Iranian Journal of Mathematical Sciences and Informatics
Vol. 9, No. 2 (2014), pp 37-43
DOI: 10.7508/ijmsi.2014.02.004

The p-median and p-center Problems on Bipartite Graphs

Jafar Fathali*, Nader Jafari Rad, Sadegh Rahimi Sherbaf

Department of Mathematics, Shahrood University of Technology,
Shahrood, Iran.

Email: fathali@shahroodut.ac.ir
Email: n.jafarirad@shahroodut.ac.ir
Email: srahimi@shahroodut.ac.ir

Abstract. Let G be a bipartite graph. In this paper we consider the two kind of location problems namely p-center and p-median problems on bipartite graphs. The p-center and p-median problems ask to find a subset of vertices of cardinality p, so that respectively the maximum and sum of the distances from this set to all other vertices in G is minimized. For each case we present some properties to find exact solutions.

Keywords: Location theory, p-median, p-center, Bipartite graphs.

2000 Mathematics subject classification: 90B80.

1. Introduction

The p-center and p-median problems are two important problems in the location theory. Let $G = (V, E)$ be an undirected graph with vertex set V, $|V| = n$, and edge set E, $|E| = m$. Each vertex v_i has a weight w_i and the edges of graph have positive lengths. In the p-center and p-median problems we want to find a subset $X \subseteq V$ of cardinality p such that respectively the sum of the weighted distances from X to all vertices and the maximum weighted distances from X to all vertices is minimized.

The p-Center and p-median problem has been known to be NP-Hard [8, 9].

*Corresponding Author

Received 22 April 2012; Accepted 20 April 2013

In the case p-center Lan et al. [10] showed a linear-time algorithm for solving the 1-center problem on weighted cactus graphs. Frederickson [3] solved this problem for trees in optimal linear-time (without necessarily restricting the location of the facilities to the vertices of the tree) using parametric search. Bespamyatnikh et al. [1] gave an $O(pn)$ time algorithm for this problem on circular-arc graphs. Kariv and Hakimi [8] addressed the p-center problem on general graphs. In [14], Tamir showed that the weighted and unweighted p-center problems in networks can be solved in $O(n^p m^p log^2 n)$ time and $O(n^{p-1} m^p log^3 n)$ time, respectively.

For the case p-median problem on a tree Kariv and Hakimi [9] showed that this problem can be solved in $O(p^2 n^2)$ time. Tamir [15] improved the time complexity to $O(pn^2)$. For the case $p = 1$ Hua et al. [7] and Goldman [5] presented linear time algorithms. For $p = 2$ an $O(n \log n)$ algorithm was given by Gavish and Sridhar [4]. Hassin and Tamir [6] proposed an $O(pn)$ algorithm for the p-median problem on a path with positive weights.

For further literature on the p-center and p-median problems and their applications the reader is referred to the books of Mirchandani and Francis [12] and Drezner and Hamacher [2]. Maimani [11] considered the median and center of a special graph.

In this paper we consider the p-center and p-median problems on bipartite graphs and present some properties of them to find their solutions.

2. Problem formulation

Let $G = (V, E)$ be a graph, with V the set of vertices, $|V| = n$, and E the set of edges. Every edge with the end vertices u and v is presented by e_{uv} and every vertex $v_i \in V$ has a real weight $w(v_i)$ that for simplicity we use w_i. $d(u, v)$ is the length of shortest path between vertices u and v.

In the p-median problem the sum of the weighted distances is minimized over all $X \subseteq V$ with $|X| = p$:

$$F_1(X) = \sum_{i=1}^{n} w_i d(X, v_i). \tag{2.1}$$

In the p-center problem the maximum of the weighted distances is minimized over all $X \subseteq V$ with $|X| = p$:

$$F_2(X) = \max_{i=1,\ldots,n} w_i d(X, v_i). \tag{2.2}$$

In this paper we consider the case that G is a bipartite graph. The graph $G = (V, E)$ is called a *bipartite graph* if V can be partitioned into two disjoint sets S_1 and S_2 such that if $uv \in E$, then either ($u \in S_1$ and $v \in S_2$) or ($u \in S_2$ and $v \in S_1$). A bipartite graph will be denoted by $G(S_1, S_2, E)$ hereafter. A bipartite graph is called a complete bipartite graph if for each $v \in S_1$ and $u \in S_2$ there exist an edge from v to u. A *k-bipartite graph* is a graph G

whose vertex set can be partitioned into k disjoint sets $V_1, V_2, ..., V_k$ such that the induced subgraph $G[V_i]$ is the empty graph for $i = 1, 2, ..., k$. A *complete k-partite graph* is a k-bipartite graph in which any vertex of each partite set is adjacent to all vertices of other partite sets.

A set of pair-wise independent edges in a graph G is called a *matching*. A matching is *perfect* if it is incident with every vertex of G.

3. Some properties

Lemma 3.1. *Let $G(S_1, S_2, E)$ be an unweighted complete bipartite graph and $X = \{x_1, ..., x_p\} \subseteq V$ be an optimal solution for the p-median problem. If $X \cap S_i = \emptyset$, then $|S_j| = p$, where $1 \leq i, j \leq 2$ and $i \neq j$.*

Proof. Without loss of generality let $X \cap S_1 = \emptyset$. Suppose to the contrary that $|S_2| > p$. Since for any $a \in S_2 \setminus X$, $d(a, X) = 2$ then

$$F_1(X) \geq n_1 + 2(n_2 - p)$$

where $n_1 = |S_1|$ and $n_2 = |S_2|$. Let $y_1 \in S_1$ then for $Y = \{x_2, \cdots, x_p\} \cup \{y_1\}$ we have

$$F_1(Y) = (n_1 - 1) + (n_2 - p + 1) = n_1 + n_2 - p < F_1(X)$$

contrary to the assumption that X is an optimal solution. □

Similarly we have the following lemma.

Lemma 3.2. *Let $G(S_1, S_2, E)$ be an unweighted complete bipartite graph and $X = \{x_1, ..., x_p\} \subseteq V$ be an optimal solution for the p-center problem. If $X \cap S_i = \emptyset$, then $|S_j| = p$, where $1 \leq i, j \leq 2$ and $i \neq j$.*

Proposition 3.3. *Let $G(S_1, S_2, E)$ be an unweighted complete bipartite graph and $p < min\{|S_1|, |S_2|\}$. Then the optimal solution of the p-median and p-center problems is a set $X = \{x_1, ..., x_p\} \subseteq V$ such that $x_1 \in S_1$ and $x_2 \in S_2$. The other $p - 2$ medians and centers can be chosen arbitrary from S_1 or S_2. The value of objective functions of the p-median and p-center problems are $F_1(X) = n - p$ and $F_2(X) = 1$, respectively.*

Proof. Suppose that $|S_1| = n_1$, $|S_2| = n_2$ and $p < min\{n_1, n_2\}$. Let $X = \{x_1, ..., x_p\} \subseteq V$ be an optimal solution for the p-median problem. If $X \subseteq S_1$, then $S_1 \setminus X \neq \emptyset$. Let $x_1 \in S_1 \setminus X$. It follows that $F_1(X) \geq n_1 + n_2 - p + 1$, since $d(x_1, X) = 2$. Let $y_1 \in S_2$, and let $Y = \{x_2, ..., x_p\} \cup \{y_1\}$. Then $F_1(Y) < F_1(X)$, a contradiction. We deduce that $X \not\subseteq S_1$. Similarly, $X \not\subseteq S_2$. So $X \cap S_1 \neq \emptyset$ and $X \cap S_2 \neq \emptyset$. Without loss of generality assume that $x_1 \in S_1$ and $x_2 \in S_2$. Now it is straightforward to see that the value of the objective function is $n - p$. This completes the proof. The proof for p-center problem is similar, and therefore is omitted. □

An extension of the Lemma 3.3 can be stated on the complete k-partite graphs. The proof is similar to the proof of 3.3, and therefore is omitted.

Theorem 3.4. *Let $G(S_1, ..., S_k, E)$ be an unweighted complete k-partite graph. Suppose the optimal solution of the p-median and p-center problems is a set $X = \{x_1, ..., x_p\} \in V$, then*

(1) *if $k \geq p$ then $x_1 \in S_1, x_2 \in S_2, ...,$ and $x_p \in S_p$.*
(2) *if $k < p$ then $x_1 \in S_1, x_2 \in S_2, ...,$ and $x_k \in S_k$. The other $p-k$ medians and centers can be chosen arbitrary from any S_i for $i = 1, \cdots, k$.*

The value of objective function of the p-median and p-center problems are $F_1(X) = n - p$ and $F_2(X) = 1$, respectively.

For an unweighted complete bipartite graph $G(S_1, S_2, E)$ if $|S_i| = 1$ for $i \in \{1, 2\}$, then we can easily see that any optimal solution of the p-median (p-center) problem contains S_i. If $|S_i| = 2$ and $p > 2$, then any solution of the p-median (p-center) problem contains S_i. If $|S_i| = 2$ and $p = 2$, then for a solution X of the p-median (p-center) problem $|X \cap S_i| \leq 2$.

We next study the p-median and the p-center problem in graphs obtaining from complete bipartite graphs by removing some perfect matchings.

Proposition 3.5. *Let $G(S_1, S_2, E)$ be an unweighted complete bipartite graph with $min\{|S_1|, |S_2|\} \geq 3$, and M be a perfect matching of G. The optimal solution of the 2-median and 2-center problems on $G \backslash M$ is a set $X = \{x_1, x_2\} \in V$ such that $x_1 \in S_1$ and $x_2 \in S_2$ and $e_{x_1 x_2} \in M$. The value of objective function of the 2-median and 2-center problems are $F_1(X) = n-2$ and $F_2(X) = 1$, respectively.*

Proof. Let $G_1 = G \setminus M$. We state a proof for 2-median. The proof for the 2-center is similar. Let $X = \{x_1, x_2\}$ be an optimal solution for the 2-median problem. If $X \cap S_1 = \emptyset$, then we let $Y = \{x_1, y_1\}$, where $y_1 \in S_1$. Then $F_1(Y) < F_1(X)$. This is a contradiction. So $X \cap S_1 \neq \emptyset$, and similarly $X \cap S_2 \neq \emptyset$. Suppose that $x_1 \in S_1$ and $x_2 \in S_2$. Since M is a perfect matching, there is a vertex $y \in S_2$ such that $x_1 y \in M$. If $y \neq x_2$, then for $Z = \{x_1, y\}$ we have $F_1(Z) < F_1(X)$ which is a contradiction. Hence, $y = x_2$, and the proof is completed. □

Similarly, the following is verified.

Proposition 3.6. *Let $G(S_1, S_2, E)$ be an unweighted complete bipartite graph with $|S_1| = 2$, $|S_2| \geq 2$. Let M be a perfect matching of G. Then any optimal solution of the 2-median and 2-center problems on $G \backslash M$ is a set $X = \{x_1, x_2\} \in V$ such that either (1) $X = S_1$, or (2) $x_1 \in S_1$ and $x_2 \in S_2$ and $e_{x_1 x_2} \in M$. The value of objective function of the 2-median and 2-center problems are $F_1(X) = n - 2$ and $F_2(X) = 1$, respectively.*

Also in the general case we have the following result which its proof is straightforward.

Theorem 3.7. *Let $G(S_1, S_2, E)$ be an unweighted complete bipartite graph on n vertices and $|S_1| \leq |S_2|$. Let $M_1, M_2, ..., M_l$ be l perfect matchings of G, where $l \leq \lceil \frac{|S_1|-1}{2} \rceil$, and let $G_1 = G \setminus (\cup_{i=1,...,l} M_i)$. Then the optimal solution of the 2-median (2-center) problem on G_1 is a set $X = \{x_1, x_2\} \in V$ such that $x_1 \in S_1$ and $x_2 \in S_2$ and $e_{x_1 x_2} \in M_j$ for some j. Further,*
(1) The value of objective function for 2-median problem is $F_1(X) = n - 2 + 2(l-1)$,
(2) The value of objective function for 2-center problem is $F_2(X) = 1$ if $l = 1$ and $F_2(X) = 2$ otherwise.

Proof. Let $X = \{x_1, x_2\} \in V$ be a solution for the 2-median problem, where $x_1 \in S_1$. If $e_{x_1 x_2} \notin M_i$ for some $i = 1, 2, .., l$, then $X_1 = \{x_1, y_1\}$ is a solution for the 2-median problem on G_1, where $y_1 \in S_2$ and $e_{x_1 y_1} \in M_1$. Then $F_1(X_1) < F_1(X)$, a contradiction. Thus $x_2 \in S_2$ and $e_{x_1 x_2} \in M_j$ for some j. Let x be a vertex of $G_1 - X$. If x is not adjacent to x_1 or x_2, then $d(x, X) = 2$, since $l \leq \lceil \frac{|S_1|-1}{2} \rceil$. This implies that $F_1(X) = n - 2 + 2(l-1)$. The proof for the 2-center problem is similar. □

Example 3.8. Consider the bipartite graph depicted in Figure 1 which all of its weights are equal to one. Table 1 contains some matchings that can be deleted from this bipartite graph without losing connectivity for this graph.

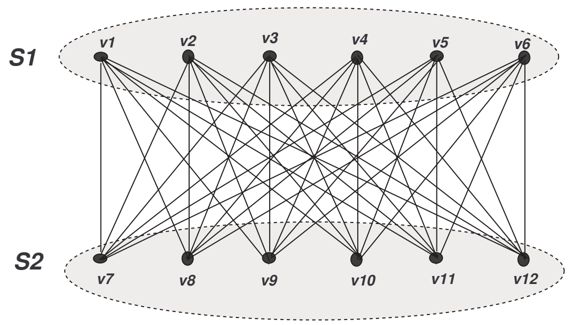

Figure 1. The bipartite graph for Example 3.8.

Figure 2 show the bipartite graph after deleting all possible matching. The results of using Theorem 3.7 after deleting each of matching from the bipartite graph for the 2-median and 2-center are shown in Table 2.

4. Summary and conclusion

We considered the unweighted p-median and p-center problems on bipartite graphs. We showed that the solutions of these problems on complete bipartite

matching	edges
M_1	$\{v_1v_7, v_2v_8, v_3v_9, v_4v_{10}, v_5v_{11}, v_6v_{12}\}$
M_2	$\{v_1v_8, v_2v_9, v_3v_{10}, v_4v_{11}, v_5v_{12}, v_6v_7\}$
M_3	$\{v_1v_9, v_2v_{10}, v_3v_{11}, v_4v_{12}, v_5v_7, v_6v_8\}$

Table 1. The perfect matchings.

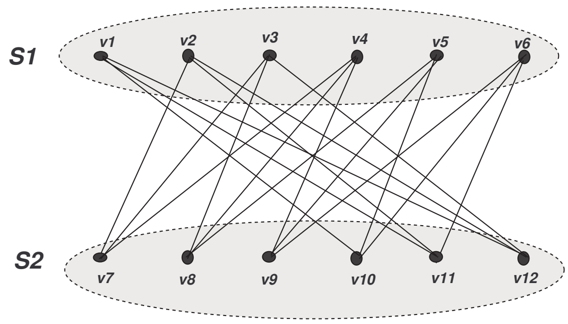

Figure 2. The bipartite graph after deleting perfect matchings.

deleted matchings	X^*	objective function	
		2-median	2-center
$\emptyset$	$\{v_1, v_7\}$	10	1
M_1	$\{v_1, v_7\}$	10	1
M_1, M_2	$\{v_1, v_7\}$	12	2
M_1, M_2, M_3	$\{v_1, v_7\}$	14	2

Table 2. The solutions of unweighted problems on bipartite graphs.

graphs can be found in $O(1)$ time. For the case $p = 2$ the solution also can be find on any connected subgraph of complete bipartite graphs that obtained by deleting perfect matchings. As the future works one may extend the considered problems on line graphs. For the center problem the solution may be computed by finding diameter of graph. For study of the diameter of line graphs see [13].

Acknowledgments

The authors would like to express their sincere gratitude to the anonymous referee for his/her careful reading of the manuscript and his/her comments which resulted in the improvement of the paper.

References

1. S. Bespamyatnikh, B. Bhattacharya, M. Keil, D. Kirkpatrick, M. Segal, Efficient algorithms for centers and medians in interval and circular-arc graphs, *Networks*, **39**, (2002), 144-152.
2. Z. Drezner, H. Hamacher, *Facility location: Applications and Theory*, Springer-Verlag, Berlin, 2002.
3. G. Frederickson, Parametric search and locating supply centers in trees, *Proceedings of Workshop on Algorithms and Data Structures*, (1991), 299-319.
4. B. Gavish, S. Sridhar, Computing the 2-median on tree networks is $O(n \log n)$ time, *Networks*, **26**, (1995), 305-317.
5. A. J. Goldman, Optimal center location in simple networks, *Transportation Sci.*, **5**, (1971), 212-221.
6. R. Hassin, A. Tamir, Improved complexity bounds for location problems on the real line, *Operations Research Letters*, **10**, (1991), 395-402.
7. L. K. Hua, Applications of mathematical methods to wheat harvesting, *Chinese Mathematics*, **2**, (1962), 77-91.
8. O. Kariv, S. L. Hakimi, An algorithmic approach to network location problems. Part I: the p-centers, *SIAM J. Appl. Math.*, **37**, (1979), 513-538.
9. O. Kariv, S. L. Hakimi, An algorithmic approach to network location problems. Part II: p-medians, *SIAM J. Appl. Math.*, **37**, (1979), 539-560.
10. Y. F. Lan, Y. L. Wang, H. Suzuki, A linear-time algorithm for solving the center problem on weighted cactus graphs, *Information Processing Letters*, **71**, (1999), 205-212.
11. H. R. Maimani, Median and Center of Zero-Divisor Graph of Commutative Semigroups, *Iranian Journal of Mathematical Sciences and Informatics*, **3**, (2008)), 69-76.
12. P.B. Mirchandani and R. Francis, *Discrete Location Theory*, J.Wiley, 1990.
13. H. S. Ramanea, I. Gutman, A. B. Ganagia, On Diameter of Line Graphs, *Iranian Journal of Mathematical Sciences and Informatics*, **8**, (2013), 105-109.
14. A. Tamir, Improved complexity bounds for center location problems on networks by using dynamic data structures, *SIAM Journal of Discrete Mathematics*, **1**, (1988), 377-396.
15. A. Tamir, An $O(pn^2)$ algorithm for the p-median and related problems on tree graphs,*Operations Research Letters*, **19**, (1996), 59-64.