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Chromaticity of Turán Graphs with At Most Three Edges Deleted

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Abstract. Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. A graph G is chromatically unique if for any graph H, $P(H, \lambda) = P(G, \lambda)$ implies H is isomorphic to G. In this paper, we determine the chromaticity of all Turán graphs with at most three edges deleted. As a by product, we found many families of chromatically unique graphs and chromatic equivalence classes of graphs.

Keywords: Chromatic polynomial, Chromatic uniqueness, Turán graph.

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1. Introduction

All graphs considered in this paper are finite and simple. Graph polynomials are a well-developed area useful for analyzing properties of graphs. Chromatic polynomial, characteristic polynomial and domination polynomial are some examples of these polynomials (see [1, 2, 6]). For a graph G, we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (simply *χ-equivalent*), denoted $G \sim H$ if $P(G) = P(H)$. Let $[G]$ denote the equivalence class determined by the graph G. A graph G is said to be *chromatically unique* (simply *χ-unique*) if $[G] = \{G\}$. A family $\mathcal{G}$ of graphs is said to be *chromatically closed* (simply *χ-closed*) if for any graph $G \in \mathcal{G}$, $P(H) = P(G)$ implies that $H \in \mathcal{G}$. For two families of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, if $P(G_1) \neq P(G_2)$ for each $G_1 \in \mathcal{G}_1$ and each $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *chromatically disjoint* (simply χ-disjoint). Many families of χ-unique graphs are known (see [7, 8, 11, 12, 13]).

For a graph G, let $e(G)$, $v(G)$, $t(G)$, $Q(G)$, $K(G)$ and $\chi(G)$ respectively be the number of vertices, edges, triangles, induced 4-cycle subgraph, complete subgraph K_4 and chromatic number of G. By $\overline{G}$, we denote the complement of G. Let O_n be an edgeless graph with n vertices. Suppose S is a set of s edges of G. Denote by $G - S$ the graph obtained from G by deleting all edges in S and by $\langle S\rangle$ the graph induced by S. For $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \cdots \leq p_t$, let $F = K(p_1, p_2, \ldots, p_t)$ be a complete t-partite graph with partition sets V_i such that $|V_i| = p_i$ for $i = 1, 2, \ldots, t$. The Turán graph, denoted $T = K(t_1 \times p, t_2 \times (p+1))$, is the unique complete t-partite graph having $t_1 \geq 1$ partite sets of size p and t_2 partite sets of size $p+1$. In this paper, we determine the chromaticity of all Turán graphs with at most s (≤ 3) edges deleted for $p \geq s+2$. As a by product, we found many families of chromatically unique graphs and equivalent classes of graphs.

2. Preliminary results and notations

Let $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ be the family $\{K(p_1, p_2, \ldots, p_t) - S \,|\, S \subset E(G)$ and $|S| = s\}$. For $p_1 \geq s + 1$, we denote by $K^{-K(1,s)}_{i,j}(p_1, p_2, \ldots, p_t)$ the graph in

$\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ where the s edges in S induced a $K(1,s)$ with center in V_i and all the end-vertices in V_j, and by $K_{i,j}^{-sK_2}(p_1, p_2, \ldots, p_t)$ the graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ where the s edges in S induced a matching with one end-vertex in V_i and another end-vertex in V_j. For convenience, we let $\mathcal{T}^{-s} = \mathcal{K}^{-s}(t_1 \times p, t_2 \times (p+1))$, $T_{i,j}^{-K(1,s)} = K_{i,j}^{-K(1,s)}(t_1 \times p, t_2 \times (p+1))$ and $T_{i,j}^{-sK_2} = K_{i,j}^{-sK_2}(t_1 \times p, t_2 \times (p+1))$. Hence, each graph in $\mathcal{T}^{-s}$ has $t_1 \geq 1$ partite sets of size p and t_2 partite sets of size $p+1$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. If G is of order n, then $P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$ (see [14]). Therefore, $\alpha(G,k) = \alpha(H,k)$ for each $k = 1, 2, \ldots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G,x) = \sum_{k=1}^{p} \alpha(G,k)x^k$ is called the *σ-polynomial* of G (see [3]), and the polynomial $h(G,x) = \sum_{k=1}^{p} \alpha(\overline{G},k)$ x^k is called the *adjoint polynomial* of G (see [9]). If $h(G,x) = h(H,x)$ then we say G is adjointly equivalent to H, denoted $G \sim_h H$. A family $\mathcal{G}$ of graphs is said to be *adjoint closed* (simply *χ_h-closed*) if for any graph $G \in \mathcal{G}$, $h(H,x) = h(G,x)$ implies that $H \in \mathcal{G}$. Clearly, the conditions $P(G,\lambda) = P(H,\lambda)$, $\sigma(G,x) = \sigma(H,x)$ and $h(\overline{G},x) = h(\overline{H},x)$ are equivalent for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H, and mG the disjoint union of m copies of G; $G \vee H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy | x \in V(G)$ and $y \in V(H)\} \cup E(G) \cup E(H)$. Throughout this paper, all the t-partite graphs G under consideration are 2-connected with $\chi(G) = t$. For terms used but not defined here we refer to [16].

Lemma 2.1. (see [7]) *Let G and H be two graphs with $H \sim G$, then $v(G) = v(H)$, $e(G) = e(H)$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,k) =$*

$\alpha(H,k)$ *for each* $k = 1, 2, \ldots,$ *and*

$$-Q(G) + 2K(G) = -Q(H) + 2K(H).$$

Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.

Lemma 2.2. (Brenti [3]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \ldots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

The above lemma is equivalent to the following:

Remark. *Let G and H be two disjoint graphs. Then*

$$h(\overline{G} + \overline{H}, x) = h(\overline{G}, x)h(\overline{H}, x).$$

In particular,

$$h(\overline{K(n_1, n_2, \ldots, n_t)}, x) = \prod_{i=1}^{t} h(K_{n_i}, x).$$

For an edge $e = v_1v_2$ of a graph G, then $G * e$ is defined as follows: the vertex set of $G * e$ is $(V(G)\backslash\{v_1, v_2\}) \cup \{u\}$, and the edge set of $G * e$ is $\{e' \,|\, e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv \,|\, v \in N_G(v_1) \cap N_G(v_2)\}$. For example, let e_1 be an edge of C_4 and e_2 an edge of K_4. Then $C_4 * e_1 = K_1 + K_2$ and $K_4 * e_2 = K_3$.

Lemma 2.3. (Liu [10]) *Let G be a graph with $e \in E(G)$. Then*

$$h(G, x) = h(G - e, x) + h(G * e, x),$$

In particular, if $e = uv$ does not belong to any triangle of G, then

$$h(G, x) = h(G - e, x) + xh(G - \{u, v\}, x),$$

where $G - e$ (respectively $G - \{u, v\}$) denote the graph obtained by deleting the edge e (respectively the vertices u and v) from G.

Denote by $\beta(G)$ the minimum real root of $h(G)$.

Lemma 2.4. (Zhao [17]) *Let G be a connected graph such that G contains H as a proper subgraph. Then*

$$\beta(G) < \beta(H).$$

Lemma 2.5. (Teo and Koh [15]) *The graph $K(p,q)$ is χ-unique for all $q \geq p \geq 2$.*

Suppose $F = K(p_1, p_2, \ldots, p_t)$ and $G = F - S$ for a set S of s edges of G. A non-empty independent set of G is *improper* if it is not a subset of a partite set of G. Otherwise, it is proper. A k-independent partition in G is improper if it contains at least one improper independent set. Define $\alpha_k(G) = \alpha(G,k) - \alpha(F,k)$ for $k \geq t+1$. Hence, $\alpha_k(G)$ is the number of k-independent partitions in G that contains at least one improper independent set.

Lemma 2.6. (Zhao [17]) *Let $F = K(p_1, p_2, \ldots, p_t)$ and $G = F - S$. If $p_1 \geq s+1$, then*

$$s \leq \alpha_{t+1}(G) = \alpha(G, t+1) - \alpha(F, t+1) \leq 2^s - 1,$$

$\alpha_{t+1}(G) = s$ if and only if the subgraph induced by any $r \geq 2$ edges in S is not a complete multipartite graph, and $\alpha_{t+1}(G) = 2^s - 1$ if and only if all s edges in S share a common end-vertex and the other end-vertices belong to the same V_i for some i.

Lemma 2.7. (Dong et al. [5]) *Let p_1, p_2 and s be positive integers with $3 \leq p_1 \leq p_2$, then*

(i) *$K_{1,2}^{-K(1,s)}(p_1, p_2)$ is χ-unique for $1 \leq s \leq p_2 - 2$,*
(ii) *$K_{2,1}^{-K(1,s)}(p_1, p_2)$ is χ-unique for $1 \leq s \leq p_1 - 2$, and*
(iii) *$K^{-sK_2}(p_1, p_2)$ is χ-unique for $1 \leq s \leq p_2 - 1$.*

Lemma 2.8. (Zhao [17]) *Let $s \geq 1$, $p \geq 2$ and $t_1 \geq 1$. If $p \geq s+2$, then $\mathcal{T}^{-s}$ is χ-closed.*

Lemma 2.9. (Zhao [17]) *Suppose $s \geq 1$ and $p \geq s+2$, then*

(i) *every* $T_{i,j}^{-K(1,s)}$ *is* χ*-unique for any* (i,j) *where* $1 \leq i \neq j \leq t$ *and* $|V_i| = |V_j| = p$*, or* $|V_i| = p$*,* $|V_j| = p+1$*, or* $|V_i| = p+1$*,* $|V_j| = p$*, or* $|V_i| = |V_j| = p+1$*.*

(ii) $T_{1,2}^{-sK_2}$ *is* χ*-unique if* $t_1 = 2$*.*

3. Graphs in $\mathcal{T}^{-s}$ for $|S| = s \leq 2$ and $p \geq s+2$

It is well known that the Turán graph is χ-unique [4]. By Lemma 2.9, all graphs in $\mathcal{T}^{-1}$ are χ-unique for $p \geq 3$. We shall now determine the chromaticity of all graphs in $\mathcal{T}^{-2}$ for $p \geq 4$. Let G be a graph in $\mathcal{T}^{-2}$. Denote by G' the (disjoint union of all) non-complete component(s) of $\overline{G}$. It is then easy to verify by exhaustive construction that $\mathcal{T}^{-2}$ contains 25 non-isomorphic graphs, named $G_{2,i}$ $(1 \leq i \leq 25)$, with the graph $G'_{2,i}$ shown in Table 1. Note that each "circle" associated with $G'_{2,i}$ is a complete graph of order p or $p+1$ as indicated.

Suppose $F = K(p_1, p_2, \ldots, p_t)$. For $G = F - S$, denote by $t_i(G)$ the number of triangles in G that contains i deleted edges in S for $i = 1, 2, 3$. Suppose $G \in \mathcal{T}^{-s}$. An edge $e = uv$ in S is of Type A (respectively, Type B and Type C) if $u \in V_i$, $v \in V_j$ for $1 \leq i < j \leq t_1$ (respectively, for $1 \leq i \leq t_1$, $t_1 + 1 \leq j \leq t$, and for $t_1 + 1 \leq i < j \leq t$). Denote by $s_1(G)$ (respectively, $s_2(G)$ and $s_3(G)$) the number of Type A (respectively, Type B and Type C) edges in S.

By Lemma 2.6, we have $2 \leq \alpha_{t+1}(G) \leq 3$. Note that for $1 \leq i \leq 25$, $t(G_{2,i}) = t(F) - 2(t_1 p + t_2(p+1)) + 4p + k$ where $0 \leq k = s_2(G_{2,i}) + 2s_3(G_{2,i}) + t_2(G_{2,i}) \leq 5$. We compute the ordered pair $(\alpha_{t+1}(G_{2,i}), k)$ for each $G_{2,i}$ $(1 \leq i \leq 25)$, and the results are shown in Table 1.

We then partition the family $\mathcal{T}^{-2}$ according to the value $(\alpha_{t+1}(G_{2,i}), k)$, and each part of the partition is denoted by $\mathcal{G}(\alpha_{t+1}(G_{2,i}), k)$. Hence, we have the following classification of the graphs.

Table 1 (1 of 2): Non-complete component(s) of $\overline{G_{2,i}}$ $(1 \leq i \leq 25)$ for $G_{2,i} \in \mathcal{K}^{-2}(t_1 \times p, t_2 \times (p+1))$

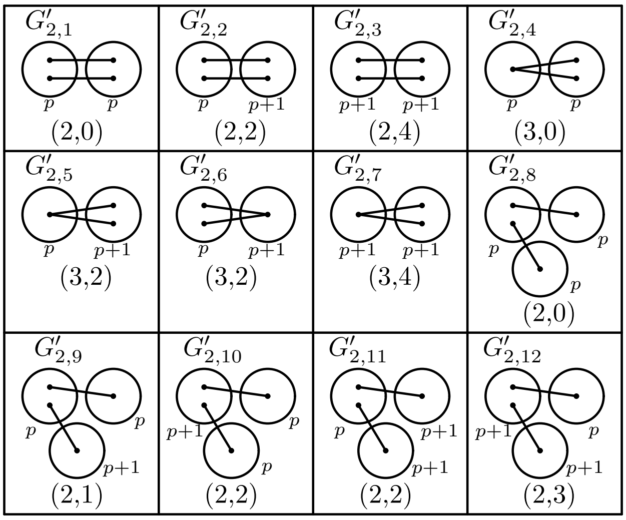
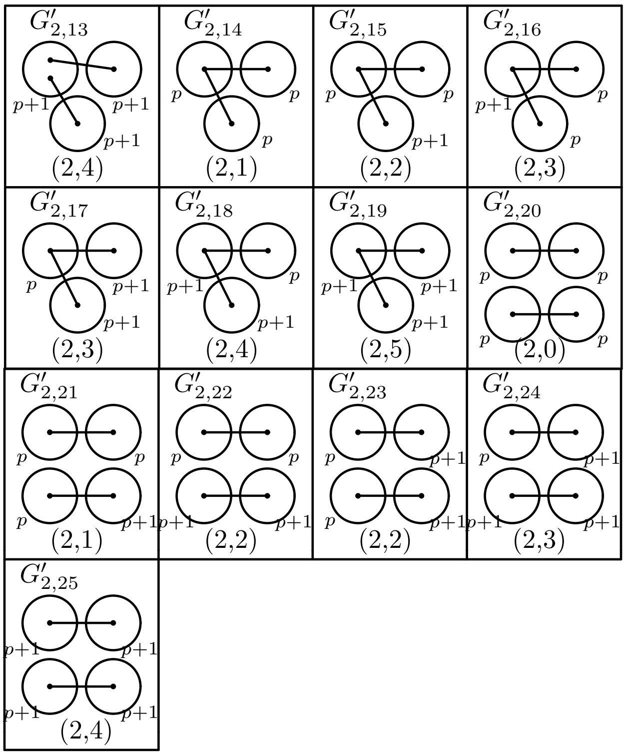


Table 1 (2 of 2): Non-complete component(s) of $\overline{G_{2,i}}$ $(1 \le i \le 25)$ for $G_{2,i} \in \mathcal{K}^{-2}(t_1 \times p, t_2 \times (p+1))$



$$
\begin{aligned}
\mathcal{G}_1 = \mathcal{G}(2,0) &= \{G_{2,1}; G_{2,8}; G_{2,20}\};\\
\mathcal{G}_2 = \mathcal{G}(2,1) &= \{G_{2,9}; G_{2,14}; G_{2,21}\};\\
\mathcal{G}_3 = \mathcal{G}(2,2) &= \{G_{2,2}; G_{2,10}; G_{2,11}; G_{2,15}; G_{2,22}; G_{2,23}\};\\
\mathcal{G}_4 = \mathcal{G}(2,3) &= \{G_{2,12}; G_{2,16}; G_{2,17}; G_{2,24}\};\\
\mathcal{G}_5 = \mathcal{G}(2,4) &= \{G_{2,3}; G_{2,13}; G_{2,18}; G_{2,25}\};\\
\mathcal{G}_6 = \mathcal{G}(2,5) &= \{G_{2,19}\};\\
\mathcal{G}_7 = \mathcal{G}(3,0) &= \{G_{2,4}\};\\
\mathcal{G}_8 = \mathcal{G}(3,2) &= \{G_{2,5}; G_{2,6}\};\\
\mathcal{G}_9 = \mathcal{G}(3,4) &= \{G_{2,7}\}.
\end{aligned}
$$

Note that for an edgeless graph O_n, $\alpha(O_n, 2) = 2^{n-1} - 1$ and $\alpha(O_n, 3) = \frac{1}{3!}(3^n - 3 \cdot 2^n + 3)$. We now present our main theorem of this section.

Theorem 3.1. *For integers $p \geq 4$, $t_1 \geq 1$ and $t_1 + t_2 \geq 3$, all the graphs in $\mathcal{T}^{-2}$ are χ-unique except that $\{G_{2,16}, G_{2,17} \,|\, t_1, t_2 \geq 2\}$ is a χ-equivalence class.*

Proof. By Lemma 2.8, $\mathcal{T}^{-2}$ is χ-closed. Observe that for $1 \leq i < j \leq 9$ and each graph $H' \in \mathcal{G}_i$ and each graph $H'' \in \mathcal{G}_j$, either $\alpha_{t+1}(H') \neq \alpha_{t+1}(H'')$ or $t(H') \neq t(H'')$. By Lemma 2.1, $H' \not\sim H''$. Hence, each $\mathcal{G}_i$ and $\mathcal{G}_j$ $(1 \leq i < j \leq 9)$ are χ-disjoint. Since $\mathcal{T}^{-2}$ is χ-closed, we conclude that each $\mathcal{G}_i$ $(1 \leq i \leq 9)$ is χ-closed. It follows immediately that $G_{2,4}, G_{2,7}$ and $G_{2,19}$ are χ-unique. By Lemma 2.9(i), we also know that $G_{2,5}$ and $G_{2,6}$ are χ-unique.

We now determine the chromaticity of the graphs in $\mathcal{G}_i$ for $1 \leq i \leq 5$. It suffices to show that for any two graphs G' and G'' in $\mathcal{G}_i$, either $\alpha_k(G') \neq \alpha_k(G'')$ for $k = t+2$ or $t+3$, or $\beta(\overline{G'}) \neq \beta(\overline{G''})$. Otherwise, we shall show that $h(\overline{G'}, x) = h(\overline{G''}, x)$. Since the proofs in determining the chromaticity of each graph in $\mathcal{G}_i$ are similar for $1 \leq i \leq 5$, we shall only elaborate in more details for graphs in $\mathcal{G}_2$ and $\mathcal{G}_3$. **(1).** Graphs in $\mathcal{G}_1$. We first note that by Lemma 2.9(ii), $G_{2,1}$ is χ-unique. Now observe that if $G_{2,8} \sim G_{2,20}$, then Lemma 2.2 implies that $h(K_p + G'_{2,8}, x) = h(G'_{2,20}, x)$. However, Lemma 2.4 implies that $\beta(K_p + G'_{2,8}) = \beta(G'_{2,8}) < \beta(G'_{2,20})$, a contradiction. Hence, all the graphs in $\mathcal{G}_1$ are χ-unique. **(2).** Graphs in $\mathcal{G}_2$. We compute $\alpha_{t+2}(G)$ for each $G \in \mathcal{G}_2$.

Let $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$ be the unique t-independent partition of $G \in \mathcal{T}^{-2}$. Suppose $\mathcal{V}' = \{V'_1, V'_2, \ldots, V'_t, V'_{t+1}, V'_{t+2}\}$ is an improper $(t+2)$-independent partition in G. Since $p \geq 4$, $\mathcal{V}'$ has at most two improper independent sets. If $\mathcal{V}'$ contains exactly two improper independent sets, say V'_{t+1} and V'_{t+2}, then $V_i \backslash (V'_{t+1} \cup V'_{t+2})$ is a proper independent set in $\mathcal{V}'$ for each $1 \leq i \leq t$. If $\mathcal{V}'$ contains exactly one improper independent set, say V'_{t+2}, then we may assume that $V_i \cap V'_{t+2} = \emptyset$ for $i = 1, 2, \ldots, t-2$, and $V_i \cap V'_{t+2} \neq \emptyset$ for $i = t-1, t$. Hence, there exist exactly two proper independent sets in $\mathcal{V}'$, say V'_t and V'_{t+1}

such that $V_t' \cup V_{t+1}' = V$ for $V \in \{V_1, V_2, \ldots, V_{t-2}, V_{t-1}\backslash V_{t+2}', V_t\backslash V_{t+2}'\}$. Define

$$\begin{aligned} f_1 &= (t_1+2)(2^{p-1}-1)+(t_2-2)(2^p-1), \\ f_2 &= (2^{p-2}-1)+t_1(2^{p-1}-1)+(t_2-1)(2^p-1), \\ f_3 &= 2(2^{p-2}-1)+(t_1-2)(2^{p-1}-1)+t_2(2^p-1). \end{aligned}$$

Hence,

$$\begin{aligned} \alpha_{t+2}(G_{2,i}) &= 1+f_2+f_3 \quad \text{for} \quad i=9,21, \\ \alpha_{t+2}(G_{2,14}) &= 2f_3. \end{aligned}$$

Since $p \geq 4$, $\alpha_{t+2}(G_{2,14}) - \alpha_{t+2}(G_{2,i}) = 2^{p-2}-1 > 0$ for $i = 9, 21$. We now show that $G_{2,9} \not\sim G_{2,21}$. By Lemma 2.4, $\beta(\overline{G_{2,9}}) = \beta(G_{2,9}') < \beta(G_{2,21}') = \beta(\overline{G_{2,21}})$, a contradiction. Hence, all the graphs in $\mathcal{G}_2$ are χ-unique.

(3). Graphs in $\mathcal{G}_3$. By a similar argument as in (2) above, we have for each graph $G \in \mathcal{G}_3$,

$$\begin{aligned} \alpha_{t+2}(G_{2,i}) &= 1+2f_2 \qquad \text{for} \quad i=2,10,11,23, \\ \alpha_{t+2}(G_{2,15}) &= f_2+f_3, \\ \alpha_{t+2}(G_{2,22}) &= 1+f_1+f_3. \end{aligned}$$

Since $p \geq 4$, $\alpha_{t+2}(G_{2,15}) - \alpha_{t+2}(G_{2,i}) = 2^{p-2}-1 > 0$ for $i =$ 2, 10, 11, 22, 23. We need to compute $\alpha_{t+3}(G_{2,i})$ for $i =$ 2, 10, 11, 22, 23. Suppose $\mathcal{V}' = \{V_1', V_2', \ldots, V_t', V_{t+1}', V_{t+2}', V_{t+3}'\}$ is an improper $(t+3)$-independent partition in G. If $\mathcal{V}'$ contains exactly two improper independent sets, say V_{t+2}' and V_{t+3}', then exactly one of $V_i\backslash(V_{t+2}' \cup V_{t+3}')$ $(1 \leq i \leq t)$ needs to be partitioned into two proper independent sets. Note that $\mathcal{V}'$ has at most two improper independent sets. If $\mathcal{V}'$ contains exactly one improper independent set, say V_{t+3}', then either exactly one of $V_i\backslash V_{t+3}'$ $(1 \leq i \leq t)$ needs to be partitioned into three proper independent sets, or else exactly two of $V_i\backslash V_{t+3}'$ $(1 \leq i \leq t)$

need to be partitioned into two proper independent sets respectively. Define

$$\begin{aligned}
g_1 &= \left[\frac{t_1+2}{3!}(3^p - 3\cdot 2^p + 3) + \frac{t_2-2}{3!}(3^{p+1} - 3\cdot 2^{p+1} + 3)\right] + \\
&\left[\binom{t_1+2}{2}(2^{p-1}-1)^2 + (t_1+2)(t_2-2)(2^{p-1}-1)(2^p-1) + \right. \\
&\left.\binom{t_2-2}{2}(2^p-1)^2\right], \\
g_2 &= \left[\frac{1}{3!}(3^{p-1} - 3\cdot 2^{p-1} + 3) + \frac{t_1}{3!}(3^p - 3\cdot 2^p + 3) + \right. \\
&\left.\frac{t_2-1}{3!}(3^{p+1} - 3\cdot 2^{p+1} + 3)\right] + \left[t_1(2^{p-2}-1)(2^{p-1}-1) + \right. \\
&(t_2-1)(2^{p-2}-1)(2^p-1) + \binom{t_1}{2}(2^{p-1}-1)^2 + \\
&\left. t_1(t_2-1)(2^{p-1}-1)(2^p-1) + \binom{t_2-1}{2}(2^p-1)^2\right], \\
g_3 &= \left[\frac{2}{3!}(3^{p-1} - 3\cdot 2^{p-1} + 3) + \frac{t_1-2}{3!}(3^p - 3\cdot 2^p + 3) + \right. \\
&\left.\frac{t_2}{3!}(3^{p+1} - 3\cdot 2^{p+1} + 3)\right] + \left[(2^{p-2}-1)^2 + \right. \\
&2(t_1-2)(2^{p-2}-1)(2^{p-1}-1) + 2t_2(2^{p-2}-1)(2^p-1) + \\
&\binom{t_1-2}{2}(2^{p-1}-1)^2 + (t_1-2)t_2(2^{p-1}-1)(2^p-1) + \\
&\left.\binom{t_2}{2}(2^p-1)^2\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\alpha_{t+3}(G_{2,2}) &= [(2^{p-3}-1) + (2^{p-2}-1) + (t_1-1)(2^{p-1}-1) + \\
&\quad (t_2-1)(2^p-1)] + 2g_2, \\
\alpha_{t+3}(G_{2,10}) &= [3(2^{p-2}-1) + (t_1-2)(2^{p-1}-1) + (t_2-1)(2^p-1)] + \\
&\quad 2g_2,
\end{aligned}$$

$$\begin{aligned} a_{t+3}(G_{2,11}) &= [(2^{p-3}-1)+(t_1+1)(2^{p-1}-1)+(t_2-2)(2^p-1)]+2g_2,\\ \alpha_{t+3}(G_{2,22}) &= [2(2^{p-2}-1)+t_1(2^{p-1}-1)+(t_2-2)(2^p-1)]+\\ &\quad g_1+g_3,\\ \alpha_{t+3}(G_{2,23}) &= 2(2^{p-2}-1)+t_1(2^{p-1}-1)+(t_2-2)(2^p-1)+2g_2. \end{aligned}$$

Since $p \geq 4$, by Software Maple,

$$\begin{aligned} \alpha_{t+3}(G_{2,22})-\alpha_{t+3}(G_{2,2}) &= 2^{2p-4}-3(2^{p-3})>0,\\ \alpha_{t+3}(G_{2,2})-\alpha_{t+3}(G_{2,10}) &= 2^{p-3}>0,\\ \alpha_{t+3}(G_{2,10})-\alpha_{t+3}(G_{2,11}) &= 2^{p-3}>0,\\ \alpha_{t+3}(G_{2,11})-\alpha_{t+3}(G_{2,23}) &= 2^{p-3}>0. \end{aligned}$$

Hence, all the graphs in $\mathcal{G}_3$ are χ-unique.

(4). Graphs in $\mathcal{G}_4$. For each graph $G \in \mathcal{G}_4$,

$$\begin{aligned} \alpha_{t+2}(G_{2,i}) &= 1+f_1+f_2 \quad \text{for} \quad i=12,24,\\ \alpha_{t+2}(G_{2,j}) &= 2f_2 \qquad\qquad \text{for} \quad j=16,17. \end{aligned}$$

Since $p \geq 4$, $\alpha_{t+2}(G_{2,j})-\alpha_{t+2}(G_{2,i}) = 2^{p-2}-1>0$. By Lemma 2.4, $\beta(\overline{G_{2,12}}) = \beta(G'_{2,12}) < \beta(G'_{2,24}) = \beta(\overline{G_{2,24}})$.

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned} h(\overline{G_{2,16}},x) &= [h(K_p,x)]^{t_1-2}[h(K_{p+1},x)]^{t_2-1}h(G'_{2,16},x)\\ &= [h(K_p,x)]^{t_1-2}[h(K_{p+1},x)]^{t_2-1}[(h(K_p,x))^2h(K_{p+1},x)+\\ &\quad 2xh(K_{p-1},x)(h(K_p,x))^2] \end{aligned}$$

and

$$\begin{aligned} h(\overline{G_{2,17}},x) &= [h(K_p,x)]^{t_1-1}[h(K_{p+1},x)]^{t_2-2}h(G'_{2,17},x)\\ &= [h(K_p,x)]^{t_1-1}[h(K_{p+1},x)]^{t_2-2}[h(K_p,x)(h(K_{p+1},x))^2+\\ &\quad 2xh(K_{p-1},x)h(K_p,x)h(K_{p+1},x)]. \end{aligned}$$

Clearly, $G_{2,16} \sim G_{2,17}$ for $t_1, t_2 \geq 2$. Hence, all the graphs in $\mathcal{G}_4$ are χ-unique except that $\{G_{2,16}, G_{2,17} \,|\, t_1, t_2 \geq 2\}$ is a χ-equivalence class.

(5). Graphs in $\mathcal{G}_5$. For each $G \in \mathcal{G}_5$,

$$\alpha_{t+2}(G_{2,i}) = 1+2f_1 \quad \text{for} \quad i=3,13,25,$$

$$\alpha_{t+2}(G_{2,18}) \quad = \quad f_1 + f_2.$$

Since $p \geq 4$, $\alpha_{t+2}(G_{2,18}) > \alpha_{t+2}(G_{2,i}) = 2^{p-2} - 1 > 0$. We now compare $\alpha_{t+3}(G_{2,3})$, $\alpha_{t+3}(G_{2,13})$ and $\alpha_{t+3}(G_{2,25})$. We have

$$\begin{aligned}
\alpha_{t+3}(G_{2,3}) &= [2(2^{p-2}-1) + t_1(2^{p-1}-1) + (t_2-2)(2^p-1)] + 2g_1, \\
\alpha_{t+3}(G_{2,13}) &= [(2^{p-2}-1) + (t_1+2)(2^{p-1}-1) + (t_2-3)(2^p-1)] + 2g_1, \\
\alpha_{t+3}(G_{2,25}) &= [(t_1+4)(2^{p-1}-1) + (t_2-4)(2^p-1)] + 2g_1.
\end{aligned}$$

Since $p \geq 4$, by Software Maple, $\alpha_{t+3}(G_{2,3}) - \alpha_{t+3}(G_{2,13}) = \alpha_{t+3}(G_{2,13}) - \alpha_{t+3}(G_{2,25}) = 2^{p-2} - 1 > 0$. Hence, all the graphs in $\mathcal{G}_5$ are χ-unique.

The proof is now complete. □

4. Graphs in $\mathcal{T}^{-3}$ for $p \geq 5$

Let u, v and w be a vertex of the complete graphs K_p, K_q and K_r respectively. Denote by $K_p \cdot K_q$ the graph obtained from $K_p + K_q$ by adding the edge uv. For convenience, let $K_p^+ = K_1 \cdot K_p$. Also let $K_p * K_q * K_r$ be the graph obtained from $K_p + K_q + K_r$ by adding two edges uv and vw.

We now list all the 25 possible "structures" induced by three edges deleted from the Turán graph which can be obtained by brute force construction (with respect to the partite sets) in Figure 1.

Since each circle in Figure 1 is a partite set of size p or $p+1$, we obtain from the "structures" in Figure 1, 213 non-isomorphic graphs in $\mathcal{T}^{-3}$, named $G_{3,i}$ $(1 \leq i \leq 213)$. A table of the graphs $G'_{3,i}$ similar to that of Table 1 is available upon request. Note that each "circle" associated with $G'_{3,i}$ is a complete graph of order p or $p+1$ as indicated. For example, the top left "structure" in Figure 1 will give us three non-isomorphic graphs $G_{3,1}$, $G_{3,2}$ and $G_{3,3}$ as shown in Figure 2.

By Lemma 2.6, we have $3 \leq \alpha_{t+1}(G_{3,i}) \leq 7$ for $1 \leq i \leq 213$. Also note that for $1 \leq i \leq 213$, $t(G_{3,i}) = t(F) - 3(t_1 p + t_2(p+1)) + 6p + k$ where

$0 \le k = s_2(G_{3,i}) + 2s_3(G_{3,i}) + t_2(G_{3,i}) - t_3(G_{3,i}) \le 9$. We now compute the ordered pair $(\alpha_{t+1}(G_{3,i}), k)$ for each $G_{3,i}$ $(1 \le i \le 213)$. A table of $G'(3,i)$ with $(\alpha_{t+1}(G_{3,i}), k)$ is also available upon request. We then partition the family $\mathcal{T}^{-3}$ according to the value $(\alpha_{t+1}(G_{3,i}), k)$, and each part of the partition is denoted by $\mathcal{G}(\alpha_{t+1}(G_{3,i}), k)$.

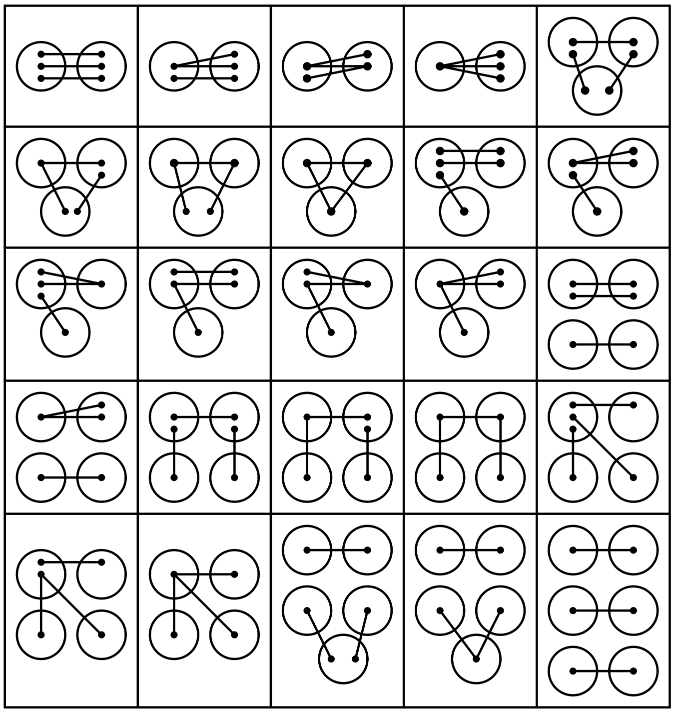


Figure 1. List of "structures" induced by three edges deleted from $K(t_1 \times p, t_2 \times (p+1))$

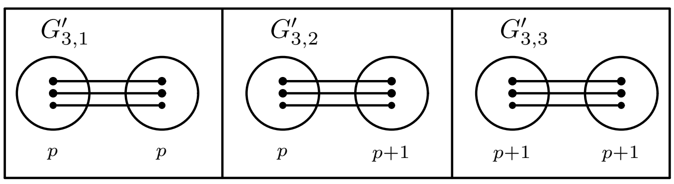


Figure 2: Graphs obtained from top left structure in Figure 1.

Hence, we have the following classification of the graphs.

$$\mathcal{G}_1 = \mathcal{G}(3,0) \quad = \quad \{G_{3,1}; G_{3,15}; G_{3,35}; G_{3,83}; G_{3,104}; G_{3,140}; G_{3,168}; G_{3,204}\};$$

$$\mathcal{G}_2 = \mathcal{G}(3,1) = \{G_{3,19}; G_{3,36}; G_{3,59}; G_{3,84}; G_{3,105}; G_{3,114}; G_{3,141}; G_{3,148}; G_{3,169}; G_{3,170}; G_{3,186}; G_{3,205}\};$$

$$\mathcal{G}_3 = \mathcal{G}(3,2) = \{G_{3,16}; G_{3,25}; G_{3,37}; G_{3,60}; G_{3,85}; G_{3,86}; G_{3,106}; G_{3,107}; G_{3,115}; G_{3,116}; G_{3,130}; G_{3,143}; G_{3,149}; G_{3,150}; G_{3,171}; G_{3,172}; G_{3,173}; G_{3,175}; G_{3,187}; G_{3,188}; G_{3,206}; G_{3,207}\}$$

$$\mathcal{G}_4 = \mathcal{G}(3,3) = \{G_{3,2}; G_{3,20}; G_{3,21}; G_{3,38}; G_{3,39}; G_{3,61}; G_{3,87}; G_{3,108}; G_{3,109}; G_{3,117}; G_{3,118}; G_{3,119}; G_{3,131}; G_{3,142}; G_{3,145}; G_{3,152}; G_{3,153}; G_{3,160}; G_{3,174}; G_{3,176}; G_{3,177}; G_{3,179}; G_{3,189}; G_{3,190}; G_{3,191}; G_{3,193}; G_{3,208}; G_{3,209}\};$$

$$\mathcal{G}_5 = \mathcal{G}(3,4) = \{G_{3,17}; G_{3,26}; G_{3,27}; G_{3,40}; G_{3,62}; G_{3,63}; G_{3,88}; G_{3,89}; G_{3,110}; G_{3,111}; G_{3,120}; G_{3,121}; G_{3,122}; G_{3,123}; G_{3,132}; G_{3,133}; G_{3,144}; G_{3,151}; G_{3,156}; G_{3,161}; G_{3,178}; G_{3,180}; G_{3,181}; G_{3,182}; G_{3,192}; G_{3,194}; G_{3,195}; G_{3,197}; G_{3,210}; G_{3,211}\};$$

$$\mathcal{G}_6 = \mathcal{G}(3,5) = \{G_{3,22}; G_{3,23}; G_{3,41}; G_{3,64}; G_{3,90}; G_{3,112}; G_{3,124}; G_{3,125}; G_{3,126}; G_{3,134}; G_{3,135}; G_{3,146}; G_{3,154}; G_{3,155}; G_{3,163}; G_{3,183}; G_{3,184}; G_{3,196}; G_{3,198}; G_{3,199}; G_{3,200}; G_{3,212}\};$$

$$\mathcal{G}_7 = \mathcal{G}(3,6) = \{G_{3,3}; G_{3,18}; G_{3,28}; G_{3,29}; G_{3,42}; G_{3,65}; G_{3,91}; G_{3,113}; G_{3,127}; G_{3,128}; G_{3,136}; G_{3,137}; G_{3,147}; G_{3,157}; G_{3,158}; G_{3,162}; G_{3,165}; G_{3,185}; G_{3,201}; G_{3,202}; G_{3,213}\};$$

$$\mathcal{G}_8 = \mathcal{G}(3,7) = \{G_{3,24}; G_{3,66}; G_{3,129}; G_{3,138}; G_{3,159}; G_{3,164}; G_{3,203}\};$$

$$\mathcal{G}_9 = \mathcal{G}(3,8) = \{G_{3,30}; G_{3,139}; G_{3,166}\};$$

$$\mathcal{G}_{10} = \mathcal{G}(3,9) = \{G_{3,167}\};$$

$$\mathcal{G}_{11} = \mathcal{G}(4,0) = \{G_{3,4}; G_{3,43}; G_{3,51}; G_{3,92}\};$$

$$\mathcal{G}_{12} = \mathcal{G}(4,1) = \{G_{3,44}; G_{3,52}; G_{3,67}; G_{3,93}\};$$

$$\mathcal{G}_{13} = \mathcal{G}(4,2) = \{G_{3,31}; G_{3,45}; G_{3,53}; G_{3,68}; G_{3,75}; G_{3,94}; G_{3,95}; G_{3,96}\};$$

$$\mathcal{G}_{14} = \mathcal{G}(4,3) = \{G_{3,5}; G_{3,6}; G_{3,46}; G_{3,47}; G_{3,54}; G_{3,55}; G_{3,69}; G_{3,76}; G_{3,97}; G_{3,98}\};$$

$$\mathcal{G}_{15} = \mathcal{G}(4,4) = \{G_{3,32}; G_{3,48}; G_{3,56}; G_{3,70}; G_{3,71}; G_{3,77}; G_{3,99}; G_{3,100}; G_{3,101}\};$$

$$\mathcal{G}_{16} = \mathcal{G}(4,5) = \{G_{3,49}; G_{3,57}; G_{3,72}; G_{3,78}; G_{3,79}; G_{3,102}\};$$

$$\mathcal{G}_{17} = \mathcal{G}(4,6) = \{G_{3,7}; G_{3,33}; G_{3,50}; G_{3,58}; G_{3,73}; G_{3,80}; G_{3,103}\};$$

$$\mathcal{G}_{18} = \mathcal{G}(4,7) = \{G_{3,74}; G_{3,81}\};$$

$$\mathcal{G}_{19} = \mathcal{G}(4,8) = \{G_{3,34}; G_{3,82}\};$$

$$\begin{aligned}
\mathcal{G}_{20} = \mathcal{G}(5,0) &= \{G_{3,8}\};\\
\mathcal{G}_{21} = \mathcal{G}(5,3) &= \{G_{3,9}\};\\
\mathcal{G}_{22} = \mathcal{G}(5,6) &= \{G_{3,10}\};\\
\mathcal{G}_{23} = \mathcal{G}(7,0) &= \{G_{3,11}\};\\
\mathcal{G}_{24} = \mathcal{G}(7,3) &= \{G_{3,12}; G_{3,13}\};\\
\mathcal{G}_{25} = \mathcal{G}(7,6) &= \{G_{3,14}\}.
\end{aligned}$$

Now we present our main result of this section.

Theorem 4.1. *For integers $p \geq 5$, $t_1 \geq 1$ and $t_1 + t_2 \geq 3$, all the graphs in $\mathcal{K}^{-3}(t_1 \times p, t_2 \times (p+1))$ are χ-unique except that*
$\{G_{3,19}, G_{3,148} \,|\, t_1 \geq 4, t_2 \geq 0\}$, $\{G_{3,20}, G_{3,118} \,|\, t_1 \geq 3, t_2 \geq 1\}$,
$\{G_{3,21}, G_{3,61}, G_{3,152} \,|\, t_1, t_2 \geq 2\}$, $\{G_{3,22},\ G_{3,124}, G_{3,155} \,|\, t_1, t_2 \geq 2\}$,
$\{G_{3,24}, G_{3,159} \,|\, t_1 \geq 1, t_2 \geq 4\}$, $\{G_{3,26}, G_{3,132} \,|\, t_1 \geq 3, t_2 \geq 1\}$,
$\{G_{3,28}, G_{3,136} \,|\, t_1, t_2 \geq 2\}$, $\{G_{3,32}, G_{3,77} \,|\, t_1 \geq 2, t_2 \geq 1\}$,
$\{G_{3,65}, G_{3,157} \,|\, t_1 \geq 1, t_2 \geq 3\}$, $\{G_{3,85}, G_{3,106} \,|\, t_1 \geq 3, t_2 \geq 1\}$,
$\{G_{3,90}, G_{3,112} \,|\, t_1 \geq 1, t_2 \geq 3\}$, $\{G_{3,117}, G_{3,189}, G_{3,193} \,|\, t_1 \geq 4, t_2 \geq 2\}$,
$\{G_{3,125}, G_{3,198} \,|\, t_1 \geq 2, t_2 \geq 3\}$, $\{G_{3,127}, G_{3,158} \,|\, t_1 \geq 1, t_2 \geq 3\}$,
$\{G_{3,128}, G_{3,202} \,|\, t_1 \geq 1, t_2 \geq 4\}$, $\{G_{3,162}, G_{3,165} \,|\, t_1, t_2 \geq 3\}$,
$\{G_{3,192}, G_{3,197} \,|\, t_1, t_2 \geq 3\}$ *and* $\{G_{3,196},\ G_{3,200} \,|\, t_1 \geq 2, t_2 \geq 4\}$
are χ-equivalence classes.

Proof. By Lemma 2.8, $\mathcal{T}^{-3}$ is χ-closed. Observe that for $1 \leq i < j \leq 25$ and each graph $H' \in \mathcal{G}_i$ and each graph $H'' \in \mathcal{G}_j$, either $\alpha_{t+1}(H') \neq \alpha_{t+1}(H'')$ or $t(H') \neq t(H'')$. By Lemma 2.1, $H' \not\sim H''$. Hence, each $\mathcal{G}_i$ and $\mathcal{G}_j$ $(1 \leq i < j \leq 25)$ are χ-disjoint. Since $\mathcal{T}^{-3}$ is χ-closed, we conclude that each $\mathcal{G}_i$ $(1 \leq i \leq 25)$ is χ-closed. It follows immediately that $G_{3,8}, G_{3,9}, G_{3,10}, G_{3,11}, G_{3,14}$ and $G_{2,167}$ are χ-unique. By Lemma 2.9(i), we also know that $G_{3,12}$ and $G_{3,13}$ are χ-unique.

We now determine the chromaticity of the graphs in $\mathcal{G}_i$ for $i \in \{1, 2, \ldots, 19\} \backslash \{10\}$. It suffices to show that for any two graphs G' and G'' in $\mathcal{G}_i$, either $\alpha_k(G') \neq \alpha_k(G'')$ for $k = t+2$, $t+3$ or $t+4$, or $\beta(\overline{G'}) \neq \beta(\overline{G''})$. Otherwise, we shall show that $h(\overline{G'}, x) = h(\overline{G''}, x)$. We use an argument similar to that in the proof of

Theorem 3.1. In what follows, f_i and g_i $(i = 1, 2, 3)$ are as defined in the proof of Theorem 3.1.

(1). Graphs in $\mathcal{G}_1$. Define

$$\begin{aligned} d_1 &= 2(2^{p-3}-1)+(t_1-2)(2^{p-1}-1)+t_2(2^p-1), \\ d_2 &= (2^{p-3}-1)+2(2^{p-2}-1)+(t_1-3)(2^{p-1}-1)+t_2(2^p-1), \\ d_3 &= 4(2^{p-2}-1)+(t_1-4)(2^{p-1}-1)+t_2(2^p-1). \end{aligned}$$

So,

$$\begin{aligned} \alpha_{t+3}(G_{3,1}) &= 1+3d_1+3g_3, \\ \alpha_{t+3}(G_{3,i}) &= 1+3d_2+3g_3 \qquad \text{for} \quad i=15,140, \\ \alpha_{t+3}(G_{3,35}) &= 1+d_1+2d_2+3g_3, \\ \alpha_{t+3}(G_{3,83}) &= 1+d_1+2d_3+3g_3, \\ \alpha_{t+3}(G_{3,104}) &= 1+2d_2+d_3+3g_3 \\ \alpha_{t+3}(G_{3,168}) &= 1+d_2+2d_3+3g_3, \\ \alpha_{t+3}(G_{3,204}) &= 1+3d_3+3g_3. \end{aligned}$$

Since $p \geq 5$,

$$\begin{aligned} \alpha_{t+3}(G_{3,1})-\alpha_{t+3}(G_{3,35}) &= 2^{p-2}>0, \\ \alpha_{t+3}(G_{3,35})-\alpha_{t+3}(G_{3,i}) &= 2^{p-3}>0 \qquad \text{for} \quad i=15,140, \\ \alpha_{t+3}(G_{3,i})-\alpha_{t+3}(G_{3,j}) &= 2^{p-3}>0 \qquad \text{for} \quad j=83,104, \\ \alpha_{t+3}(G_{3,j})-\alpha_{t+3}(G_{3,168}) &= 2^{p-3}>0, \\ \alpha_{t+3}(G_{3,168})-\alpha_{t+3}(G_{3,204}) &= 2^{p-3}>0. \end{aligned}$$

So, $\alpha_{t+3}(G_{3,1}) > \alpha_{t+3}(G_{3,35}) > \alpha_{t+3}(G_{3,15}) = \alpha_{t+3}(G_{3,140}) > \alpha_{t+3}(G_{3,83}) = \alpha_{t+3}(G_{3,104}) > \alpha_{t+3}(G_{3,168}) > \alpha_{t+3}(G_{3,204})$.

Let $\mathcal{I}_k^j(G)$ be the number of k-independent partitions with $j \geq 1$ improper independent sets. Then, $\alpha_k(G) = \sum_{j\geq 1} \mathcal{I}_k^j(G)$. We now compare $\alpha_{t+4}(G_{3,15})$

with $\alpha_{t+4}(G_{3,140})$.

$$\begin{aligned}\alpha_{t+4}(G_{3,15}) &= \mathcal{I}^1_{t+4}(G_{3,15}) + \mathcal{I}^2_{t+4}(G_{3,15}) + \\ &\quad [3(2^{p-3}-1) + (t_1-3)(2^{p-1}-1) + t_2(2^p-1)], \\ \alpha_{t+4}(G_{3,140}) &= \mathcal{I}^1_{t+4}(G_{3,140}) + \mathcal{I}^2_{t+4}(G_{3,140}) + \\ &\quad [(2^{p-4}-1) + 3(2^{p-2}-1) + (t_1-4)(2^{p-1}-1) + t_2(2^p-1)].\end{aligned}$$

Note that $\mathcal{I}^j_{t+4}(G_{3,15}) = \mathcal{I}^j_{t+4}(G_{3,140})$ for $j = 1, 2$. Since $p \geq 5$, $\alpha_{t+4}(G_{3,15}) - \alpha_{t+4}(G_{3,140}) = 2^{p-4} > 0$.

We now show that $h(\overline{G_{3,83}}, x) \neq h(\overline{G_{3,104}}, x)$. By Lemmas 2.2 and 2.3,

$$\begin{aligned}h(\overline{G_{3,83}}, x) &= \Big[h(K_p, x)\Big]^{t_1-4}\Big[h(K_{p+1}, x)\Big]^{t_2}\Big[(h(K_p \cdot K_p, x))^2 + \\ &\quad xh(K_p \cdot K_p, x)h(K_{p-1} \cdot K_{p-1}, x)\Big], \\ h(\overline{G_{3,104}}, x) &= \Big[h(K_p, x)\Big]^{t_1-4}\Big[h(K_{p+1}, x)\Big]^{t_2}\Big[(h(K_p \cdot K_p, x))^2 + \\ &\quad x(h(K_p \cdot K_{p-1}, x))^2\Big].\end{aligned}$$

By Lemma 2.4, $\beta(K_p \cdot K_p) < \beta(K_p \cdot K_{p-1})$. Therefore, $\beta(\overline{G_{3,83}}) < \beta(\overline{G_{3,104}})$. Hence, all the graphs in $\mathcal{G}_1$ are χ-unique.

The chromaticity of graphs in $\mathcal{G}_i$, $i = \{2, 3, \ldots, 19\}\backslash\{10\}$ can be similarly determined as in (1). Since it is long and rather repetitive, the proofs are omitted. The proof is now complete. □

A diagram of all the χ-equivalence classes obtained in Theorem 4.1 is presented in Figure 3 for easy reference and comparison.

Remarks. Observe that graphs $G_{3,162}$ and $G_{3,165}$ can be obtained from $G_{2,16}$ and $G_{2,17}$ respectively with one more edge deleted. For $t_1, t_2 \geq s \geq 2$, let G' (respectively, G'') be a graph in $\mathcal{T}^{-s}$ such that $\langle S \rangle$ is a star graph with the central vertex belongs to a partite set of size p (respectively, size $p+1$) and the end-vertices belong to different partite sets of size $p+1$ (respectively, size p). It is clear that $G' \not\cong G''$. By Lemmas 2.2, 2.3 and mathematical induction on s, it is easy to show that $h(\overline{G'}, x) = h(\overline{G''}, x)$ for all $p \geq 2$.

Conjecture 1. *The family $\{G', G''\}$ is a χ-equivalence class.*

For integers $m_1, m_2, m_3 \geq 0$, let G and H be graphs in $\mathcal{T}^{-s}$ such that the graph induced by all the non-complete components of $\overline{G}$ (respectively, $\overline{H}$) is $G'_{2,16} + m_1(K_p \cdot K_p) + m_2(K_p \cdot K_{p+1}) + m_3(K_{p+1} \cdot K_{p+1})$ (respectively, $G'_{2,17} + m_1(K_p \cdot K_p) + m_2(K_p \cdot K_{p+1}) + m_3(K_{p+1} \cdot K_{p+1})$) where $m_1 + m_2 + m_3 \geq 1$.

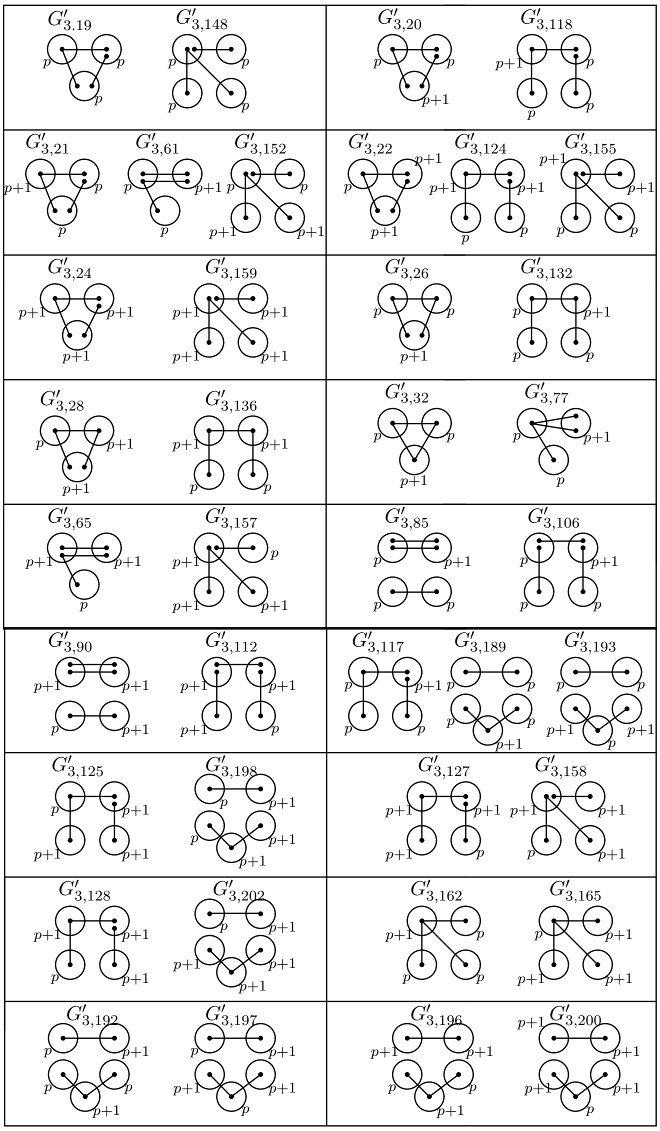


Figure 3: "Structures" of graphs in Chromatic Equivalence Classes listed in Theorem 2.

Clearly, $G \sim H$ but $G \ncong H$. Observe that $\{G, H\}$ is a χ-equivalence class if $m_1 = m_j = 0$ for $j = 2$ or 3. Hence, we end this paper with the following problem.

Problem 1. *Find all the values of* $m_i, i = 1, 2, 3$ *such that* $\{G, H\}$ *is a* χ-*equivalence class for* G *and* H *defined above.*

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