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A Semidefinite Optimization Approach to Quadratic Fractional Optimization with a Strictly Convex Quadratic Constraint

Maziar Salahi*, Saeed Fallahi

Department of Applied Mathematics, Faculty of Mathematical Sciences
University of Guilan, Rasht, Iran.

E-mail: salahim@guilan.ac.ir
E-mail: saeedf808@gmail.com

Abstract. In this paper we consider a fractional optimization problem that minimizes the ratio of two quadratic functions subject to a strictly convex quadratic constraint. First using the extension of Charnes-Cooper transformation, an equivalent homogenized quadratic reformulation of the problem is given. Then we show that under certain assumptions, it can be solved to global optimality using semidefinite optimization relaxation in polynomial time.

Keywords: Quadratic fractional optimization, Semidefinite optimization relaxation, Global optimization.

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1. Introduction

We often come across the fractional optimization problems (FOPs), which arise in various disciplines such as certain portfolio selection problems [5, 6, 7], stochastic decision making problems [12] and problems in economics [11]. For example, Total Least Squares (TLS), which is an extension of the usual Least Squares method, used in a variety of disciplines such as signal processing,

*Corresponding Author

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statistics, physics, economic, biology and medicine, requires that the following fractional problem to be solved [1, 9]:

$$\min_{x\in\mathbb{R}^n} \quad \frac{\|Ax-b\|^2}{1+\|x\|^2}$$

In order to have a meaningful solution, often a constrained or regularized version of it has been solved [10]. In the constrained version, the following problem is considered to be solved.

$$\min_{x\in\mathbb{R}^n} \quad \frac{\|Ax-b\|^2}{1+\|x\|^2}$$
$$\|Lx\|^2 \leq \beta,$$

where $L \in \mathbb{R}^{k\times n}, k \leqslant n$ is a full row rank matrix and β is a positive number. The main difficulty with FOPs is the nonconvexity. Dinkelbach studied the nonlinear model of fractional problems and showed an interesting and useful relationship between fractional and parametric optimization problems [4]. His idea has been applied by several authors, for example Beck et al. [1] transformed the regularized TLS to a parametric program and proposed an iterative algorithm to find the global optimal solution of the regularized TLS. Most recently, Beck et al. have applied semidefinite optimization (SDO) relaxation to solve the following problem:

$$\min \quad \frac{x^TA_1x+b_1^Tx+c_1}{x^TA_2x+b_2^Tx+c_2}$$
$$\|Lx\|^2 \leq \rho, \tag{1.1}$$

where L is the same as in the TLS and ρ is a positive number. They have shown that under certain conditions, the global optimal solution is achievable [2].

In this paper, we consider the following problem

$$\min \quad \frac{x^TA_1x+b_1^Tx+c_1}{x^TA_2x+b_2^Tx+c_2}$$
$$x^TA_3x+b_3^Tx+c_3 \leq 0, \qquad (QCQFO)$$

where $A_i^T = A_i \in R^{n\times n}, b_i \in R^n, c_i \in R, i = 1,\dots,n,$ $x^TA_2x+b_2^Tx+c_2 > 0$ in the feasible region, A_3 is assumed to be positive definite and $c_3 < 0$. Using the well-known Charnes - Cooper transformation, we show that (QCQFO) has an inherent hidden homogeneity and semidefinite relaxation technique can be applied to find the global optimal solution in polynomial time. The proofs are constructive and are not appeared in [2] and the conditions under which the global solution is obtained are different than those in [2]. First in Section 2 we give the homogenized version of (QCQFO). Then in Section 3, the semidefinite relaxation scheme is used to derive an optimal solution of the original problem.

2. Homogenization

Using the generalized Charnes-Cooper transformation

$$z = \frac{1}{\sqrt{x^T A_2 x + b_2^T x + c_2}},$$

and

$$y = \frac{x}{\sqrt{x^T A_2 x + b_2^T x + c_2}},$$

(QCQFO) transforms to the following equivalent minimization problem:

$$\begin{aligned} \min \quad & y^T A_1 y + b_1^T yz + c_1 z^2 \\ & y^T A_2 y + b_2^T yz + c_2 z^2 = 1, \\ & y^T A_3 y + b_3^T yz + c_3 z^2 \leq 0, \\ & z > 0. \end{aligned} \tag{2.1}$$

Obviously if (y, z) solves (2.1), then $(-y, -z)$ also solves it, thus $z > 0$ can be replaced by $z \neq 0$. The new problem is as follows:

$$\begin{aligned} \min \quad & y^T A_1 y + b_1^T yz + c_1 z^2 \\ & y^T A_2 y + b_2^T yz + c_2 z^2 = 1, \\ & y^T A_3 y + b_3^T yz + c_3 z^2 \leq 0, \\ & z \neq 0. \end{aligned} \tag{2.2}$$

Obviously, since in (QCQFO) A_3 is positive definite, then in any optimal solution (y^*, z^*) of (2.2), $z^* \neq 0$ and thus $x^* = \frac{y^*}{z^*}$ is an optimal solution for (QCQFO). Therefore we can omit $z^* \neq 0$ in (2.2).

3. SDO Relaxation

In this section we present a SDO relaxation approach to solve (2.2) globally. Problem (2.2) in the matrix form is given by

$$\begin{aligned} \min \quad & M_0 \bullet \widehat{X} \\ & M_1 \bullet \widehat{X} = 1, \\ & M_2 \bullet \widehat{X} \leq 0, \end{aligned} \tag{3.1}$$

where

$$A \bullet B = Tr(A^T B), \widehat{X} = \begin{bmatrix} z^2 & y^T z \\ yz & yy^T \end{bmatrix},$$

and

$$M_0 = \begin{bmatrix} c_1 & b_1^T/2 \\ b_1/2 & A_1 \end{bmatrix}, M_1 = \begin{bmatrix} c_2 & b_2^T/2 \\ b_2/2 & A_2 \end{bmatrix}, M_2 = \begin{bmatrix} c_3 & b_3^T/2 \\ b_3/2 & A_3 \end{bmatrix}.$$

The semidefinite relaxation of (3.1) is given by [3]:

$$\begin{aligned} \min \quad & M_0 \bullet X \\ & M_1 \bullet X = 1, \\ & M_2 \bullet X \leq 0, \\ & X \succeq 0_{(n+1)\times(n+1)}, \end{aligned} \tag{3.2}$$

where

$$X = \begin{bmatrix} X_{00} & x_0^T \\ x_0 & \overline{X} \end{bmatrix},$$

and its dual is given by

$$\begin{aligned} \max \quad & y_1 \\ & Z = M_0 - y_1 M_1 - y_2 M_2, \\ & Z \succeq 0_{(n+1)\times(n+1)}, \\ & y_2 \leq 0. \end{aligned} \tag{3.3}$$

In the next theorem we show that problems (3.2) and (3.3) satisfy slater regularity.

Theorem 3.1. *Both* (3.2) *and* (3.3) *satisfy the slater regularity conditions. Hence both problems attain their optimal values and the duality gap is zero.*

Proof. Let

$$X = \begin{bmatrix} k & 0_{1\times n} \\ 0_{n\times 1} & \lambda I_n \end{bmatrix},$$

then it is strictly feasible for (3.2) if $\lambda > 0$ and $k > 0$ exist such that:

$$\begin{aligned} kc_2 + \lambda Tr(A_2) = 1, \\ kc_3 + \lambda Tr(A_3) < 0. \end{aligned}$$

Let us consider $m > \left(\frac{-Tr(A_3)}{c_3}\right)$. Now by choosing $m^{'} > m > 0$ such that $m^{'}c_2 + Tr(A_2) > 0$, and letting $\lambda^{'} = \frac{1}{m^{'}c_2 + Tr(A_2)}$ and $k^{'} = m^{'}\lambda^{'}$ the inequalities hold. For the dual problem (3.3), first note that since $c_3 < 0$ then $c_2 > 0$ and M_2 is not positive definite. Now if M_1 is positive definite then we can choose $y_1 < 0$ and $y_2 < 0$ such that Z is positive definite, which results the strict feasibility of the dual problem (3.3). Otherwise if M_1 is indefinite, then since $x^T A_2 x + b_2^T x + c_2 > 0$ in the feasible region, thus the following system is not solvable

$$\begin{aligned} f(x) = x^T A_2 x + b_2^T x + c_2 \leq 0, \\ g(x) = x^T A_3 x + b_3^T x + c_3 \leq 0. \end{aligned} \tag{3.4}$$

Let us now introduce the following system:

$$\widetilde{f}(\tilde{x}) = \bar{x}^T A_2 \bar{x} + b_2^T \bar{x} x_1 + c_2 x_1^2 \leq 0,$$
$$\widetilde{g}(\tilde{x}) = \bar{x}^T A_3 \bar{x} + b_3^T \bar{x} x_1 + c_3 x_1^2 \leq 0, \tag{3.5}$$

where $\tilde{x} = (x_1, \bar{x})^T$. First we show that the new system has no solution. By contrary suppose that $z = (x_1, \bar{x}) \in \mathbb{R}^{n+1}$ solves (3.5). Thus if $x_1 \neq 0$, then we have

$$\widetilde{f}(z/x_1) = {f(\bar{x})}/{x_1^2} \leq 0,$$
$$\widetilde{g}(z/x_1) = {g(\bar{x})}/{x_1^2} \leq 0,$$

which is in contradiction with (3.4). For the case $x_1 = 0$ we have $\bar{x}^T A_3 \bar{x} \leq 0$ and $\bar{x}^T A_2 \bar{x} \leq 0$, which is in contradiction with the positive definiteness ofA_3. Therefore by the S-Lemma [8] there exist nonnegative multipliers λ_1', λ_2' such that

$$\lambda_1' \left(\bar{x}^T A_2 \bar{x} + b_2^T \bar{x} x_1 + c_2 x_1^2\right) + \lambda_2' \left(\bar{x}^T A_3 \bar{x} + b_3^T \bar{x} x_1 + c_3 x_1^2\right) > 0$$
$$\forall\, (x_1, \bar{x}) \in \mathbb{R}^{n+1} \setminus \{0\}\,.$$

Thus $B = \lambda_1' M_1 + \lambda_2' M_2 > 0$. Since neither M_1 nor M_2 are positive definite, then $\lambda_1', \lambda_2' \neq 0$. Now by choosing $\lambda_3 > 0$ such that $M_0 + \lambda_3 B \succ 0$ and letting $y_1 = -\lambda_1' \lambda_3$ and $y_2 = -\lambda_2' \lambda_3$, Z will be positive definite, which implies slater regularity of the dual problem. Thus both (3.2) and (3.3) are solvable and having equal objective values. □

In the next theorem it is shown that the global optimal solution of (2.1) can be derived from an optimal solution of (3.2).

Theorem 3.2. *SDO relaxation* (3.2) *gives a global optimal solution of* (2.1) *in a polynomial time.*

Proof. Suppose X^* is an optimal solution of rank r for (3.2) and (Z^*, y_1^*, y_2^*) is optimal solution for (3.3). If at the optimality $M_2 \bullet X^* < 0$, then obviously $y_2^* = 0$. Now Suppose

$$X^* = \sum_{i=1}^{r} x_i^* \left(x_i^*\right)^T,$$

be a rank one decomposition of X^* for which $\left(x_i^*\right)^T M_2 x_i^* \leq 0 \quad \forall i = 1, ..., r$ [12]. We also have

$$M_1 \bullet \left(\sum_{i=1}^{r} x_i^* \left(x_i^*\right)^T \right) = \sum_{i=1}^{r} \left(x_i^*\right)^T M_1 x_i^* = 1.$$

Thus at least for one k, $1 \leq k \leq r$, we have $\left(x_k^*\right)^T M_1 x_k^* > 0$. Now if $\left(x_k^*\right)^T M_1 x_k^* = 1$, then $x_k^* \left(x_k^*\right)^T$ is an optimal solution for (3.2), otherwise suppose $(x_k^*)^T M_1 x_k^* = \alpha > 0$. By letting $x^{**} = \left(\frac{x_k^*}{\sqrt{\alpha}}\right)$, and $X^{**} = x^{**}(x^{**})^T$ one can easily check that

$$M_1 \bullet X^{**} = 1,\ M_2 \bullet X^{**} \leq 0,\ Z^* \bullet X^{**} = 0,\ y_2^*(M_2 \bullet X^{**}) = 0.$$

Therefore X^{**} is an optimal solution for (3.2). Moreover since (3.2) is a relaxation of (2.1) and A_3 is positive definite, then one can easily construct an optimal solution for (2.1). However, if at the optimality $M_2 \bullet X^* = 0$, then it is sufficient to compute a rank one decomposition for X^* such that $(x_i^*)^T M_2 x_i^* = 0 \quad \forall i = 1, \ldots, r$ and follow as before. Since SDO is solvable in polynomial time using interior point methods, thus the global optimal solution of (2.1) is found in polynomial time. □

4. Conclusions

In this paper, first using the extended Charnes - Cooper transformation, (QCQFO) represented as a homogenized quadratic optimization with two quadratic constraints. Then it is proved that under certain assumptions, using SDO relaxation the global optimal solution of (QCQFO) can be found in polynomial time. The question whether under weaker conditions the proved goal is achievable is left for future research.

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