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Best Coapproximation in Quotient Spaces

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Abstract. A kind of approximation, called best coapproximation was introduced and discussed in normed linear spaces by C. Franchetti and M. Furi in 1972. Subsequently, this study was taken up by several researchers in different abstract spaces. In this paper, we prove some results on the existence and uniqueness of best coapproximation in quotient spaces when the underlying spaces are metric linear spaces. We also show how coproximinality is transmitted to and from quotient spaces.

Keywords: Best coapproximation, Coproximinal set, co-Chebyshev set, Boundedly compact set, Pseudo co-Chebyshev set.

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1. Introduction and Preliminaries

As a counter part to best approximation, a kind of approximation called best coapproximation was introduced in normed linear spaces by C. Franchetti and M. Furi [3] to study some characteristic properties of real Hilbert spaces. Subsequently, this theory has been developed to a large extent in normed linear spaces and in Hilbert spaces by C. Franchetti and M. Furi, H. Mazaheri, P.L. Papini and I. Singer, Geetha S. Rao and by many others (see e.g. [3, 5, 6, 12, 13] and references cited therein). In a series of papers, G. Albinus, G.G. Lorentz, T.D. Narang, G. Pantelidis, K. Schnatz, A.I. Vasilev and others (see e.g. [1, 4, 7, 11, 14, 16, 19] and references cited therein) have tried to extend various

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results on best approximation available in normed linear spaces to metric linear spaces. The situation in case of best coapproximation is somewhat different. Whereas some attempts have been made to discuss best coapproximation in metric linear spaces (see e.g. [9, 10]) but still in these spaces this theory is less developed as compared to the theory of best approximation. The present paper is also a step in this direction. The paper mainly deals with some results on the existence and uniqueness of best coapproximation in quotient spaces when the underlying spaces are metric linear spaces. We also show how coproximinality is transmitted to and from quotient spaces. The results proved in the paper extend and generalize various known results on the subject. To start with, we recall a few definitions.

Let M be a non-empty subset of a metric space (X, d). An element $m_0 \in M$ is said to be a **best approximation (best coapproximation)** to $x \in X$ if for every $m \in M$,

$$d(x, m_0) \leq d(x, m) \ (respectively,\ d(m_0, m) \leq d(x, m)).$$

The set of all such m_0 is denoted by $P_M(x)(R_M(x))$, i.e., $P_M(x) = \{m_0 \in M : d(x, m_0) \leq d(x, m) \ for\ every\ m \in M\}(R_M(x) = \{m_0 \in M : d(m_0, m) \leq d(x, m) \ for\ every\ m \in M\})$.

The set M is called **proximinal (coproximinal)**, if $P_M(x)(R_M(x))$ contains at least one element for every $x \in X$. If for each $x \in X$, $P_M(x)(R_M(x))$ has exactly one element then M is called a **Chebyshev (co-Chebyshev)** set in X. If for each $x \in X$, $P_M(x)(R_M(x))$ has atmost one element then M is called a **semi-Chebyshev (semi co-Chebyshev)** set in X

For a proximinal (coproximinal) subset M of X, the mapping $P_M(R_M) : X \to 2^M (\equiv$ the collection of all subsets of $M)$ defined by $P_M(x) = \{m_0 \in M : d(x, m_0) \leq d(x, m) \ for\ every\ m \in M\}$ $(R_M(x) = \{m_0 \in M : d(m_0, m) \leq d(x, m) \ for\ every\ m \in M\})$ is called **metric projection (metric coprojection)**.

A linear space X together with a translation invariant metric d (i.e., $d(x + z, y + z) = d(x, y) \ for\ all\ x, y, z \in X)$ such that addition and scalar multiplication are continuous in (X, d) is called a **metric linear space** (see [14], p.1).

Every normed linear space is a metric linear space but a metric linear space need not be normable (see [15], p.32).

Remarks.

(1) A proximinal subset of a metric space need not be coproximinal:
Let $X = \mathbb{R}^2$ and $M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then M is a compact subset of $\mathbb{R}^2$ and hence proximinal. However, M is not coproximinal as $(0, 0) \in \mathbb{R}^2$ does not have any best coapproximation in M.

(2) A coproximinal subset of a metric space need not be proximinal:
Let $X = \mathbb{R} - \{1\}$ and $M = (1, 2]$, then M is a coproximinal subset of X but is not proximinal.
(3) A Chebyshev subset of a metric space need not be co-Chebyshev:
Let $X = \mathbb{R}$ and $M = [1, 2]$, then M is Chebyshev but not co-Chebyshev.
(4) A co-Chebyshev subset of a metric space need not be Chebyshev:
Let $X = R^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and $M = \{(x, y) \in R^2 : x = y\}$. Then M is a proximinal subset of X. We have $P_M(x, y) = \{\alpha(x, x) + (1-\alpha)(y, y) : 0 \leq \alpha \leq 1\}$, i.e., M is not Chebyshev, but $R_M(x, y) = \{(\frac{x+y}{2}, \frac{x+y}{2})\}$, i.e., M is co-Chebyshev.
(5) If M is a non-empty subset of a metric space (X, d) then the set $P_M^{-1}(m_0)(R_M^{-1}(m_0))$ is a closed set for every $m_0 \in M$, where $P_M^{-1}(m_0) = \{x \in X : m_0 \in P_M(x)\}$ and $R_M^{-1}(m_0) = \{x \in X : m_0 \in R_M(x)\}$.
(6) If M is a subspace of a metric linear space (X, d) then $P_M^{-1}(0) \bigcap M = \{0\}$ and $R_M^{-1}(0) \bigcap M = \{0\}$.
(7) If M is a subspace of a metric linear space (X, d), then $m_0 \in P_M(x)$ ($m_0 \in R_M(x)$) if and only if $x - m_0 \in P_M^{-1}(0)$ ($x - m_0 \in R_M^{-1}(0)$) and $P_M(x+m) = P_M(x)+m$ ($R_M(x+m) = R_M(x)+m$) for every $m \in M$.
(8) If M is subspace of a metric linear space (X, d), then $d(m, 0) = d(m, R_M^{-1}(0))$ for every $m \in M$.

For a closed subspace M of a metric linear space (X, d), the quotient space

$$X/M = \{x + M : x \in X\}$$

with linear operations
(i) $(x + M) + (y + M) = (x + y) + M$ for every $x, y \in X$
(ii) $\lambda(x + M) = \lambda x + M \ for\ every\ x \in X\ and\ for\ every\ scalar\ \lambda$
is a metric linear space endowed with the translation invariant metric

$$d(x + M, y + M) = \inf_{m \in M} d(x - y, m).$$

Since M is a subspace, we have $d(x + M, y + M) = \inf_{m\in M} d(x - y, m) = \inf_{m\in M} d(-m, y - x) = \inf_{m'\in M} d(y - x, m') = d(y + M, x + M)$.

For a closed subspace M of a metric linear space (X, d), the **canonical mapping** $\pi : X \to X/M$ is defined by $\pi(x) = x + M$. This canonical mapping π is linear, continuous and open (see [15], p.29).

Let X and Y be metric spaces, then a mapping $u : X \to Y$ is called **upper semi-continuous** if the set $H = \{x \in X : u(x) \bigcap F \neq \phi\}$ is closed for each closed subset F of Y.

2. Coproximinal and Co-Chebyshev Subspaces and Quotient Spaces

In this section, we prove some results concerning existence and uniqueness of elements of best coapproximation in quotient spaces and show how coproximinality, semi co-Chebyshevity, co-Chebyshevity, pseudo co-Chebyshevity and quasi co-Chebyshevity are transmitted to and from quotient spaces. For best approximation, some of these results were proved in normed linear spaces by Cheney and Wulbert [2] (see also [18], p.23, p.36).

Theorem 2.1. *If M is a closed subspace of a metric linear space (X, d) and W a coproximinal subspace of X containing M, then W/M is coproximinal in X/M.*

Proof. Let $x+M \in X/M,\ x \in X$, and w be a best coapproximation to x in W. We show that $w + M$ is a best coapproximation to $x + M$ in W/M. Suppose it is not, then there exist $w' + M \in W/M$ such that $d(w + M, w' + M) > d(x+M, w'+M)$, *i.e.*, $\inf_{m\in M} d(x-w', m) < d(w-w', M)$. Then there exist some $m_0 \in M$ such that

$$d(x - w', m_0) < d(w - w', M) \leq d(w - w', m_0)$$

i.e., $d(x, w' + m_0) < d(w, w' + m_0)$. Thus w is not a best coapproximation to x from W, a contradiction. Hence $w + M$ is a best coapproximation to $x + M$ in W/M and consequently, W/M is coproximinal in X/M. □

Concerning the semi co-Chebyshevity of W in X, we have

Theorem 2.2. *Let M be a co-Chebyshev subspace of a metric linear space (X, d) and W a closed subspace of X containing M. If W/M is semi co-Chebyshev in X/M then W is semi co-Chebyshev in X.*

Proof. Let $x \in X$ be such that $y_1, y_2 \in R_W(x)$. Since $y_1, y_2 \in R_W(x)$, we have $y_1 + M, y_2 + M \in R_{W/M}(x + M)$. But W/M is semi co-Chebyshev in X/M, we have $y_1 + M = y_2 + M$, *i.e.*, $y_1 - y_2 \in M$. Now $y_1, y_2 \in R_W(x)$ implies that $x - y_1, x - y_2 \in R_W^{-1}(0)$. Since $x - y_1,\ x - y_2 \in R_W^{-1}(0)$, we have $x-y_1,\ x-y_2 \in R_M^{-1}(0)$ as $M \subseteq W$ and M is a subspace. Since $0 \in R_M(x - y_1)$, we have $d(0, m) \leq d(x-y_1, m)$ for every $m \in M$, i.e., $d(y_1-y_2, m+y_1-y_2) \leq d(x-y_1+y_1-y_2, m+y_1-y_2)$ for every $m \in M$, i.e., $d(y_1-y_2, m') \leq d(x-y_2, m')$ for every $m' \in M$. This implies that $y_1 - y_2 \in R_M(x - y_2)$. Moreover, $0 \in R_M(x - y_2)$ and M is co-Chebyshev in X, we have $y_1 - y_2 = 0$, i.e., $y_1 = y_2$. Hence W is semi co-Chebyshev in X. □

Before proving the next theorem, we prove the following lemma:

Lemma 2.3. *If M is a coproximinal subspace of a metric linear space (X, d) and $R_M^{-1}(0)$ is a convex set then M is co-Chebyshev in X.*

Proof. Suppose that for some $x \in X$, there exist $m_1, m_2 \in R_M(x)$. Since $m_1, m_2 \in R_M(x)$, we have $x - m_1, x - m_2 \in R_M^{-1}(0)$. We first claim that $m_1 - x \in R_M^{-1}(0)$. Since $0 \in R_M(x - m_1)$, we have $d(0, m) \leq d(x - m_1, m)$ for every $m \in M$. This implies $d(-m, 0) \leq d(-m, m_1 - x)$ for every $m \in M$, i.e., $d(0, m') \leq d(m_1 - x, m')$ for every $m' \in M$. Therefore, $m_1 - x \in R_M^{-1}(0)$.

This proves our claim. Now, $x - m_2, m_1 - x \in R_M^{-1}(0)$ and $R_M^{-1}(0)$ is convex, we have $\frac{1}{2}[(x - m_2) + (m_1 - x)] \in R_M^{-1}(0)$, i.e., $\frac{1}{2}[m_1 - m_2] \in R_M^{-1}(0)$. Also $\frac{1}{2}[m_1 - m_2] \in M$ and so $\frac{1}{2}[m_1 - m_2] \in R_M^{-1}(0) \bigcap M = \{0\}$. This implies $m_1 = m_2$. Hence M is co-Chebyshev. $\square$

Remark 2.4. If we take M to be only a subset of metric linear space (X, d), then the convexity of $R_M^{-1}(0)$ need not imply the co-Chebyshevity of M.

Example 2.5. Let $X = \mathbb{R}$ and $M = [0, \infty)$, then $R_M^{-1}(0) = (-\infty, 0]$ and $R_M(-1) = [0, 1]$ i.e. $R_M^{-1}(0)$ is a convex set but M is not co-Chebyshev.

A subset M of a metric space (X, d) is said to be **boundedly compact**, if every bounded sequence in M has a convergent subsequence in M. It is well-known (see [17], p.383) that a boundedly compact subset of a metric space is proximinal and hence closed.

For boundedly compact subsets of a metric space, the following result was proved in [8]:

Lemma 2.6. *Let M be a coproximinal and boundedly compact subset of a metric space (X, d). Then,*
(i)R_M is upper semi-continuous.
(ii)$R_M(x)$ is compact for each $x \in X$.

Using the above lemma, we prove the following:

Theorem 2.7. *Let M be a closed linear subspsce of a metric linear space (X, d) and W a coproximinal subspace of X containig M. Then the following are true:*
(i)$\pi(R_W^{-1}(0)) \subseteq R_{W/M}^{-1}(M)$.
(ii)If W is boundedly compact and M is proximinal, then $R_{W/M}$ is upper semi-continuous.

Proof. (i) Let $x \in R_W^{-1}(0)$ and $g \in W$. Then for each $h \in M$,

$$d(g + M, M) \leq d(g + h, 0) \leq d(x, g + h)$$

i.e., $d(g+M, M) \leq \inf_{h \in M} d(x, g+h) = d(x+M, g+M)$ for every $g+M \in W/M$. Hence $M \in R_{W/M}(x + M)$ and so $x + M \in R_{W/M}^{-1}(M)$, i.e., $\pi(x) \in R_{W/M}^{-1}(M)$. Therefore, $\pi(R_W^{-1}(0)) \subseteq R_{W/M}^{-1}(M)$.
(ii) Let $\{g_n + M\}$ be a bounded sequence in W/M, i.e., $\sup_{n \in \mathbb{N}} d(g_n + M, 0) < \infty$. Since M is proximinal, there exist a sequence $\{m_n\}$ in M such that $\{g_n + m_n\}$ is a bounded sequence in W. Since W is boundedly compact, $\{g_n + m_n\}$ has a subsequence $\{g_{n_k} + m_{n_k}\} \to w_0 \in W$. Consider

$$d(g_{n_k} + M, w_0 + M) = d(g_{n_k} - w_0, M) \leq d(g_{n_k} + m_{n_k}, w_0) \to 0 \ as\ k \to \infty.$$

Hence $\{g_{n_k} + M\} \to w_0 + M$. Therefore, W/M is boundedly compact. Since W/M is coproximinal, it follows from Lemma 2.6 that $R_{W/M}$ is upper semi-continuous. $\square$

Remark 2.8. Results analogous to Theorems 2.1, 2.2 and 2.7 can be found in [6] for normed linear spaces.

Concerning the compactness of $R_{F/G}(x+G)$, we have

Theorem 2.9. *Let M be a proximinal subspace of a metric linear space (X, d) and W a coproximinal subspace of X containing M such that $R_W(x)$ is boundedly compact for every $x \in X$. Then $R_{W/M}(x+M)$ is compact for every $x+M \in X/M$.*

Proof. Since W is coproximinal in X, W/M is coproximinal in X/M by Theorem 2.1. Let $x+M \in X/M$ and $\{g_n+M\}$ be a sequence in $R_{W/M}(x+M)$. Since $g_n+M \in R_{W/M}(x+M)$, we have

$$d(g_n+M, g+M) \leq d(x+M, g+M) \ for \ every \ g \in W.$$

Since M is proximinal in X, there exist $m_n \in M$ such that $d(g_n-g, m_n) = d(g_n-g, M) \leq d(x, g)$ for every $g \in W$, i.e., $d(g_n-m_n, g) \leq d(x, g)$ for every $g \in G$ and so $g_n - m_n \in R_W(x)$. Since $R_W(x)$ is boundedly compact and $R_W(x)$ is a bounded set, $\{g_n - m_n\}$ has a subsequence $\{g_{n_k} - m_{n_k}\} \to w_0 \in R_W(x)$, as $R_W(x)$ is closed. Consider

$$d(g_{n_k}+M, w_0+M) = \inf_{m \in M} d(g_{n_k}-w_0, m) \leq d(g_{n_k}-m_{n_k}, w_0) \to 0$$

implies that $\{g_{n_k}+M\} \to w_0+M \in R_{W/M}(x+M)$, as $w_0 \in R_W(x)$. Hence $R_{W/M}(x+M)$ is compact. $\square$

A closed subspace M of a metric linear space (X, d) is said to be **quasi co-Chebyshev** if $R_M(x)$ is non-empty and compact for every $x \in X$. Therefore, we have

Corollary 2.10. *Let M be a proximinal subspace of a metric linear space (X, d) and W a coproximinal subspace of X containing M. If W is quasi co-Chebyshev in X then W/M is quasi co-Chebyshev in X/M.*

Remark 2.11. Taking $R_W(x)$ to be compact, Theorem 2.9 was proved for normed linear spaces in [6].

If we take G to be boundedly compact then we have the following result:

Theorem 2.12. *Let M be a boundedly compact subspace of a metric linear space (X, d) and W/M a coproximinal subspace of X/M where $W \supseteq M$. If $R_{W/M}(x+M)$ is boundedly compact for every $x+M \in X/M$, then $R_W(x)$ is compact for every $x \in X$.*

Proof. Let $\{w_n\}$ be a sequence in $R_W(x)$. Since $R_W(x)$ is bounded, $\{w_n\}$ is a bounded sequence. Then $\{w_n+M\}$ is a bounded sequence in $R_{W/M}(x+M)$. Since $R_{W/M}(x+M)$ is closed and boundedly compact subset of W/M, $\{w_n+M\}$ has a subsequence $\{w_{n_i}+M\} \to w_0+M \in R_{W/M}(x+M)$. Then there exist a sequence $\{m_{n_i}\}$ *in* M such that $\{w_{n_i} + m_{n_i}\} \to w_0$. We claim that $\{m_{n_i}\}$ is a bounded sequence in M. Consider $d(m_{n_i}, 0) = d(m_{n_i} + w_{n_i}, w_{n_i}) \leq d(m_{n_i} + w_{n_i}, 0) + d(0, w_{n_i})$. This implies $\sup_{i\in N} d(m_{n_i}, 0) \leq \sup_{i\in N} d(m_{n_i} + w_{n_i}, 0) + \sup_{i\in N} d(0, w_{n_i}) < \infty$, as $\{w_{n_i} + m_{n_i}\}$ is a convergent sequence and $\{w_{n_i}\}$ is a bounded sequence. This proves our claim.

Therefore, $\{m_{n_i}\}$ is a bounded sequence in M. Since M is boundedly compact, $\{m_{n_i}\}$ has a subsequence $m_{n_{i,l}} \to m_0$. Also $\{w_{n_{i,l}} + m_{n_{i,l}}\} \to w_0$ implies that $\{w_{n_{i,l}}\} \to w_0 - m_0$. Since $d(w_{n_{i,l}}, w) \leq d(x, w)$ for every $w \in W$, we have $d(w_0 - m_0, w) \leq d(x, w)$ for every $w \in W$, i.e., $w_0 - m_0 \in R_W(x)$ and hence $R_W(x)$ is compact. $\square$

Corollary 2.13. *Let M be a boundedly compact subspace of a metric linear space (X, d) and W/M a coproximinal subspace of X/M where $W \supseteq M$. If W/M is quasi co-Chebyshev then so is W.*

Let A be a convex subset of a metric linear space (X, d) and $l(A)$ the linear manifold spanned by A i.e.

$$l(A) = \{\alpha y + \beta z : y, z \in A,\ \alpha + \beta = 1\}.$$

For any fixed $y \in Y$, the set $l(A) - y = \{x - y : x \in l(A)\}$ is then a linear subspace of X satisfying $l(A) - y = l(A - y)$. The dimension of A is defined as:

$$dim\ A = \begin{cases} dim\ l(A) & \text{if } A \neq \phi \\ -1 & \text{if } A = \phi \end{cases}$$

For every $y \in A$, $dim\ A = dim\ l(A) = dim\ [l(A) - y] = dim\ [l(A - y)] = dim\ (A - y)$.

For a subset A of a metric linear space (X, d) we have, $l(\pi(A)) = \pi(l(A))$.

A closed subspace W of a metric linear space (X, d) is called **pseudo co-Chebyshev** if $R_W(x)$ is non-empty and finite dimensional for every $x \in X$.

Concerning the pseudo co-Chebyshevity, we have

Theorem 2.14. *Let M be a finite dimensional subspace of a metric linear space (X, d) and W a subspace of X containing M. If W is pseudo co-Chebyshev in X then W/M is pseudo co-Chebyshev in X/M.*

Proof. Let $x \in X$. Since W is pseudo co-Chebyshev in X, $R_W(x)$ is non-empty and finite dimensional in X. In view of Theorem 2.1, we have W/M is coproximinal in X/M. Thus, we have

$$dim\ [R_{W/M}(x + M)] = dim\ l[R_{W/M}(x + M)] = dim\ [l(\pi(R_W(x))].$$

But $dim\ l[\pi(R_W(x))] = dim\ \pi[l(R_W(x))] = dim\ [l(R_W(x))]/M$ implies that $dim\ [\pi(l(R_W(x))] < dim\ l[R_W(x)] < \infty$ and so $dim\ [R_{W/M}(x + M)] < \infty$. Hence W/M is pseudo co-Chebyshev in X/M. $\square$

Remark 2.15. For normed linear spaces, analogous result was proved in [5].

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References

1. G. Albinus: *Approximation in Metric Linear Spaces*, Approximation Theory, Banach Center Publications, Vol. 4, PWN- Polish Scientific Publishers, Warsaw, 1979.
2. E.W. Cheney, D.E. Wulbert, The Existence and Uniqueness of Best Approximation, *Math. Scand.*, **24**, (1969), 113-140.
3. C. Franchetti, M. Furi: Some Characteristic Properties of Real Hilbert Spaces, *Rev. Roumaine Math. Pures Appl.*, **17**(1972), 1045-1048.
4. G. G. Lorentz: Operations in Linear Metric Spaces, *Duke Math. J.*, **15**, (1948), 755-761.
5. H. Mazheri, S.M.S. Modarres: Some Results Concerning Proximinality and co-Proximinality, *Nonlinear Anal.*, **62**, (2005), 1123-1126.
6. H. Mazheri: Best Coapproximation in Quotient Spaces, *Nonlinear Anal.*, **68**, (2008), 3122-3126.
7. T.D. Narang: On Some Approximation Problems in Metric Linear Spaces, *Indian J. Pure Appl. Math.*, **14**, (1983), 253-256.
8. T.D. Narang: Best Coapproximation in Metric Spaces, *Publications de l'Institut Mathematique*, **51**, (1992), 71-76.
9. T.D. Narang, S.P. Singh, Best Coapproximation in Metric Linear Spaces, *Tamkang J. Math.*, **30**, (1999), 241-252.
10. T.D. Narang, S. Gupta, On Best Approximation and Best Coapproximation, *Thai J. Math.*, **14**, (2016), 505-516.
11. G. Pantelidis, Approximationstheorie für Metrische Lineare Räme, *Math. Ann.*, **184**,(1969), 30-48.
12. P.L. Papini, I. Singer, Best Coapproximation in Normed Linear Spaces, *Mh. Math.*, **88**, (1979), 27-44.
13. G.S. Rao, R. Sarvanan, Some Results Concerning Uniform Best Coapproximation, *J. Inequal. Pure Appl. Math.*, **3**, (2002),1-13.
14. S. Rolewicz, *Metric Linear Spaces*, Monografie Matematyczne 56 (Warszawa 1972)
15. W. Rudin, *Functional Analysis*, McGraw-Hill, Inc., 1973.
16. K. Schnatz, Nonlinear Duality and Best Approximation Metric Linear Spaces *J. Approx. Theory*, **49**, (1987), 201-218.
17. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, New York, 1970.
18. I. Singer, *The Theory of Best Approximation and Functional Analysis*, SIAM Philadelphia, Pennslvania, 1974.
19. A. I. Vasilev: Bounded Compactness of Sets in Linear Metric Spaces, *Math. Notes*, **11**, (1972), 396-401.