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On Twin-good Rings

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Abstract. In this paper, we investigate various kinds of extensions of twin-good rings. Moreover, we prove that every element of an abelian neat ring R is twin-good if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. The main result of [25] states some conditions that any right self-injective ring R is twin-good. We extend this result to any regular Baer ring R by proving that every element of a regular Baer ring is twin-good if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Also we illustrate conditions under which extending modules, continuous modules and some classes of vector space are twin-good.

Keywords: Twin-good ring, Neat ring, Regular Baer ring, π-Regular baer ring.

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1. Introduction

Many authors have studied rings generated additively by their unit elements, (See [1], [2], [3], [8], [9], [13], [14], [15], [23],). The rings in which each element is the sum of k units were called (s, k)-rings by Henriksen [13]. *Vámos* has

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called such rings *k-good* rings [27]. Particularly, a ring R is called 2-*good* if each element of R can be expressed as the sum of 2 units in R . A ring R is said to be *twin-good* if for each $x \in R$ there exists a unit $u \in R$ such that both $x + u$ and $x - u$ are units in R (See [22]). Clearly every twin-good ring is 2-good, but the reverse doesn't always hold. For example, $\mathbb{Z}_3$ is 2-good but not twin-good. For the first time this concept was discussed by Chen. Chen [6, Theorem 3], proved that for an exchange ring with primitive factors artinian, there exists a $u \in U(R)$ such that $1_R \pm u \in U(R)$ if and only if, for any $a \in R$, there exists a $u \in U(R)$ such that $a \pm u \in U(R)$. In other words, He proved that an exchange ring with primitive factors artinian is twin-good if and only if 1_R is twin-good.

In this paper in section 2, we give some examples of twin-good rings and their related properties. In particular, we investigate some of extensions of twin-good rings. Also, we prove that each element of any abelian neat ring R is twin-good if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Srivastava and Siddique ([25]) proved that every right self-injective ring R is twin-good if and only if no factor ring of R isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Since every regular right self-injective ring is a Baer ring the natural question which arises from [25] is that: which regular Baer rings are twin-good? In this paper we investigate conditions that under which regular Baer and π-regular Baer rings are twin-good. Also we will discuss on extending modules, continuous modules and some classes of victor space which are twin-good.

2. Examples and Basic Properties

Throughout this paper all rings are considered associative with identity element. For a ring R, $J(R)$ will denote the Jacobson radical of R and $M_n(R)$ denotes the n by n matrix ring over R. We use $|X|$ and c to denote the cardinality of a set X and the cardinality of the continuum, respectively. Before discussing the main results we bring some example and properties of twin-good rings.

Example 2.1. (i) Every divisor ring D, which is not isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$, is twin-good. For $a \in D$ if $a = 0$, then $a \pm 1_D \in U(D)$. If $a \neq 0$, then there exists $b \neq 0, 1$ such that $a(1 \pm b) \in U(D)$.
(ii) For every ring R, $J(R)$ is twin good.
(iii) If $R \neq 0$ is a local ring and $2, 3 \in U(R)$, then R is twin-good.

The following observations were noted in [22], and their proofs are straightforward.

Lemma 2.2. *For a ring R, we have the following:*
(i) *If R is twin-good then for any proper ideal I of R, the factor ring R/I is also twin-good.*

(ii) If a factor ring R/I is twin-good and $I \subseteq J(R)$, then R is twin-good. Thus, in particular, it follows that a ring R is twin-good if and only if $R/J(R)$ is twin-good.
(iii) If R is a direct product of rings R_i where each R_i is twin-good, then R is also twin-good.

Lemma 2.3. *If R is an abelian regular ring, then $M_n(R)$ is twin-good for each $n \geq 2$.*

Proof. See [25, Corollary 7]. □

Remark 2.4. By Lemma 2.2 ,and Lemma 2.3 it is obvious that every semisimple ring is twin-good.

Let $S(R)$ be the nonempty set of all proper ideals of R generated by central idempotents. Recall that the factor ring R/P is called a *Pierce stalk* of R if P is a maximal element in $S(R)$ (see [26]).

Proposition 2.5. *For a ring R, the following statements are equivalent:*
(1) *R is twin-good.*
(2) *All homomorphism images of R are twin-good.*
(3) *All indecomposable factor rings of R are twin-good.*
(4) *R/I is twin-good for every ideal I of R contained in $J(R)$.*
(5) *A/I is twin-good for every proper ideal I of R generated by central idempotents of R.*
(6) *All Pierce stalks of R are twin-good.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (2) $\Rightarrow$ (5) $\Rightarrow$ (6) are trivial.
(4) $\Rightarrow$ (1) See Lemma 2.2.
(6) $\Rightarrow$ (1): If R is not twin-good, put
$\Omega =\{I \lhd R | I$ is a proper ideal generated by central idempotents of R such that R/I is not twin-good$\}$.
Then $\Omega \neq \phi$ since $0 \in \Omega$. It is clear that Ω contains a maximal element M. We next prove that R/M is a Pierse stalk. Assume the contrary, so there is a central idempotent e such that $M + eR$ and $M + (1 - e)R$ are proper ideals of R and properly contain M. Since $(M + eR) \cap (M + (1 - e)R) = M$, $(M + eR) + (M + (1 - e)R) = R$, by Chinese Remainder Theorem, $R/M \cong R/(M + eR) \times R/(M + (1 - e)R)$. The maximality of M implies that $M + eR$ and $M + (1 - e)R$ are not in Ω, hence $R/(M + eR)$ and $R/(M + (1 - e)R)$ are twin-good. So R/M is twin-good, and it yields a contradiction. Thus R/M is a Pierce stalk, but R/M also is twin-good, which is a contradiction with hypothesis.
(3) $\Rightarrow$ (1) It is similar to (6) $\Rightarrow$ (1), so we omit the proof. □

Let $R[[x, \sigma]]$ denote the ring of skew formal power series over R, that is all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. We have:

Proposition 2.6. *Let R be a ring. Then the ring $R[[x,\sigma]]$ is twin-good if and only if R is twin-good. In particular, $R[[x]]$ is twin-good if and only if R is twin-good.*

Proof. If $R[[x,\sigma]]$ is twin-good, by $R \cong R[[x,\sigma]]/(x)$ and Proposition 2.5, R is twin-good. Conversely, suppose that R is twin-good. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x,\sigma]]$. There exist $u, v, w \in U(R)$, such that, $a_0 = u + v$, $a_0 = -u + w$. Then $f(x) = u + (v + a_1x + a_2x^2 + \ldots)$, $f(x) = -u + (w + a_1x + a_2x^2 + \ldots)$, where $(v + a_1x + a_2x^2 + \ldots)$, and $(w + a_1x + a_2x^2 + \ldots) \in U(R[[x,\sigma]])$. Thus $R[[x,\sigma]]$ is twin-good. □

Recall that a ring R is said to be *semicommutative* if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. Commutative rings, symmetric rings, reversible rings and one-sided duo rings are all semicommutative (See [5]).

Remark 2.7. If R is semicommutative, then the polynomial ring $R[x]$ is not twin-good. Therefore; if R is commutative, symmetric, reversible, and one-sided duo, then the polynomial ring $R[x]$ is not twin-good.

Remark 2.8. A subring of a twin-good ring and the polynomial ring over a twin-good ring need not be twin-good.
Indeed, if $\mathbb{Q}$ is the rational number field, then $\mathbb{Q}$ and $\mathbb{Q}[[x]]$ are both twin-good, but the polynomial ring $\mathbb{Q}[x]$ over $\mathbb{Q}$, as a subring of $\mathbb{Q}[[x]]$ is not twin-good.

Following Goodearl-Menal [11], an associative ring R is said to have *unit* 1-*stable range* if $aR + bR = R$ with $a, b \in R$ implies that there exists some $u \in U(R)$ such that $a + bu \in U(R)$. For example algebraic algebras over infinite field satisfies unit 1-stable range. Here we have:

Proposition 2.9. *Every ring R satisfying unit 1-stable range is twin-good.*

Proof. For any $a \in R$, there exists $u \in U(R)$ such that $a \pm 1.u \in U(R)$ since $aR + 1.R = R$. □

Corollary 2.10. *If R is an algebraic algebra over an infinite field F, then R is twin-good.*

Corollary 2.11. *If R satisfies unit 1-stable range condition, and e is any idempotent in R, then eRe is twin-good.*

Proof. The result follows from [28, Theorem 2.8] and Proposition 2.9. □

Corollary 2.12. *Let $e^2 = e \in R$. Then eR is twin-good if and only if so is eRe.*

Proof. Put $\sigma : eR \to eRe, \sigma(x) = xe$. It is easy to see that σ is an epimorphism, $ker\sigma = eR(1-e), eR/ker\sigma \cong eRe$, and $ker\sigma \subseteq J(eR)$ since $(ker\sigma)^2 = 0$. So by Lemma 2.2, the result follows. □

Proposition 2.13. (1) *Let $e^2 = e \in R$. If eRe and $(1-e)R(1-e)$ are both twin-good, then R is twin-good.*
(2) *Let e be a central idempotent of R. Then R is twin-good if and only if so are eR and $(1-e)R$.*

Proof. (1) See [22, Theorem 6.11].
(2) Since e is a central idempotent the result follows from (1) and Proposition 2.5 (2). □

Corollary 2.14. *If $\{e_1, e_2, \dots, e_m\}$ is a complete set of pairwise orthogonal idempotents in a ring R and each e_iRe_i is twin-good, then so is R.*

Proof. By Proposition 2.13(1), and induction on m, R is twin-good. □

Corollary 2.15. *If R is twin-good, then so is the matrix ring $M_n(R)$ for any positive integer n.*

Corollary 2.16. *Let $M_1, M_2, \dots, M_n$ be submodules of M. If $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$ are modules and $End(M_i)$ is twin-good for each i, then $End(M)$ is twin-good.*

Proposition 2.17. *Let $\{e_1, \dots, e_n\}$ be a set of idempotents in a ring R. If e_iRe_i is twin-good, for each $i = 1, \dots, n$. Then the ring,* $\begin{pmatrix} e_1Re_1 & \dots & e_1Re_n \\ \dots & \dots & \dots \\ e_nRe_1 & \dots & e_nRe_n \end{pmatrix}$ *is twin-good.*

Proof. By Proposition 2.13, the result holds for $n = 2$. Assume inductively that the result holds for $n = k \geq 2$. Let $n = k+1$, and let

$$C = \begin{pmatrix} e_2Re_2 & \dots & e_2Re_{k+1} \\ \dots & \dots & \dots \\ e_{k+1}Re_2 & \dots & e_{k+1}Re_{k+1} \end{pmatrix}_{k\times k},$$

$$B = \begin{pmatrix} e_2Re_1 \\ \vdots \\ e_{k+1}Re_1 \end{pmatrix}_{k\times 1}, A = \begin{pmatrix} e_1Re_2 & \dots & e_1Re_{k+1} \end{pmatrix}_{1\times k}.$$

Then C is twin-good. Now take $D = \begin{pmatrix} r & a \\ b & c \end{pmatrix} \in \begin{pmatrix} e_1Re_1 & A \\ B & C \end{pmatrix}$, similar to the proof of Proposition 2.13(1), we can see that D is twin-good. □

Theorem 2.18. *Let R be twin-good. Then the following statements hold:*
(1) *For any $n \in N$, the ring $T_n(R)$ of $(n \times n)$ upper triangular matrices over R is twin-good.*
(2) *Put $QM_2(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a+b = c+d, a, b, c, d \in R\}$. Then $QM_2(R)$ is twin-good.*

(3) *For any* $n \in N$, $S_n(R) = \{ \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_0 & \dots & a_{n-3} \\ \dots & & \dots & & \dots \\ 0 & 0 & 0 & \dots & a_0 \end{pmatrix} | a_i \in R, i = 0, 1, \dots, n-1\}$ *is twin-good.*
(4) *For any* $n \in N$, $R[x]/(x^n)$ *is twin-good, where* (x^n) *is the ideal generated by* x^n.

Proof. (1) See [22, Corollary 6.15].
(2) Put $\psi : QM_2(R) \to T_2(R)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & b \\ 0 & d-b \end{pmatrix}$. Then ψ is a monomorphism of rings. Also, for any $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in T_2(R)$, we have

$$\psi(\begin{pmatrix} x-y & z \\ x-y-z & y+z \end{pmatrix}) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$$

Hence ψ is an isomorphism of rings. This completes the proof by (1).
(3) Let $A = (a_{ij}) \in S_n(R)$, where $a_{ij} = 0$ if $i > j$. By hypothesis there exist $u_i, v_i, w_i \in U(R)$ such that $a_0 = u_i + v_i$, $a_0 = -u_i + w_i$, for each $1 \le i \le n$. Then $A = diag(u_1, u_2, \dots, u_n) + B$, $A = diag(-u_1, -u_2, \dots, -u_n) + C$ where $B = (b_{ij}), C = (c_{ij})$ with $b_{ii} = v_i, c_{ii} = w_i (1 \le i \le n)$ and $b_{ij} = c_{ij} = a_{ij}$ for $(i \ne j)$. It is clear that $diag(u_1, \dots, u_n), B, C \in U(S_n(R))$.
(4) Note that $R[x]/(x_n) \cong S_n(R)$, we obtain the result by (3). □

A ring R is called right *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular, such that $ab_1 = ba_1$. It is a well-known fact that R is a right Ore ring if and only if the classical right quotient ring of R exists.

Proposition 2.19. *Let R be a right Ore ring and Q be the classical right quotient ring of R. If R is twin-good, then so is Q.*

Proof. For any $r = ab^{-1} \in Q$, where $a, b \in R$ with b regular. By hypothesis there exist $u, v, w \in U(R)$ such that $a = u+v$ and $a = -u+w$. Hence $r = ub^{-1} + vb^{-1}$ and $r = -ub^{-1} + wb^{-1}$. It is clear that $(ub^{-1})^{-1} = bu^{-1}, (vb^{-1})^{-1} = bv^{-1}$ and $(wb^{-1})^{-1} = bw^{-1}$ thus ub^{-1}, vb^{-1} and $wb^{-1} \in U(Q)$. Therefore Q is twin-good. □

The converse of Proposition 2.19 is not true. For example, the rational number field $\mathbb{Q}$ is the classical right quotient ring of $\mathbb{Z}$, but $\mathbb{Z}$ is not twin-good.

3. Some Classes Of Twin-good Rings and Modules

Recall that a ring R is said to be clean if every element of R is the sum of a unit and an idempotent. McGovern [19], defined that R is a *neat ring* if every proper homomorphic image of R is clean. In particular, the ring of integers, $\mathbb{Z}$

and any nonlocal PID are examples of neat rings. A ring R is called *Baer* if the left annihilator of every nonempty subset of R is generated by an idempotent. The concept of a Baer ring was introduced by Kaplansky to abstract properties of rings of operators on a Hilbert space in his 1965 book [16]. In this section we will discuss some conditions under which, abelian neat rings, regular and π-regular Baer rings are twin-good.

Lemma 3.1. *A ring $R \neq 0$ is local if and only if it is clean and 0 and 1 are the only idempotents in R.*

Proof. See [19, Lemma 14]. □

Theorem 3.2. *Let R be an abelian neat ring then the following conditions are equivalent:*

(1) Every element of R is twin-good.
(2) Identity of R is twin-good.
(3) R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
Now, we try to show (3) $\Rightarrow$ (1).
Suppose that R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Now we show that each element of R is twin-good. Suppose, to the contrary, there exists $a \in R$ which is not twin-good.
Let $\Omega =\{\mathrm{I} \mid I$ is an ideal of R and $a+I$ is not twin-good in $R/I\}$ and L be the maximal element of Ω. Clearly R/L is an indecomposable ring and hence has no non-trivial idempotent. Since R/L is a clean ring and has no non-trivial idempotent, by Lemma 3.1, R/L is a local ring. Now let $T = R/L$ then $T/J(T)$ is a division ring. Let $x = a + L$. Since $x + J(T)$ is not twin-good in $T/J(T)$; $T/J(T) \cong \mathbb{Z}_2$ or $T/J(T) \cong \mathbb{Z}_3$, this contradicts the assumption. Thus, each element of R is twin-good. □

Corollary 3.3. *If R is an abelian neat ring and $2, 3 \in U(R)$, then R is twin-good.*

Corollary 3.4. *Every abelian clean ring R is twin-good if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

It is not necessary that every neat ring is twin-good and contrariwise. In other words, the concepts of neat rings and twin-good ring are independent of each other. This is illustrated by examples below.

Example 3.5. (*i*) The ring of integers $\mathbb{Z}$, is a neat ring but it is not twin-good.
(*ii*) every Boolean ring with more than two elements is a neat ring but not twin-good.
(*iii*) Let $R = \{diag(a_1, a_2, \ldots, a_n)|a_1, a_2, \ldots, a_n \in \mathbb{Z}\}$. R is twin-good but $\mathbb{Z}$ is a homomorphic image of R that is not clean; therefore, R is not a neat ring.

Now we will investigate the circumstances that a regular Baer ring is twin-good.

Theorem 3.6. *A ring R is a Baer ring if and only if R itself, regarded as a regular R-module, is a Baer semisimple module.*

Proof. See [12, Theorem 4] $\square$

Theorem 3.7. *Let R be a regular Baer ring then the following conditions are equivalent:*

(1) Every element of R is twin-good.
(2) Identity of R is twin-good.
(3) R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
Now, we try to show (3) $\Rightarrow$ (1).
Suppose that no factor ring of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Now we show that each element of R is a sum of two units. By previous Theorem R itself, regarded as a regular (left) right R-module, is a regular Baer semisimple module; therefore, by [12, Proposition 2] R is the direct sum of a family of Baer simple submodules. This family is not empty. We have $R_R = \bigoplus_{i=1}^{n} M_i$ while the M_i are Baer simple R-submodules of R. Let $R_R = \bigoplus_{j=1}^{r} M_{i_j}^{n_j}$, where $\{M_{i_1}, \ldots, M_{i_r}\}$ is a set of representatives of the isomorphism classes of M_i for $i = 1, \ldots, n$ such that $n_1 + n_2 + \ldots + n_r = n$. Then

$$R \cong End_R(R) \cong End_R(M_{i_1}^{n_1} \oplus \ldots \oplus M_{i_r}^{n_r})$$

$$\cong \begin{pmatrix} Hom(M_{i_1}^{n_1}, M_{i_1}^{n_1}) & Hom(M_{i_1}^{n_1}, M_{i_2}^{n_2}) & \ldots & Hom(M_{i_1}^{n_1}, M_{i_r}^{n_r}) \\ Hom(M_{i_2}^{n_2}, M_{i_1}^{n_1}) & Hom(M_{i_2}^{n_2}, M_{i_2}^{n_2}) & \ldots & Hom(M_{i_2}^{n_2}, M_{i_r}^{n_r}) \\ \vdots & \vdots & \vdots & \\ Hom(M_{i_r}^{n_r}, M_{i_1}^{n_1}) & Hom(M_{i_r}^{n_r}, M_{i_2}^{n_2}) & \ldots & Hom(M_{i_r}^{n_r}, M_{i_r}^{n_r}) \end{pmatrix}$$

Now by this fact that $M_{i_l} \ncong M_{i_{\acute{l}}}$ for $l \neq \acute{l}$ and regularity of $Hom(M_{i_l}, M_{i_{\acute{l}}})$ we have $Hom(M_{i_l}, M_{i_{\acute{l}}}) = 0$; therefore, $Hom(M_{i_l}^{n_l}, M_{i_{\acute{l}}}^{n_{\acute{l}}}) = 0$. So

$$R \cong \begin{pmatrix} Hom(M_{i_1}^{n_1}, M_{i_1}^{n_1}) & 0 & \ldots & 0 \\ 0 & Hom(M_{i_2}^{n_2}, M_{i_2}^{n_2}) & \ldots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \ldots & Hom(M_{i_r}^{n_r}, M_{i_r}^{n_r}) \end{pmatrix}$$

Thus $R \cong \prod_{j=1}^{r} End_R(M_{i_j}^{n_j}) \cong \prod_{j=1}^{r} M_{n_j}(End_R(M_{i_j}))$. As M_{i_j} is a Baer simple $R-$module for each $1 \leq j \leq r$, so $End_R(M_{i_j})$ is a domain by [12, Theorem 2]. In the other hand $D_j := End_R(M_{i_j})$ is a regular domain , thus is a division ring. Since R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$, so each element of $M_{n_j}(D_j)$ for all $1 \leq j \leq r$ is a twin-good. Therefore, R is twin-good. $\square$

Corollary 3.8. *Let R be a regular ring and A its lattice of principal right ideals. If A is a complete lattice, then R is twin-good if has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Corollary 3.9. *Let R be a regular ring with finite Goldie dimension then $M_n(R)$ is twin-good.*

Proof. By [18, Theorems 6.59, 6.62, 7.55, 7.63], $M_n(R)$ is a regular Baer ring, thus $M_n(R)$ is twin-good. □

A ring R is called *π-regular* if for each element $a \in R$ there exists a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^n x a^n$. Since the class of π-regular ring properly contains the class of regular rings, it is interesting to investigate the twin-goodness of π-regular rings.

Theorem 3.10. *Let R be a π-regular Baer ring with $|Id(R)| < c$. If R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ then R is twin-good.*

Proof. Since R is a semilocal ring, $R/J(R)$ is a semisimple ring and $R/J(R) \cong \prod_{l=1}^{r} M_{n_l}(D_l)$ while D_l is division ring for each l, so it is clear that R is twin-good. □

4. Some Class of Twin-good Modules

We recall that the module M_R is twin-good if its endomorphism ring is twin-good. In this Section, we will give necessary and sufficient conditions for some class of modules to be twin-good.

Corollary 4.1. *Let R be a regular ring with finite Goldie dimension. Then every finitely generated free R-module F, is twin-good.*

Corollary 4.2. *If R is a twin-good ring, then every free R-module of finite rank is twin-good.*

Recall that the module M is called an extending module if every closed submodule is a direct summand. Among examples of extending modules, we would mention semisimple modules, injective modules and uniform modules.

Proposition 4.3. *Let M_R is an extending module such that its endomorphism ring S is a regular ring. Then M is twin-good if and only if S has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. If M be an extending module such that its endomorphism ring S is a regular ring, then M is a Baer module, and subsequently S is a Baer ring (See [24, Proposition 4.12]). Therefore, the result follows from Theorem 3.7. □

Srivastava and Siddique [25] proved that every element of a right self-injective ring is twin-good if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ and they extend this result to endomorphism ring of right quasi-continuous module with finite exchange property. As a continuous modules is quasi-continuous with finite exchange property [20, Theorem 3.24], they proved that every element in the endomorphism ring of a continuous module, is twin-good if no factor of its endomorphism ring is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Since every regular self-injective ring is regular Baer ring, as an application of this fact and Theorem 3.7, we give a shorter proof of some results of [25]. As a consequence we get the following result:

Corollary 4.4. *A right self-injective ring R is twin good if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proposition 4.5. *Let M_R be a continuous module. Then each element of the endomorphism ring of M_R is twin-good if and only if it has no factor isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. Let $S = End_R(M)$. If M is a continuous module, by [20, Theorem 3.11 and Proposition 3.5], $\overline{S} = S/J(S)$ is a regular right continuous ring, thus $\overline{S}_{\overline{S}}$ is an extending module with regular endomorphism ring. Therefore, by Proposition 4.3, $\overline{S}$ is twin-good, so is M. □

Recall that if V is a right vector space over a division ring D, then $End_D(V)$ is a regular Baer ring, in fact we have:

Corollary 4.6. *Every element of $End_D(V)$ is twin-good, except when $dim(V_D) = 1$ and $D = \mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Corollary 4.7. *If V is a vector space of finite dimension $n > 1$ over the field $\mathbb{Z}_2$, then V is twin-good.*

Corollary 4.8. *If V is a vector space of countably infinite dimension over an arbitrary field F, then V is twin-good.*

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