Iranian Journal of Mathematical Sciences and Informatics
Vol. 10, No. 1 (2015), pp 149-155
DOI: 10.7508/ijmsi.2015.01.012

Epi-Cesaro Convergence

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Abstract. Since the turn of the century there have been several notions of convergence for subsets of metric spaces appear in the literature. Appearing in as a subset of these notions is the concepts of epi-convergence. In this paper we peresent definitions of epi-Cesaro convergence for sequences of lower semicontinuous functions from X to $[-\infty, \infty]$ and Kuratowski Cesaro convergence of sequences of sets. Also we characterize the connection between epi-Cesaro convergence of sequences of functions and Kuratowski Cesaro convergence of their epigarphs.

Keywords: Cesaro convergence, Epi-convergence, Epi-Cesaro convergence, Lower semicontinuous function.

2000 Mathematics subject classification: 40A05, 40A35.

1. Introduction and Background

During the past five decades new concepts of convergence for sequences of functions have been appearing in mathematical analysis. These concepts are especially designed to approach the limit of sequences of variational problems and are called variational convergence. With each type of variational problem

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Received 02 November 2014; Accepted 16 February 2015

is associated a particular concept of convergence. In [3], Attouch developed a convergence theory for sequences of functions, called epi-convergence. This concepts of convergence has natural applications in all branches of optimization theory. In this paper, we will introduce a new convergence kind for sequences of function sequences and call it epi-Cesaro convergence.

To facilitate this process we recall the basic definitions and concepts (see [1]-[16]). The Cesaro limit superior and Cesaro limit inferior a real sequence (x_n) are defined as follow:

$$(C,1) - \limsup_n x_n = \inf_{n\geq 1} \sup_{m\geq n} \frac{1}{m} \sum_{k=1}^{m} x_k$$

and

$$(C,1) - \liminf_n x_n = \sup_{n\geq 1} \inf_{m\geq n} \frac{1}{m} \sum_{k=1}^{m} x_k.$$

The sequence $x = (x_n)$ is Cesaro convergent if and only if

$$(C,1) - \limsup_n x_n = (C,1) - \liminf_n x_n.$$

The following characterization may be found in [9].
A sequence (x_n) is Cesaro convergent to ℓ, provided

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_k = \ell.$$

In this case we shall write $(C,1) - \lim x_n = \ell$.
The notion of Cesaro convergence extend the usual concept of convergence in a non-trivial fashion. We know that a convergent sequence is a Cesaro convergent sequence. But the converse does not holds in general. For example, the sequence $x = (x_n) = (1,0,1,0,...)$ is Cesaro convergent $\frac{1}{2}$, however this sequence is not convergent.

Let (X,d) be a metric space. An extended real-valued function $f : X \to [-\infty,\infty]$ on a metric space X is called lower semicontinuous provided its epigraph

$$epif \equiv \{(x,\alpha) : x \in X,\ \alpha \in \mathbb{R} \text{ and } \alpha \geq f(x)\}$$

is closed subset of $X \times \mathbb{R}$. Given a sequence (f_n) of lower semicontinuous functions from X into $[-\infty,\infty]$, we say that (f_n) is epi-convergent to f, and we write $f = lim_e f_n$, provided at each $x \in X$, the following two conditions both hold:

(i) whenever (x_n) is convergent to x, we have $f(x) \leq \liminf f_n(x_n)$;
(ii) there exists a sequence (x_n) convergent to x such that $f(x) = \lim f_n(x_n)$.

Although closely connected to the notion of pointwise convergence it is neither stronger nor weaker. In fact, certain of functions have different pointwise and epi-limits. Consider the sequence

$$f_n(x) = \begin{cases} 0, & \text{if } x = \frac{1}{n} \\ \\ 1, & \text{if } x \neq \frac{1}{n} \end{cases}$$

that pointwise convergent to the function $h(x) = 1$ for all x and epi-convergent to

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0. \end{cases}$$

The epi-limit takes into account the behaviour of the f in the neighborhood of 0, whereas the pointwise limit restricts attention to what happens with the f_n at the point 0.

2. Main Results

Definition 2.1. Let (X, d) be a metric space, for every $x \in X$, let us denote the system of the neighbourhood of x by $U(x)$. With any sequence (f_n) of lower semicontinuous functions from X into $[-\infty, \infty]$ are associated two Cesaro limit functions:

(iii) The epi-Cesaro limit inferior of the sequence (f_n), denoted by $(C, 1) - li_e f_n$ is defined by

$$(C, 1) - li_e f_n(x) = \sup_{V \in U(x)} (C, 1) - \liminf_n \inf_{u \in V} f_n(u).$$

(iv) The epi-almost limit superior of the sequence (f_n), denoted by $(C, 1) - ls_e f_n$ is defined

$$(C, 1) - ls_e f_n(x) = \sup_{V \in U(x)} (C, 1) - \limsup_n \inf_{u \in V} f_n(u).$$

Definition 2.2. Let (X, d) be a metric space and (f_n) be a sequence of lower semicontinuous functions from X into $[-\infty, \infty]$. This sequence (f_n) is said to be epi-Cesaro convergent at x, if the following equality holds:

$$(C, 1) - li_e f_n(x) = (C, 1) - ls_e f_n(x).$$

This common value is then denoted $(C, 1) - lim_e f_n(x)$:

$$(C, 1) - lim_e f_n(x) = (C, 1) - li_e f_n(x) = (C, 1) - ls_e f_n(x).$$

For lower semicontinuous functions, equivalent definition can be given as following.

Definition 2.3. Given a sequence (f_n) of lower semicontinuous function on a metric space (X, d), we say that (f_n) is epi-Cesaro convergent to f provided at each $x \in X$, the following two conditions both hold:

(v) whenever (x_n) is Cesaro convergent to x,we have $f(x) \leq (C,1)-\liminf f_n(x_n)$;
(vi) there exists a sequence (x_n) Cesaro convergent to x such that $f(x) = (C,1)-\lim f_n(x_n)$.

In this case we write $(C,1)-lim_e f_n = f$.

The notion of pointwise Cesaro convergence it is neither stronger nor weaker than epi-Cesaro convergence. In fact, there exist some functions that have different pointwise Cesaro and epi-Cesaro limits.

Example 2.4. Let

$$a_k = \begin{cases} (-1)^n, & \text{if} \quad k = n^2 \quad k = 1,2,3,... \\ 0, & \quad otherwise. \end{cases}$$

Define the following function sequence:

$$f_n(x) = \sum_{k=1}^{n} a_k.$$

Since

$$\frac{1}{k^2}\sum_{i=1}^{k^2}\sum_{l=1}^{i} a_l = \frac{k^2-(k-1)^2+(k-2)^2-(k-3)^2+...+2^2-1}{k^2}$$

$$= \frac{(1^2+3^2+5^2+...+k^2)-(2^2+4^2+6^2+...+(k-1)^2)}{k^2} = \frac{1}{2}\frac{k+1}{k}$$

if k is odd and

$$\frac{1}{k^2}\sum_{i=1}^{k^2}\sum_{l=1}^{i} a_l = \frac{(k-1)^2-(k-2)^2+(k-3)^2-(k-4)^2+...+2^2-1}{k^2}$$

$$= \frac{(1^2+3^2+5^2+...+(k-1)^2)-(2^2+4^2+6^2+...+(k-2)^2)}{k^2} = \frac{1}{2}\frac{k-1}{k}$$

if k is even, the sequence $(f_n(x))$ is Cesaro convergent to the function $f(x) = \frac{1}{2}$. However this sequence is epi-Cesaro convergent to the function $f(x) = -1$.

Example 2.5. If

$$f_n(x) = \begin{cases} x^2, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even,} \end{cases}$$

then the sequence $(f_n(x))$ is Cesaro convergent to the function $f(x) = \frac{x^2}{2}$ but epi-Cesaro convergent to the function $f(x) = 0$ that is $(C,1)-lim_e f_n(x) = 0$.

Example 2.6. Let

$$a_k = \begin{cases} (-1)^n, & \text{if} \quad k = 2^n \quad k = 1,2,3,... \\ 0, & \quad otherwise. \end{cases}$$

Define the following function sequence:

$$f_n(x) = \sum_{k=1}^{n} a_k.$$

Since

$$\frac{1}{2^k}\sum_{i=1}^{2^k}\sum_{l=1}^{i} a_l = \frac{2^k - 2^{k-1} + 2^{k-2} - ... + 2 - 1}{2^k}$$
$$= 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + ... + \frac{1}{2^{k-1}} - \frac{1}{2^k} = \frac{1}{3}(2 - \frac{1}{2^k})$$

if k is odd and

$$\frac{1}{2^k}\sum_{i=1}^{2^k}\sum_{l=1}^{i} a_l = \frac{2^{k-1} - 2^{k-2} + 2^{k-3} - ... + 2 - 1}{2^k}$$
$$= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^3} + ... + \frac{1}{2^{k-1}} - \frac{1}{2^k} = \frac{1}{3}(1 - \frac{1}{2^k})$$

if k is even, we have $(C,1) - \liminf f_n(x) = \frac{1}{3}$ and $(C,1) - \limsup f_n(x) = \frac{2}{3}$, that is the sequence $(f_n(x))$ is not Cesaro convergent. However this sequence is epi-Cesaro convergent to the function $f(x) = -1$.

Definition 2.7. Let (A_n) be a sequence of closed subsets of metric space (X,d). We say that (A_n) is Kuratowski Cesaro convergent to a closed subset A of X provided $A = (C,1) - LiA_n = (C,1) - LsA_n$ where

$$(C,1) - LiA_n = \{x \in X : \text{ there exist a sequence } \quad (a_n) \quad \text{Cesaro}$$
$$\text{convergent to} \quad x \text{ with } \quad a_n \in A_n \quad \text{ for all but finitely integers } \quad n\}$$
$$(C,1) - LsA_n = \{x \in X : \text{there exists positive integers}$$
$$n_1 < n_2 < n_3 < ..., \text{and} \quad a_k \in A_{n_k} \text{ such that } \quad (C,1) - \lim_{k\to\infty} a_k = x\}$$

in this case we write $A = (C,1) - LimA_n$.

Theorem 2.8. *Let (X,d) be a metric space and (f_n) be a sequence of lower semicontinuous functions from X into $[-\infty,\infty]$. The Cesaro limit sets $(C,1) - Li(epif_n)$ and $(C,1) - Ls(epif_n)$ are still epigraphs. They are equal respectively to the epigraphs of $(C,1) - lif_n$ and $(C,1) - lif_n$ that is,*

$$(C,1) - Li(epif_n) = epi((C,1) - ls_e f_n) \tag{2.1}$$

and

$$(C,1) - Ls(epif_n) = epi((C,1) - li_e f_n) \tag{2.2}$$

Proof. Let us first prove (2.1). By definition $(C,1) - Li$, $(x,\alpha) \in (C,1) - Li(epif_n)$ if and only if for all $V \in U(x)$ and for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ such that there exists $x_k \in V$ satisfying

$$\alpha + \epsilon > \frac{1}{m}\sum_{k=1}^{m} f_k(x_k).$$

for $m \geq n$.

This can be reformulated in the following way:

$$\alpha > \sup_{V \in U(x)} \inf_{n} \sup_{m \geq n} \inf_{u \in V} \frac{1}{m} \sum_{k=1}^{m} f_k(u)$$

that is

$$\alpha > \sup_{V \in U(x)} \limsup_{n} \inf_{u \in V} \frac{1}{m} \sum_{k=1}^{m} f_k(u) = ((C,1) - ls_e f_n)(x)$$

which means $(x, \alpha) \in epi((C,1) - ls_e f_n)$.

In view of the definition of $(C,1) - Li(epi f_n)$, the proof of (2.2) follows from exactly the same argument as above. □

We are now able to state the main result of this paper and establish the equivalence between epi-Cesaro convergence of a sequence of functions and the Kuratowski Cesaro convergence of their epigraphs. It is direct consequence of Definition 2.3 and Theorem 2.8.

Theorem 2.9. *Let (X, d) be a metric space and (f_n) a sequence of lower semi-continuous functions from X into $[-\infty, \infty]$. The sequence (f_n) is epi-Cesaro convergent if and only if the sequence of sets $(epi f_n)$ is Cesaro convergent in the Kuratowski sense. In that case following equality holds:*

$$epi((C,1) - lim_e f_n) = (C,1) - Lim(epi f_n).$$

Theorem 2.9 allows us to view epigraphs, as epi-Cesaro convergence of a sequence of functions in terms of set Cesaro convergence.

Acknowledgments

The authors are indebted to an anonymous referee for his/her suggestions and helpful remarks.

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