_n^{'l'} z_n - \Delta_n^{'l'} z_n\| \le \|S'_n \Delta_n^{'l'} z_n - x_n\| + \|x_n - \Delta_n^{'l'} z_n\|,$$

we derive the desired conclusion.
Step 7. Here, we observe that $x^* \in \bigcap_{i=1}^{\infty} F(S'_i) \cap \bigcap_{i=1}^{\infty} F(T'_i)$. We know $\{z_n\}$ is bounded, then there exists a bounded subset B of C such that $\{z_n\} \subset B$. From

$$\frac{1}{2}\|z_n - T' z_n\| \le \frac{1}{2}\|z_n - T'_n z_n\| + \frac{1}{2}\|T'_n z_n - T' z_n\|,$$

and since h_B is an increasing, continuous and convex function from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $h_B(0) = 0$, we see that

$$\begin{aligned} h_B(\frac{1}{2}\|z_n - T' z_n\|) &\le \frac{1}{2} h_B(\|z_n - T'_n z_n\|) + \frac{1}{2} h_B(\|T'_n z_n - T' z_n\|) \\ &\le \frac{1}{2} h_B(\|z_n - T'_n z_n\|) + \frac{1}{2} \sup\{h_B(\|T'_n z - T' z\|) :\ z \in B\}.\end{aligned}$$

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Assumption of (3.1), continuity of h_B, Lemma 2.14 and (3.9), imply that

$$\lim_{n\to\infty} h_B(\frac{1}{2}\|z_n - T'z_n\|) = 0.$$

From this and the properties of h_B, we are able to derive that

$$\lim_{n\to\infty} \|z_n - T'z_n\| = 0. \tag{3.25}$$

The statement

$$\lim_{n\to\infty} \|{\Delta'}_n^{l'} z_n - S'{\Delta'}_n^{l'} z_n\| = 0, \tag{3.26}$$

follows from the similar path. Since subsequences $\{{\Delta'}_{n_k}^{l'} z_{n_k}\}$ of $\{{\Delta'}_n^{l'} z_n\}$ and $\{z_{n_k}\}$ of $\{z_n\}$ converges weakly to $x^* \in C$ and from (3.25), (3.26),

$$F(T') = \bigcap_{i=1}^{\infty} F(T'_i) = \bigcap_{i=1}^{\infty} \widehat{F}(T'_i) = \widehat{F}(T'),$$
$$and$$
$$F(S') = \bigcap_{i=1}^{\infty} F(S'_i) = \bigcap_{i=1}^{\infty} \widehat{F}(S'_i) = \widehat{F}(S'),$$

we conclude that $x^* \in F(T') \cap F(S') = \bigcap_{i=1}^{\infty} F(T'_i) \cap \bigcap_{i=1}^{\infty} F(S'_i)$.

Step 8. We show that $x^* \in \bigcap_{i=1}^{m} \Omega(f_i, A_i, \phi_i)$. Denote $\Psi_n^i = K_{r_i,n}^{\Theta_i} \circ K_{r_{i-1},n}^{\Theta_{i-1}} \circ \ldots K_{r_1,n}^{\Theta_1}$, for any $i \in \{1, 2, .., m\}$, so $x_n = \Psi_n^m u_n$. By (3.6), we see for any $i \in \{1, 2, ..., m\}$

$$\begin{aligned}\phi(u^*, x_n) &= \phi(u^*, \Psi_n^m u_n) \leq \ldots \leq \phi(u^*, \Psi_n^i u_n)\\ &\leq \ldots \leq \phi(u^*, K_{r_1,n}^{\Theta_1} u_n) \leq \phi(u^*, u_n)\\ &\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + \phi(u^*, x_{n-1}).\end{aligned}$$

From this, conditions (ii) and (iii) and convergence $\{\phi(u^*, x_n)\}$ to p, we realize that for any $i \in \{1, 2, ..., m\}$,

$$\lim_{n\to\infty} \phi(u^*, \Psi_n^i u_n) = p. \tag{3.27}$$

Let $r'_i = \sup_{n\geq 1}\{\|\Psi_n^i u_n\|, \|\Psi_n^{i-1} u_n\|\}$. By Lemma 2.5 there exists continuous straictly increasing and convex function g'_i with $g'_i(0) = 0$ such that $g'_i(\|x - y\|) \leq \phi(x, y)$, $\forall x, y \in B_{r'_i}(0)$ and $i \in \{1, 2, ..., m\}$. By Lemma 2.11 and for $u^* \in F$ we have

$$g'_i(\|\Psi_n^i u_n - \Psi_n^{i-1} u_n\|) \leq \phi(\Psi_n^i u_n, \Psi_n^{i-1} u_n) \leq \phi(u^*, \Psi_n^{i-1} u_n) - \phi(u^*, \Psi_n^i u_n).$$

Now (3.27) gives us $\lim_{n\to\infty} g'_i(\|\Psi_n^i u_n - \Psi_n^{i-1} u_n\|) = 0$. Then from the property of g'_i, for any $i \in \{1, 2, ..., m\}$ we have

$$\lim_{n\to\infty} \|\Psi_n^i u_n - \Psi_n^{i-1} u_n\| = 0. \tag{3.28}$$

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Since J is uniformly norm to norm continuous on bounded subset of E and $\{r_{i,n}\} \subset [a, \infty)$ for $i \in \{1, 2, ..., m\}$ we obtain

$$\lim_{n\to\infty} \frac{\|J\Psi_n^i u_n - J\Psi_n^{i-1} u_n\|}{r_{i,n}} = 0. \tag{3.29}$$

From (3.28) and since for $i \in \{1, 2, ..., m\}$

$$\begin{aligned}\|\Psi_n^i u_n - x_n\| = \|\Psi_n^i u_n - \Psi_n^m u_n\| &\le \|\Psi_n^m u_n - \Psi_n^{m-1} u_n\| \\ &+ \|\Psi_n^{m-1} u_n - \Psi_n^{m-2} u_n\| + ... + \|\Psi_n^{i+1} u_n - \Psi_n^i u_n\|,\end{aligned}$$

it is clear that $\lim_{n\to\infty} \|\Psi_n^i u_n - x_n\| = 0$. Since subsequence $\{x_{n_k}\}$ converges weakly to x^*, so

$$\Psi_{n_k}^i u_{n_k} \rightharpoonup x^* \ as \ n_k \to \infty. \tag{3.30}$$

From Lemma 2.11 we can quickly deduce that for any $i \in \{1, 2, ..., m\}$

$$\Theta_i(\Psi_{n_k}^i u_{n_k}, y) + \frac{1}{r_{i,n_k}} \langle y - \Psi_{n_k}^i u_{n_k}, J\Psi_{n_k}^i u_{n_k} - J\Psi_{n_k}^{i-1} u_{n_k}\rangle \ge 0, \ \forall \ y \in C,$$

where $\Theta_i(x_{n_k}, y) = f_i(x_{n_k}, y) + \langle A_i x_{n_k}, y - x_{n_k}\rangle + \phi_i(y) - \phi_i(x_{n_k})$ for all $x_{n_k}, y \in C$. By monotonicity of f_i and A_i for any $i \in \{1, 2, ..., m\}$, we see

$$\frac{1}{r_{i,n_k}} \langle y - \Psi_{n_k}^i u_{n_k}, J\Psi_{n_k}^i u_{n_k} - J\Psi_{n_k}^{i-1} u_{n_k}\rangle \ge \Theta_i(y, \Psi_{n_k}^i u_{n_k}), \ \forall \ y \in C. \tag{3.31}$$

We know since $y \to f_i(x, y) + \langle A_i x, y - x\rangle + \phi_i(y) - \phi_i(x)$ is convex and lower semicontinuous, it is weakly lower semicontinuous. This, (3.29), (3.30) and (3.31) allow us to deduce that

$$\Theta_i(y, x^*) \le 0, \ \forall \ y \in C. \tag{3.32}$$

For any $t \in [0, 1]$ and $y \in C$, let $y_t = ty + (1 - t)x^*$. Therefore $y_t \in C$ and by (3.32), it is obtained that for any $i \in \{1, 2, ..., m\}$

$$\Theta_i(y_t, x^*) \le 0. \tag{3.33}$$

Then from (C_1), convexity of $y \to \Theta_i(x, y)$ for any $y \in C$ and $i \in \{1, 2, ..., m\}$ and (3.33) we get

$$0 = \Theta_i(y_t, y_t) \le t\Theta_i(y_t, y) + (1 - t)\Theta_i(y_t, x^*) \le t\Theta_i(y_t, y) \le \Theta_i(y_t, y). \tag{3.34}$$

When $t \to 0$ in (3.34) for any $i \in \{1, 2, ..., m\}$ and condition (C_3), we derive that

$$0 \le \lim_{t\to 0} \Theta_i(ty + (1 - t)x^*, y) \le \Theta_i(x^*, y), \ \forall \ y \in C.$$

It follows that $x^* \in \Omega(f_i, A_i, \phi_i)$ for all $i \in \{1, 2, ..., m\}$, i.e., $x^* \in \bigcap_{i=1}^m \Omega(f_i, A_i, \phi_i)$.
Step 9. In this step, we observe that $x^* \in VI(C, A)$. Denote $\tilde{T} \subset E \times E^*$ as follows:

$$\tilde{T}v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \neq C. \end{cases}$$

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By Lemma 2.10, $\tilde{T}$ is maximal monotone and $\tilde{T}^{-1}(0) = VI(C, A)$. For $(v, w) \in G(\tilde{T})$ we have $w \in \tilde{T}v = Av + N_C(v)$, hence $w - Av \in N_C(v)$. From $z_n \in C$, we deduce that

$$\langle v - z_n, w - Av\rangle \geq 0. \tag{3.35}$$

By definition of z_n and Lemma 2.7, we find that

$$\langle v - z_n, Jz_n - (Jx_n - \lambda_n Ax_n)\rangle \geq 0 \Rightarrow \langle v - z_n, (\frac{Jx_n - Jz_n}{\lambda_n}) - Ax_n)\rangle \leq 0 \tag{3.36}$$

Statements (3.35) and (3.36) imply that

$$\begin{aligned}
\langle v - z_{n_k}, w\rangle &\geq \langle v - z_{n_k}, Av\rangle \\
&\geq \langle v - z_{n_k}, Av\rangle + \langle v - z_{n_k}, \frac{Jx_{n_k} - Jz_{n_k}}{\lambda_{n_k}} - Ax_{n_k})\rangle \\
&= \langle v - z_{n_k}, Av - Ax_{n_k}\rangle + \langle v - z_{n_k}, \frac{Jx_{n_k} - Jz_{n_k}}{\lambda_{n_k}}\rangle \\
&= \langle v - z_{n_k}, Av - Az_{n_k}\rangle + \langle v - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle \\
&+ \langle v - z_{n_k}, \frac{Jx_{n_k} - Jz_{n_k}}{\lambda_{n_k}}\rangle \\
&\geq -\|v - z_{n_k}\|\frac{\|x_{n_k} - z_{n_k}\|}{\alpha} - \|v - z_{n_k}\|\frac{\|Jx_{n_k} - Jz_{n_k}\|}{a} \\
&\geq -M(\frac{\|x_{n_k} - z_{n_k}\|}{\alpha} + \frac{\|Jx_{n_k} - Jz_{n_k}\|}{a}),
\end{aligned}$$

where $M := \sup_{n\geq 1}\{\|v - z_{n_k}\|\}$. By tending $n_k \to \infty$ and (3.7), we conclude that $\langle v - x^*, w\rangle \geq 0$. Then $x^* \in \tilde{T}^{-1}0$ follows from the maximality of $\tilde{T}$, i.e., $x^* \in VI(C, A)$.

Step 10. It need only be demonstrated that x^* is unique cluster point of $\{x_n\}$ to complete the proof. Assume that $\overline{x}$ is another cluster point of $\{x_n\}$. Let $q(\overline{x}) = \lim_{n\to\infty} \phi(\overline{x}, x_n)$ and $q(x^*) = \lim_{n\to\infty} \phi(x^*, x_n)$. It is easy to show that

$$\phi(\overline{x}, x_n) = \phi(x^*, x_n) + \phi(\overline{x}, x^*) + 2\langle x^* - \overline{x}, Jx_n - Jx^*\rangle. \tag{3.37}$$

By tending $n_j \to \infty$ in above such that $x_{n_j} \rightharpoonup \overline{x}$ and since J is weakly sequentially continuous, it follows that

$$q(\overline{x}) - q(x^*) = \phi(\overline{x}, x^*) + 2\langle x^* - \overline{x}, J\overline{x} - Jx^*\rangle. \tag{3.38}$$

Similarly tending $n_k \to \infty$ in (3.37) such that $x_{n_k} \rightharpoonup x^*$, we have

$$q(\overline{x}) - q(x^*) = \phi(\overline{x}, x^*). \tag{3.39}$$

Then from (3.38) and (3.39), yields that

$$\langle x^* - \overline{x}, Jx^* - J\overline{x}\rangle = 0,$$

so we deduced that $x^* = \overline{x}$, because J is strictly monotone. □

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Example 3.2. The following sequences are satisfied in conditions of Theorem 3.1.

$\{\alpha_n\} = \{\beta_n\} = \{\frac{1}{4n^3}\}, \quad \{\gamma_n\} = \{\frac{1}{3} - \frac{1}{4n^3}\}, \quad \{\eta_n\} = \{\frac{2}{3} - \frac{1}{4n^3}\}, \quad \{\mu_n\} = \{\frac{1}{2}\}.$

It is easy to see

$$\alpha_n + \beta_n + \gamma_n + \eta_n = 1,$$
$$\sum_{n=1}^{\infty} \mu_n \alpha_n = \sum_{n=1}^{\infty} \mu_n \beta_n = \sum_{n=1}^{\infty} \frac{1}{8n^3} < \infty,$$
$$\liminf_{n\to\infty} \mu_n \gamma_n (1-\mu_n) = \frac{1}{4}(\frac{1}{3} - \frac{1}{4}) = \frac{1}{48} > 0,$$
$$\liminf_{n\to\infty} \mu_n \eta_n (1-\mu_n) = \frac{1}{4}(\frac{2}{3} - \frac{1}{4}) = \frac{5}{48} > 0.$$

Remark 3.3. If in Theorem 3.1, $\alpha_n = \beta_n = 0 \ \forall n \geq 1$, then similar to proof of Theorem 3.2 [3] we can prove that $\{x_n\}$ converges weakly to $x^* = \lim_{n\to} \Pi_F x_n$.

Proof. Let $y_n = \Pi_F x_n$. By (3.6) and since $y_n \in F$ we see

$$\phi(y_n, x_{n+1}) \leq \phi(y_n, x_n). \tag{3.40}$$

From Lemma 2.8 and above inequality, it is obtained that

$$\begin{aligned}\phi(y_{n+1}, x_{n+1}) = \phi(\Pi_F x_{n+1}, x_{n+1}) &\leq \phi(y_n, x_{n+1}) - \phi(y_n, y_{n+1})\\ &\leq \phi(y_n, x_n).\end{aligned}$$

Then $\{\phi(y_n, x_n)\}$ is convergent and by (3.40) we have

$$\phi(y_n, x_{n+m}) \leq \phi(y_n, x_n), \ \forall \ m \in N.$$

Since $y_{n+m} = \Pi_F x_{n+m}$ by Lemma 2.8, one deduce that

$$\begin{aligned}\phi(y_n, y_{n+m}) &\leq \phi(y_n, x_{n+m}) - \phi(y_{n+m}, x_{n+m})\\ &\leq \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m}).\end{aligned}$$

By Lemma 2.5 there exists a continuous, strictly increasing convex function $\tilde{g}$ with $\tilde{g}(0) = 0$ such that

$$\tilde{g}(\|y_n - y_{n+m}\|) \leq \phi(y_n, y_{n+m}) \leq \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m}), \ \forall y_n \in B_{\tilde{r}}(0),$$

where $\tilde{r} = \sup_{n\geq 1}\{\|y_n\|\}$. Since $\{\phi(y_n, x_n)\}$ is convergent, we derive that $\lim_{n\to\infty} \tilde{g}(\|y_n - y_{n+m}\|) = 0$. Now property of $\tilde{g}$ yields $\lim_{n\to\infty}(\|y_n - y_{n+m}\|) = 0$, i.e., $\{y_n\}$ is a cauchy sequence, so there exists $\tilde{x} \in F$ such that $\{y_n\}$ converges strongly to $\tilde{x}$ because F is closed. From Lemma 2.7 and since $x^* \in F$, we see

$$\langle y_{n_k} - x^*, Jx_{n_k} - Jy_{n_k}\rangle \geq 0.$$

J is weakly sequentially continuous, hence tending $n \to \infty$ in above inequality, we derive that

$$\langle \tilde{x} - x^*, Jx^* - J\tilde{x}\rangle \geq 0. \tag{3.41}$$

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From the monotonicity of J, we reach

$$\langle x^* - \tilde{x}, Jx^* - J\tilde{x}\rangle \geq 0. \tag{3.42}$$

With regard to (3.41) and (3.42), we observe that $\langle x^* - \tilde{x}, Jx^* - J\tilde{x}\rangle = 0$. Since J is strictly monotone, we have $x^* = \tilde{x}$ and proof is complete. □

With the change of conditions on the operators T, S, T', and S' in Theorem 3.1, we provide the following theorem.

Theorem 3.4. *Let E be a 2-uniformly convex and smooth Banach space, let C be a nonempty closed convex subset of E. For any $i = 1, 2, 3, ..., m$, let f_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(C1) - (C4)$ and let $\{A_i\}$ be a finite family of continuous and monotone mappings from C to E^* and $\{\phi_i\}$ be a finite family of proper lower semicontinuous and convex functions from C to $\mathbb{R}$. Let $A : C \to E^*$ be an α-inverse strongly monotone operator. Let $T_j, S_k \subset E \times E^*$ be maximal monotone operators satisfying $D(T_j), D(S_k) \subset C$ and $J^{T_j}_{\rho_{j,n}} = (J + \rho_{j,n}T_j)^{-1}J$ for all $\rho_{j,n} > 0$ and $j = 1, 2, ..., l$, also $J^{S_k}_{\rho'_{k,n}} = (J + \rho'_{k,n}S_k)^{-1}J$ for all $\rho'_{k,n} > 0$ and $k = 1, 2, ..., l'$. Let $\{T'_n\}$ and $\{S'_n\}$ be two sequences of closed relative quasi-nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^{\infty} F(T'_i) \cap \bigcap_{i=1}^{\infty} F(S'_i) \cap \bigcap_{i=1}^{l} T_i^{-1}0 \cap \bigcap_{i=1}^{l'} S_i^{-1}0 \cap \bigcap_{i=1}^{m} \Omega(f_i, A_i, \phi_i) \cap VI(C, A) \neq \emptyset$ and $\|Ax\| \leq \|Ax - Av\|$ for all $x \in C$ and $v \in F$. Let $T' : C \to E$ such that $T' := J^{-1}(\sum_{j=0}^{\infty} \xi_j JT'_j)$ with $T'_0 = I$ and $\sum_{j=0}^{\infty} \xi_j = 1$. Let $S' : C \to E$ such that $S' = J^{-1}(\sum_{j=0}^{\infty} \zeta_j JS'_j)$ with $S'_0 = I$ and $\sum_{j=0}^{\infty} \zeta_j = 1$.*
Assume that $u, e_n \in E$ are given, $\{\alpha_n\}, \{\mu_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\} \subseteq [0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$ such that
i. $\alpha_n + \beta_n + \gamma_n + \eta_n = 1$, ii. $\sum_{n=1}^{\infty} \mu_n\alpha_n < \infty$, iii. $\sum_{n=1}^{\infty} \mu_n\beta_n < \infty$,
iv. $\liminf_{n\to\infty} \eta_n\mu_n(1 - \mu_n) > 0$, v. $\liminf_{n\to\infty} \mu_n\gamma_n(1 - \mu_n) > 0$,
vi. $\liminf_{n\to\infty} \rho_{j,n} > 0$ for each $j \in \{1, 2, ..., l\}$,
vii. $\liminf_{n\to\infty} \rho'_{j,n} > 0$ for each $j \in \{1, 2, ..., l'\}$,
viii. $\{r_{i,n}\} \subset [a, \infty)$ for some $a > 0$ and for any $i \in \{1, 2, ..., m\}$,
ix. $\{\lambda_n\} \subset [d, e]$ for some d, e with $0 < d < e < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E,
x. the sequence $\{e_n\}$ is bounded.
For arbitrary $u \in E$ generate a sequence $\{x_n\}$ by following method

$$\begin{aligned}
x_n &= K^{\Theta_m}_{r_m,n} \circ K^{\Theta_{m-1}}_{r_{m-1},n} \circ ...K^{\Theta_1}_{r_1,n} u_n,\\
z_n &= \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n),\\
u_{n+1} &= J^{T_l}_{\rho_l,n} \circ J^{T_{l-1}}_{\rho_{l-1},n} \circ ...J^{T_1}_{\rho_1,n} J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n\\
&\quad + \gamma_n JS' J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ...J^{S_1}_{\rho'_1,n} z_n + \eta_n JT' z_n) + (1 - \mu_n)Jx_n].
\end{aligned} \tag{3.43}$$

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If J is weakly sequentially continuous and $I-T'$ and $I-S'$ are demiclosed at 0, then the sequence $\{x_n\}$ converges weakly to a point $x^* \in F$.

Proof. Let $u^* \in F$. By (3.5) and similar to the proof of (3.6), we see

$$\begin{aligned}
\phi(u^*, x_{n+1}) &= \phi(u^*, K^{\Theta_m}_{r_m,n+1} \circ K^{\Theta_{m-1}}_{r_{m-1},n+1} \circ ...K^{\Theta_1}_{r_1,n+1} u_{n+1}) \leq \phi(u^*, u_{n+1}) \\
&= \phi(u^*, J^{T_l}_{\rho_l,n} \circ J^{T_{l-1}}_{\rho_{l-1},n} \circ ...J^{T_1}_{\rho_1,n} J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n \\
&+ \gamma_n JS' J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ...J^{S_1}_{\rho'_1,n} z_n + \eta_n JT' z_n) + (1-\mu_n) Jx_n]) \\
&\leq \phi(u^*, J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS' J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ...J^{S_1}_{\rho'_1,n} z_n \\
&+ \eta_n JT' z_n) + (1-\mu_n) Jx_n]) \\
&= V(u^*, \mu_n(\alpha_n Ju + \beta_n Je_n + \gamma_n JS' J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ...J^{S_1}_{\rho'_1,n} z_n \\
&+ \eta_n JT' z_n) + (1-\mu_n) Jx_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, S' J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ...J^{S_1}_{\rho'_1,n} z_n) \\
&+ \mu_n \eta_n \phi(u^*, T' z_n) + (1-\mu_n)\phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, J^{S_{l'}}_{\rho'_{l'},n} \circ J^{S_{l'-1}}_{\rho'_{l'-1},n} \circ ...J^{S_1}_{\rho'_1,n} z_n) \\
&+ \mu_n \eta_n \phi(u^*, z_n) + (1-\mu_n)\phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + (\mu_n\gamma_n + \mu_n\eta_n)\phi(u^*, z_n) + (1-\mu_n)\phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + (1 - \mu_n(1-\gamma_n-\eta_n))\phi(u^*, x_n) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n L + \phi(u^*, x_n),
\end{aligned}$$

where $\phi(u^*, e_n) \leq L$. By conditions (ii) and (iii) and Lemma 2.1, we deduce that there exists $\lim_{n\to\infty} \phi(u^*, x_n)$. Then $\{x_n\}$ and $\{z_n\}$ are bounded. Hence there exist $x^* \in C$ and subsequence $\{z_{n_k}\}$ such that $z_{n_k} \rightharpoonup x^*$ as $n_k \to \infty$. Set

$$\begin{aligned}
\Delta^j_n &= J^{T_j}_{\rho_j,n} \circ J^{T_{j-1}}_{\rho_{j-1},n} \circ ...J^{T_1}_{\rho_1,n}, \ \forall j \in \{1,2,...,l\}, \\
\Delta'^j_n &= J^{S_j}_{\rho'_j,n} \circ J^{S_{j-1}}_{\rho'_{j-1},n} \circ ...J^{S_1}_{\rho'_1,n}, \ \forall j \in \{1,2,...,l'\}.
\end{aligned}$$

From Lemma 2.3, we have

$$\begin{aligned}
\phi(u^*, x_{n+1}) &\leq \phi(u^*, u_{n+1}) \\
&= \phi(u^*, \Delta^l_n J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n \\
&+ \gamma_n JS' \Delta'^{l'}_n z_n + \eta_n JT' z_n) + (1-\mu_n) Jx_n]) \\
&\leq \phi(u^*, J^{-1}[\mu_n(\alpha_n Ju + \beta_n Je_n \\
&+ \gamma_n JS' \Delta'^{l'}_n z_n + \eta_n JT' z_n) + (1-\mu_n) Jx_n]) \\
&\leq \mu_n \alpha_n \phi(u^*, u) + \mu_n \beta_n \phi(u^*, e_n) + \mu_n \gamma_n \phi(u^*, S' \Delta'^{l'}_n z_n) \\
&+ \mu_n \eta_n \phi(u^*, T' z_n) + (1-\mu_n)\phi(u^*, x_n) - \mu_n\gamma_n(1-\mu_n)\overline{g}(\|JS'\Delta'^{l'}_n z_n - Jx_n\|).
\end{aligned}$$

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This gives

$$\mu_n\gamma_n(1-\mu_n)\overline{g}(\|JS'\Delta_n^{'l'}z_n - Jx_n\|) \leq \mu_n\alpha_n\phi(u^*,u) + \mu_n\beta_n\phi(u^*,e_n)$$
$$+ \mu_n\gamma_n\phi(u^*,S'\Delta_n^{'l'}z_n) + \mu_n\eta_n\phi(u^*,T'z_n) + (1-\mu_n)\phi(u^*,x_n) - \phi(u^*,x_{n+1})$$
$$\leq \mu_n\alpha_n\phi(u^*,u) + \mu_n\beta_n L + \phi(u^*,x_n) - \phi(u^*,x_{n+1}).$$

From (ii) and (iii) and since $\{\phi(u^*,x_n)\}$ is convergent, it follows that

$$\lim_{n\to\infty}\mu_n\gamma_n(1-\mu_n)\overline{g}(\|JS'\Delta_n^{'l'}z_n - Jx_n\|) = 0.$$

By condition (v) and property of $\overline{g}$, we see

$$\lim_{n\to\infty}\|JS'\Delta_n^{'l'}z_n - Jx_n\| = 0.$$

Since J^{-1} is uniformly norm to norm continuous on bounded subsets of E, we derive that

$$\lim_{n\to\infty}\|S'\Delta_n^{'l'}z_n - x_n\| = 0. \tag{3.44}$$

From the proof of Theorem 3.1, we know

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.45}$$

Therefore

$$\lim_{n\to\infty}\|S'\Delta_n^{'l'}z_n - z_n\| = 0,$$

follows from (3.44) and (3.45). Also by similar path of the proof of (3.13), we have

$$\lim_{n\to\infty}\|\Delta_n^{'l'}z_n - z_n\| = 0. \tag{3.46}$$

Hence $\lim_{n\to\infty}\|S'\Delta_n^{'l'}z_n - \Delta_n^{'l'}z_n\| = 0$. Then $S'\Delta_{n_k}^{'l'}z_{n_k} \rightharpoonup x^*$ as $n_k\to\infty$. By (3.46) and since $I-S'$ is demiclosed at 0, we conclude that $x^*\in S'x^*$, i.e., $x^*\in F(S') = \bigcap_{i=1}^{\infty}F(S_i')$. In a similar path, we have

$$\lim_{n\to\infty}\|T'z_n - z_n\| = 0,$$

so from demiclosedness of $I-T'$, we obtain $x^*\in F(T') = \bigcap_{i=1}^{\infty}F(T_i')$. The rest of the proof is similar to the proof of Theorem 3.1, so we ignore from the presentation of it. □

Remark 3.5. If in Theorem 3.4, $\alpha_n = \beta_n = 0\ \forall n\geq 1$, similar to proof of Remark 3.3, we can observe the generated sequence $\{x_n\}$ of method (3.43) converges weakly to $x^* = \lim_{n\to}\Pi_F x_n$.

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