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On the Zero-divisor Cayley Graph of a Finite Commutative Ring

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Abstract. Let R be a finite commutative ring. Let $Z(R)$ and $J(R)$ be the set of all zero-divisor elements and the Jacobson radical of R, respectively. The zero-divisor Cayley graph of R, denoted by $Z\mathbb{CAY}(R)$, is the graph obtained by setting all the elements of $Z(R)$ to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x-y \in Z(R)$. The induced subgraph of $Z\mathbb{CAY}(R)$ on the vertex set $Z(R) \setminus J(R)$ is denoted by $Z\mathbb{CAY}^*(R)$. In this paper, the basic properties of $Z\mathbb{CAY}(R)$ and $Z\mathbb{CAY}^*(R)$ are investigated and some characterization results regarding connectedness, girth and planarity of $Z\mathbb{CAY}(R)$ and $Z\mathbb{CAY}^*(R)$ are given. Finally, we study the clique number of $Z\mathbb{CAY}(R)$.

Keywords: Clique, Connectivity, Diameter, Girth, Planar graph.

2010 Mathematics subject classification: 13A99, 05C25, 05C40, 13M05, 16U60.

1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade, see for example [1, 2, 4, 10]. The Cayley graph introduced by Arthur Cayley in 1878 is a useful tool for connection between group theory and the theory of algebraic graphs. Let G be an abelian group and S be a subset of G. The Cayley graph $\mathbb{CAY}(G, S)$ is

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a graph whose vertices are elements of G and in which two distinct vertices x and y are joined by an edge if and only if $x - y \in S$. We refer the reader to [8] for general properties of Cayley graphs. Let R be a commutative ring with identity and R^+ and $Z(R)$ be the additive group and the set of all zero-divisors of R, respectively. The authors in [1] have studied $\mathbb{CAY}(R^+, Z(R))$ and its subgraph Reg$\mathbb{CAY}(R)$ the induced subgraph on the regular elements of R. We denote by $Z\mathbb{CAY}(R)$ the induced subgraph on zero-divisor elements of R. In this paper, following [4], we are interested in studying $Z\mathbb{CAY}(R)$.

Let $J(R)$ denote the Jacobson radical of R. It is easy to see that every $x \in J(R)$ is adjacent to each vertex of $Z\mathbb{CAY}(R)$. Thus the main part of the graph $Z\mathbb{CAY}(R)$ is the induced subgraph of $Z\mathbb{CAY}(R)$ on the vertex set $Z(R) \setminus J(R)$. We denote it by $Z\mathbb{CAY}^*(R)$. The graphs in Figure 1 are the zero-divisor Cayley graphs of the rings indicated. In the figures of this paper, the vertices in Jacobson radical are shown by circle.

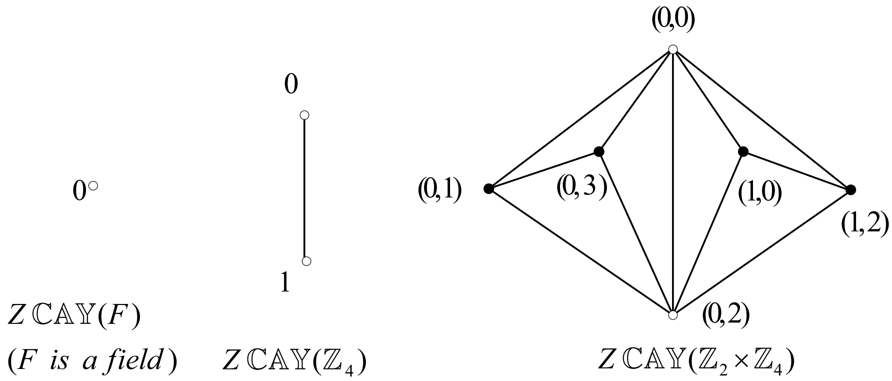

Figure 1. The zero-divisor Cayley graphs of some specific rings.

The plan of the paper is as follows: In Section 2 of this paper, we bring some preliminaries and notations about graph and ring theory. In Section 3, we state some basic properties of $Z\mathbb{CAY}(R)$. In Section 4, we study the connectivity, diameter and girth of $Z\mathbb{CAY}(R)$ and $Z\mathbb{CAY}^*(R)$. In Section 5, the planarity of $Z\mathbb{CAY}(R)$ and $Z\mathbb{CAY}^*(R)$ are investigated. In the final section, we study the clique number of $Z\mathbb{CAY}(R)$.

2. Preliminaries and Notations

The graphs in this paper are simple, that is they have no loops or multiple edges. For a graph G, let $V(G)$ denote the set of vertices, and let $E(G)$ denote the set of edges. For $x \in V(G)$ we denote by $N(x)$ the set of all vertices of G

adjacent to x. Also, the *degree* of x, denoted $d_G(x)$, is the size of $N(x)$. The maximum and minimum degree of vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The *union* of two simple graphs G and H is the graph $G \cup H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. If $V(G)$ and $V(H)$ are disjoint, we refer to their union as a *disjoint union*, and denote it by $G + H$. The *join* of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding edges joining every vertex of G to every vertex of H.

Let G be a graph. For two vertices x and y of G, a *walk (path)* of length n between x and y is an ordered list of (distinct) vertices $x = x_0, x_1, \ldots, x_n = y$ such that x_{i-1} is adjacent to x_i for $i = 1, \ldots, n$. The *distance* between x and y, denoted by $d(x, y)$, is the length of shortest path between x and y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path between x and y). The largest distance among all distances between pairs of the vertices of a graph G is called the *diameter* of G and is denoted by $\mathrm{diam}(G)$. A *cycle* in G is a path that begins and ends at the same vertex. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G has no cycle). A graph G is called *connected* if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected* (a singleton graph is connected with zero diameter). The *null graph* is the graph whose vertex set and edge set are empty.

A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We denote the complete graph on n vertices by K_n. A *clique* of a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. The complement of a graph G, denoted by $\overline{G}$, is the graph with the same vertex set as G such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

For a set X, $|X|$ denotes the cardinal number of X. Also, $\mathbb{F}_{p^n}$ denotes the field with p^n elements and $\mathbb{Z}_n$ denotes for the ring of integers modulo n. Following the literature, we write

$$D_2(R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

In this paper, for convenience, we denote the elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $D_2(R)$ by A and B, respectively.

It is well known that every finite commutative ring can be expressed as a direct product of finite local rings, and this decomposition is unique up to permutations of such local rings (see [5, Theorem 8.7]). In this paper, we assume that R is a finite commutative ring with identity and we accept the following notations: Let $R = R_1 \times \cdots \times R_n$ be a ring, where R_is are local rings. We set $e_0 := (0,0,\dots,0)$, $e_1 := (1,0,\dots,0)$, $e_2 := (0,1,0,\dots,0)$, $\dots$, $e_n := (0,\dots,0,1)$.

For a ring R, we denote by $U(R)$ the set of all unit elements of R. We note that if R is a finite commutative ring, then $U(R) = R \setminus Z(R)$. In other words, the set of all zero-divisors and the set of all nonunit elements of R coincide. If $R = R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$, then

$$Z(R) = \{(x_1, x_2, \dots, x_n) \in R | x_i \in \mathfrak{m}_i \text{ for some } 1 \le i \le n\}.$$

This fact is frequently used in this paper.

3. Some Basic Properties of $Z\mathbb{CAY}(R)$

In this section, we study the basic properties of $Z\mathbb{CAY}(R)$. We begin with the following proposition.

Proposition 3.1. *The following are equivalent:*
(1) $Z\mathbb{CAY}^*(R)$ is a null graph,
(2) $Z(R) = J(R)$,
(3) $Z\mathbb{CAY}(R)$ is a complete graph,
(4) R is a local ring.

Proof. (1)$\Leftrightarrow$(2) is trivial.

(2)$\Rightarrow$(3) Suppose that $J(R) = Z(R)$. Then $Z(R)$ is an ideal and hence $Z\mathbb{CAY}(R)$ is a complete graph.

(3)$\Rightarrow$(4) By [9, Lemma 3.13], it is enough to show that $Z(R)$ is an ideal of R. It is easy to see that $Z(R)$ is closed under scalar multiplication. Now let $x, y \in Z(R)$. Since $Z\mathbb{CAY}(R)$ is a complete graph, we have $x - y \in Z(R)$. So $Z(R)$ is an ideal of R.

(4)$\Rightarrow$(2) Let R be a local ring with maximal ideal $\mathfrak{m}$. Then $J(R) = Z(R) = \mathfrak{m}$. □

In the rest of this section, we study the maximum and minimum degree of $Z\mathbb{CAY}(R)$.

Proposition 3.2. *Let $R = R_1 \times \cdots \times R_n$ be a ring, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Let $(a_1, ..., a_n) \in Z(R)$ and $G = Z\mathbb{CAY}(R)$. Then $d_G(a_1, ..., a_n) = d_G(\delta_1, ..., \delta_n)$, where*

$$\delta_i = \begin{cases} 1 & \textit{if } a_i \in U(R_i), \\ 0 & \textit{if } a_i \in \mathfrak{m}_i. \end{cases}$$

Proof. Let $G = Z\mathbb{CAY}(R)$ and let $a = (a_1, ..., a_k, \alpha, \beta, a_{k+3}, ..., a_n)$ be an element of $Z(R)$, where $(\alpha, \beta) \in (\mathfrak{m}_{k+1}, U(R_{k+2}))$. Then

$$\begin{aligned} d_{\overline{G}}(a) &= d_{\overline{G}}(a_1, ..., a_k, \alpha, \beta, a_{k+3}, ..., a_n) \\ &= |\{(d_1, ..., d_k, x, y, d_{k+3}, ..., d_n) \in Z(R) | d_i - a_i \in U(R_i) \\ &\quad \text{for } 1 \leq i \leq n \text{ with } i \neq k+1, k+2, x - \alpha \in U(R_{k+1}), y - \beta \in U(R_{k+2})\}| \\ &= |\{(d_1, ..., d_k, x, y, d_{k+3}, ..., d_n) \in Z(R) | d_i - a_i \in U(R_i) \\ &\quad \text{for } 1 \leq i \leq n \text{ with } i \neq k+1, k+2, x \in U(R_{k+1}), y - 1 \in U(R_{k+2})\}| \\ &= d_{\overline{G}}(a_1, ..., a_k, 0, 1, a_{k+3}, ..., a_n). \end{aligned}$$

A similar argument works if $(\alpha, \beta) \in (U(R_{k+1}), \mathfrak{m}_{k+2}) \cup (\mathfrak{m}_{k+1}, \mathfrak{m}_{k+2}) \cup (U(R_{k+1}), U(R_{k+2}))$. Now the assertion follows by repeating this argument. $\square$

Theorem 3.3. *Let $R = R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Let $G = Z\mathbb{CAY}(R)$ and $\delta = (\delta_1, ..., \delta_n) \in Z(R)$, where $\delta_i \in \{0, 1\}$ for all $i = 1, 2, \ldots, n$. Then*

$$d_G(\delta) = |Z(R)| - |U(R)| - 1 + [\prod_{\substack{1 \leq i \leq n \\ \delta_i = 0}} (|R_i| - |\mathfrak{m}_i|) \prod_{\substack{1 \leq i \leq n \\ \delta_i = 1}} (|R_i| - 2|\mathfrak{m}_i|)].$$

Proof. Let $N := \{(x_1, x_2, \ldots, x_n) \in R | x_i - \delta_i \in U(R_i) \text{ for all } 1 \leq i \leq n\}$. Then $N = \{(\delta_1 + u_1, \delta_2 + u_2, \ldots, \delta_n + u_n) \in R | u_i \in U(R_i) \text{ for all } 1 \leq i \leq n\}$ and hence $|N| = |U(R)|$. On the other hand,

$$\begin{aligned} |N \cap U(R)| &= |\{(x_1, x_2, \ldots, x_n) \in U(R) | x_i - \delta_i \in U(R_i) \text{ for all } 1 \leq i \leq n\}| \\ &= |\{(x_1, x_2, \ldots, x_n) \in U(R) | x_i - \delta_i \notin \mathfrak{m}_i \text{ for all } 1 \leq i \leq n\}| \\ &= |\{(x_1, x_2, \ldots, x_n) \in R | x_i - \delta_i \notin \mathfrak{m}_i \text{ and } x_i \notin \mathfrak{m}_i \text{ for all } 1 \leq i \leq n\}| \\ &= \prod_{\substack{1 \leq i \leq n \\ \delta_i = 0}} (|R_i/\mathfrak{m}_i| - 1)|\mathfrak{m}_i|) \prod_{\substack{1 \leq i \leq n \\ \delta_i = 1}} (|R_i/\mathfrak{m}_i| - 2)|\mathfrak{m}_i|) \\ &= \prod_{\substack{1 \leq i \leq n \\ \delta_i = 0}} (|R_i| - |\mathfrak{m}_i|) \prod_{\substack{1 \leq i \leq n \\ \delta_i = 1}} (|R_i| - 2|\mathfrak{m}_i|). \end{aligned}$$

Since $N = (N \cap U(R)) \cup (N \cap Z(R))$ and $(N \cap U(R)) \cap (N \cap Z(R)) = \emptyset$, we have

$$\begin{aligned} d_{\overline{G}}(\delta) &= |N \cap Z(R))| \\ &= |N \setminus (N \cap U(R))| \\ &= |N| - |(N \cap U(R))| \\ &= |U(R)| - [\prod_{\substack{1 \leq i \leq n \\ \delta_i = 0}} (|R_i| - |\mathfrak{m}_i|) \prod_{\substack{1 \leq i \leq n \\ \delta_i = 1}} (|R_i| - 2|\mathfrak{m}_i|)]. \end{aligned}$$

Now the assertion follows from the fact that $d_G(\delta) + d_{\overline{G}}(\delta) = |Z(R)| - 1$. $\square$

The following result is an immediate consequence of the proof of Proposition 3.2 and Theorem 3.3.

Corollary 3.4. *Let $R = F_1 \times \cdots \times F_n$, where F_is are fields and $|F_i| = p_i^{\alpha_i}$ and $p_1^{\alpha_1} \leq p_2^{\alpha_2} \leq \cdots \leq p_n^{\alpha_n}$. If $G = Z\mathbb{CAY}(R)$, then*

$$\begin{aligned}\Delta(G) &= d_G(0,0,\ldots,0) = |Z(R)| - 1,\\ \delta(G) &= d_G(1,1,\ldots,1,0) = |Z(R)| - 1 - (p_n^{\alpha_n} - 1)[\prod_{i=1}^{n-1}(p_i^{\alpha_i} - 1) - \prod_{i=1}^{n-1}(p_i^{\alpha_i} - 2)].\end{aligned}$$

4. Connectivity, Diameter and Girth

The following theorem determines the diameter of $Z\mathbb{CAY}(R)$.

Theorem 4.1. *Let R be a ring. Then $Z\mathbb{CAY}(R)$ is connected and*

$$\operatorname{diam}(Z\mathbb{CAY}(R)) = \begin{cases} 0 & \text{if } R \text{ is a field},\\ 1 & \text{if } R \text{ is local which is not a field},\\ 2 & \text{otherwise}.\end{cases}$$

Proof. Let $R = R_1 \times \cdots \times R_n$, where R_i's are local rings. First suppose that $n = 1$. In this case we may assume R is a local ring with maximal ideal $\mathfrak{m}$. If R is a field, then $Z\mathbb{CAY}(R)$ has only one vertex and hence $\operatorname{diam}(Z\mathbb{CAY}(R)) = 0$. If $n = 1$ and R is not a field, then $Z\mathbb{CAY}(R) \cong K_{|\mathfrak{m}|}$. Hence $\operatorname{diam}(Z\mathbb{CAY}(R)) = 1$.

Second suppose that $n \geq 2$. Let a, b be two distinct elements of $Z(R) \setminus J(R)$ and let $c \in J(R)$. Then $a, b \in N(c)$. It follows that $\operatorname{diam}(Z\mathbb{CAY}(R)) \leq 2$. On the other hand, if $x = (1, 0, 0, ..., 0)$ and $y = (0, 1, 1, ..., 1)$, then x and y are not adjacent, and so $d(x, y) \geq 2$. Therefore $\operatorname{diam}(Z\mathbb{CAY}(R)) = 2$ and the proof is complete. □

In the following theorem, we completely characterize the girth of $Z\mathbb{CAY}(R)$.

Theorem 4.2. *Let R be a ring. Then $\operatorname{gr}(Z\mathbb{CAY}(R)) \in \{3, \infty\}$ and $\operatorname{gr}(Z\mathbb{CAY}(R)) = \infty$ if and only if R is isomorphic to the one of the following rings.*

$$\mathbb{F}_{p^n}, \mathbb{Z}_4, D_2(\mathbb{Z}_2), \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Proof. Let $R = R_1 \times \cdots \times R_n$, where R_i's are local rings. We consider the following cases:

Case 1: $n = 1$. In this case we may assume R is a local ring with maximal ideal $\mathfrak{m}$. If $|\mathfrak{m}| = 1$, then R is a field and hence $Z\mathbb{CAY}(R)$ has only one vertex and hence $\operatorname{gr}(Z\mathbb{CAY}(R)) = \infty$ and $R = \mathbb{F}_{p^n}$. If $|\mathfrak{m}| = 2$, then $Z\mathbb{CAY}(R) \cong K_2$ and hence $\operatorname{gr}(Z\mathbb{CAY}(R)) = \infty$. Since $\mathfrak{m}$ is a nonzero finite dimensional vector space over the field $R/\mathfrak{m}$, we must have $|R/\mathfrak{m}| \leq |\mathfrak{m}|$ and hence $|R| = 4$. Therefore $R = \mathbb{Z}_4$ or $R = D_2(\mathbb{Z}_2)$ (see [6, Page 687]). If $|\mathfrak{m}| \geq 3$, then every three distinct elements of $\mathfrak{m}$ form a triangle and so $\operatorname{gr}(Z\mathbb{CAY}(R)) = 3$.

Case 2: $n = 2$. If $|R_1| \geq 3$, then $(r_1, 0), (r_2, 0), (r_3, 0)$, where r_1, r_2, r_3 are distinct elements of R_1, form a triangle and so $\mathrm{gr}(Z\mathbb{CAY}(R)) = 3$. A similar argument shows that if $|R_2| \geq 3$, then $\mathrm{gr}(Z\mathbb{CAY}(R)) = 3$. Otherwise $R = R_1 \times R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and hence $\mathrm{gr}(Z\mathbb{CAY}(R)) = \infty$.

Case 3: $n \geq 3$. In this case, e_1, e_2, e_3 form a triangle in $Z\mathbb{CAY}(R)$ and so $\mathrm{gr}(Z\mathbb{CAY}(R)) = 3$. □

Proposition 4.3. *Let R be a ring such that $Z\mathbb{CAY}^*(R)$ is not a null graph. Then the following are equivalent:*

(1) $Z\mathbb{CAY}^*(R)$ is disconnected,

(2) R is a direct product of two local rings,

(3) $Z\mathbb{CAY}^*(R)$ is disjoint union of two complete graphs.

Proof. Let $R = R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$.

(1)⇒(2) By Proposition 3.1, it is enough to show that $n \leq 2$. Suppose on the contrary that $n \geq 3$. Now let $a = (a_1, a_2, a_3, ..., a_n)$ and $b = (b_1, b_2, b_3, ..., b_n)$ be two arbitrary distinct vertices of $Z\mathbb{CAY}^*(R)$. Set $c = (1, 0, a_3, ..., a_n)$ and $d = (1, 0, b_3, ..., b_n)$. Then a, c, d, b is a walk and hence $Z\mathbb{CAY}^*(R)$ is connected, which is a contradiction.

(2)⇒(3) Let $R = R_1 \times R_2$, where R_1 and R_2 are local rings with maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. Then $Z\mathbb{CAY}^*(R)$ is disjoint union of two complete graphs with vertex sets $\mathfrak{m}_1 \times U(R_2)$ and $U(R_1) \times \mathfrak{m}_2$.

(3)⇒(1) is trivial. □

In the following figures the graphs $Z\mathbb{CAY}(\mathbb{F}_m \times \mathbb{F}_n)$, $Z\mathbb{CAY}(\mathbb{F}_n \times \mathbb{Z}_4)$ and $Z\mathbb{CAY}(\mathbb{F}_n \times D_2(\mathbb{Z}_2)$ are presented.

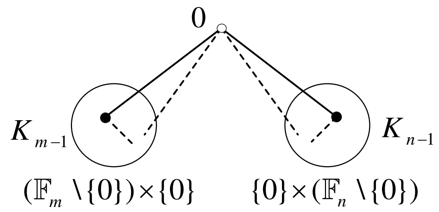

Figure 2. The graph $Z\mathbb{CAY}(\mathbb{F}_m \times \mathbb{F}_n) = (K_{m-1} + K_{n-1}) \vee K_1$.

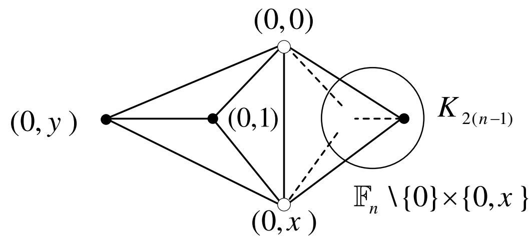

Figure 3. The graph $Z\mathbb{CAY}(\mathbb{F}_n \times \mathbb{Z}_4)$, if $x = 2$ and $y = 3$; and the graph $Z\mathbb{CAY}(\mathbb{F}_n \times D_2(\mathbb{Z}_2))$, if $x = A$ and $y = B$.

We note that $Z\mathbb{CAY}(\mathbb{F}_n \times \mathbb{Z}_4) = Z\mathbb{CAY}(\mathbb{F}_n \times D_2(\mathbb{Z}_2)) = (K_{2(n-1)} + K_2) \vee K_2$, by Figure 3.

The following theorem characterizes the diameter of $Z\mathbb{CAY}^*(R)$.

Theorem 4.4. *Let R be a ring such that $Z\mathbb{CAY}^*(R)$ is not a null graph. Then* $\operatorname{diam}(Z\mathbb{CAY}^*(R)) \in \{2, \infty\}$ *and* $\operatorname{diam}(Z\mathbb{CAY}^*(R)) = \infty$ *if and only if R is a direct product of two local rings.*

Proof. Let $R = R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Since $Z\mathbb{CAY}^*(R)$ is not a null graph, by Proposition 3.1, we have two following cases:

Case 1: $n = 2$. In this case, Proposition 4.3 implies that $\operatorname{diam}(Z\mathbb{CAY}^*(R)) = \infty$.

Case 2: $n \geq 3$. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two arbitrary distinct vertices of $Z\mathbb{CAY}^*(R)$. Then there are $i, j \in \{1, 2, \ldots, n\}$ such that $a_i \in \mathfrak{m}_i$ and $b_j \in \mathfrak{m}_j$. Let $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$. Then a, e_k, b form a path in $Z\mathbb{CAY}^*(R)$ and so $\operatorname{diam}(Z\mathbb{CAY}^*(R)) \leq 2$. On the other hand, if $x = (1, 0, ..., 0)$ and $y = (0, 1, 1, ..., 1)$, then x and y are not adjacent, and so $d(x, y) \geq 2$. Therefore $\operatorname{diam}(Z\mathbb{CAY}^*(R)) = 2$ and the proof is complete. $\square$

In the following theorem, we completely characterize the girth of $Z\mathbb{CAY}^*(R)$.

Theorem 4.5. *Let R be a ring such that $Z\mathbb{CAY}^*(R)$ is not a null graph. Then* $\operatorname{gr}(Z\mathbb{CAY}^*(R)) \in \{3, \infty\}$ *and* $\operatorname{gr}(Z\mathbb{CAY}^*(R)) = \infty$ *if and only if* $R \in \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times D_2(\mathbb{Z}_2), \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times D_2(\mathbb{Z}_2)\}$.

Proof. Let $R = R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Since $Z\mathbb{CAY}^*(R)$ is not a null graph, we only have two following cases:

Case 1: $n = 2$. If $|U(R_1)| = |R_1 \setminus \mathfrak{m}_1| \geq 3$ or $|U(R_2)| = |R_2 \setminus \mathfrak{m}_2| \geq 3$, then $Z\mathbb{CAY}^*(R)$ has a triangle and so $\operatorname{gr}(Z\mathbb{CAY}^*(R)) = 3$. Now suppose that $|U(R_1)| \leq 2$ and $|U(R_2)| \leq 2$. Then [7, Corollary 4.5] implies that $R = R_1 \times R_2$, where $R_1, R_2 \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, D_2(\mathbb{Z}_2)\}$. In view of Figure 2, we have $\operatorname{gr}(Z\mathbb{CAY}^*(R)) = \infty$ if $R \in \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3\}$. Figure 3 implies that $\operatorname{gr}(Z\mathbb{CAY}^*(R)) = \infty$ if $R \in \{\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_2 \times D_2(\mathbb{Z}_2), \mathbb{Z}_3 \times D_2(\mathbb{Z}_2)\}$. If $R = \mathbb{Z}_4 \times \mathbb{Z}_4$, then $(1, 0), (3, 0), (1, 2)$ form a triangle in $Z\mathbb{CAY}^*(R)$. If $R = D_2(\mathbb{Z}_2) \times D_2(\mathbb{Z}_2)$, then $(1, 0), (B, 0), (1, A)$ form a triangle in $Z\mathbb{CAY}^*(R)$ and if $R = \mathbb{Z}_4 \times D_2(\mathbb{Z}_2)$, then $(1, 0), (3, 0), (1, A)$ form a triangle in $Z\mathbb{CAY}^*(R)$.

Case 2: $n \geq 3$. In this case, e_1, e_2, e_3 form a triangle in $Z\mathbb{CAY}^*(R)$ and so $\operatorname{gr}(Z\mathbb{CAY}^*(R)) = 3$. $\square$

5. Planarity

A graph is said to be *planar* if it can be drawn in the plane such that its edges intersect only at their ends. A *subdivision* of an edge is obtained by inserting some new vertices of degree two into this edge. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (see [11, Theorem 6.2.2]). We recall the following proposition.

Proposition 5.1. *(*[3, Proposition 2.1]*) If R is a local ring with maximal ideal $\mathfrak{m}$, then there exists a prime number p such that $|R/\mathfrak{m}|$, $|R|$ and $|\mathfrak{m}|$ are all powers of p.*

The following theorem gives a necessary and sufficient condition for the planarity of $Z\mathbb{CAY}^*(R)$.

Theorem 5.2. *Let $R = R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Then $Z\mathbb{CAY}^*(R)$ is planar if and only if $Z\mathbb{CAY}^*(R)$ is a null graph or R is isomorphic to one of the following rings:*

(1) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
(2) $R = R_1 \times R_2$, *where R_1, R_2 are fields with $|R_1| \leq 5$ and $|R_2| \leq 5$.*

Proof. By Proposition 3.1, $n \geq 2$. We consider the following cases:

Case 1: $n \geq 4$. In this case the set of vertices $\{e_1, e_2, e_3, e_4, e_1 + e_2\}$ is a clique and hence $Z\mathbb{CAY}^*(R)$ is not planar, by Kuratowski's Theorem.

Case 2: $n = 3$. If $|R_i| \leq 2$ for all $i = 1, 2, 3$, then $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and Figure 4 shows $Z\mathbb{CAY}^*(R)$ is planar. Now suppose that $|R_i| \geq 3$ for some $1 \leq i \leq 3$, say $i = 1$. Therefore $|U(R_1)| \geq 2$, by Proposition 5.1. Let H be the subgraph induced by the vertices $\{(1,0,0), (a,0,0), (0,1,1), (0,0,1), (0,1,0), (1,1,0)\}$, where $a \in U(R_1) \setminus \{1\}$. Then it is easy to see that $K_{3,3}$ is a subgraph of H and hence $Z\mathbb{CAY}^*(R)$ is not planar, by Kuratowski's Theorem.

Case 3: $n = 2$ and $R = R_1 \times R_2$ with $|R_i| > 5$ for some $1 \leq i \leq 2$. Without loss of generality, we may assume that $|R_1| > 5$. By Proposition 5.1, we have $|R_1 \setminus \mathfrak{m}_1| \geq 6$. Now the set $U(R_1) \times \{0\}$ has a clique of order 6 and hence $Z\mathbb{CAY}^*(R)$ is not planar, by Kuratowski's Theorem.

So, if $Z\mathbb{CAY}^*(R)$ is planar, then $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R = R_1 \times R_2$, where R_1, R_2 are local rings with $|R_1| \leq 5$ and $|R_2| \leq 5$.

On the other hand, in view of Figures 2, 3 and 4, it is easy to see that if R is isomorphic to one of the above rings, then $Z\mathbb{CAY}^*(R)$ is planar. This completes the proof. $\square$

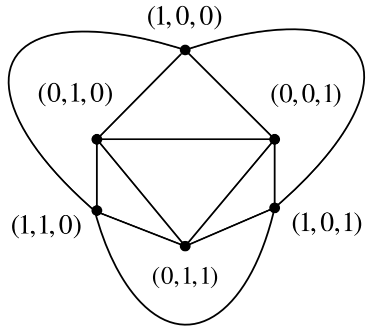

Figure 4. The graph $Z\mathbb{CAY}^*(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

The following theorem gives a necessary and sufficient condition for the planarity of $Z\mathbb{CAY}(R)$.

Theorem 5.3. *Let R be a ring. Then $Z\mathbb{CAY}(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{F}_{p^n}$, $\mathbb{Z}_4$, $D_2(\mathbb{Z}_2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$, $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times D_2(\mathbb{Z}_2)$.*

Proof. Let $R = R_1 \times \cdots \times R_n$, where R_i's are local rings. Let $Z\mathbb{CAY}(R)$ be a planar graph. We consider the following cases:

Case 1: $n = 1$. In this case, we may assume R is a local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{m} = 0$, then R is a field and $Z\mathbb{CAY}(R)$ has only one vertex and hence $Z\mathbb{CAY}(R)$ is planar. Now let $\mathfrak{m} \neq 0$. Since $\mathfrak{m}$ is a nonzero finite dimensional vector space over the field $R/\mathfrak{m}$, we must have $|R/\mathfrak{m}| \leq |\mathfrak{m}|$. Since $Z\mathbb{CAY}(R)$ is planar, Kuratowski's Theorem implies that $|\mathfrak{m}| \leq 4$ and hence $|R| \leq 16$. In view of [6, Page 687], we have $R \in \{\mathbb{Z}_4, \mathbb{F}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2+x+1)\}$.

Case 2: $n = 2$. If $|R_1| \geq 5$, then the set $R_1 \times \{0\}$ contains a subgraph isomorphic to K_5. Thus by Kuratowski's Theorem, $Z\mathbb{CAY}(R)$ is not planar. A similar argument shows that $|R_2| \leq 4$. Therefore [7, Corollary 4.5] implies that $R = R_1 \times R_2$, where $R_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{F}_4, D_2(\mathbb{Z}_2)\}$. We note that $Z\mathbb{CAY}(R)$ is not planar if $R \in \{\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times D_2(\mathbb{Z}_2), D_2(\mathbb{Z}_2) \times D_2(\mathbb{Z}_2)\}$, since $Z\mathbb{CAY}(R) = (K_4 + K_4) \vee K_4$.

Case 3: $n \geq 3$. In this case the elements e_0, e_1, e_2, e_3, $e_1 + e_2$ and $e_2 + e_3$ obtain a graph $K_{3,3}$ in the structure of $Z\mathbb{CAY}(R)$. Thus by Kuratowski's Theorem, $Z\mathbb{CAY}(R)$ is not planar.

Let $R = \mathbb{F}_4[x]/I$ and $S = \mathbb{Z}_4[x]/J$, where $I = (x^2)$ and $J = (x^2+x+1)$. Since $Z(R) = \{I, x+I, ax+I, bx+I\}$, where a, b are distinct elements of $\mathbb{F}_4 \setminus \{0,1\}$ and $Z(S) = \{J, 2+J, 2x+J, 2+2x+J\}$, we must have $Z\mathbb{CAY}(R) = Z\mathbb{CAY}(S) = K_4$. We also have $Z\mathbb{CAY}(\mathbb{F}_{p^n}) = K_1$ and $Z\mathbb{CAY}(\mathbb{Z}_4) = Z\mathbb{CAY}(D_2(\mathbb{Z}_2)) = K_2$. So, in view of Figures 2 and 3, it is easy to see that, if R is one of the rings that appears in the statement of the theorem, then $Z\mathbb{CAY}(R)$ is a planar graph. This completes the proof.

□

6. Clique Number

In the following, we denote the set of maximal ideals of a ring R by $\max(R)$.

Lemma 6.1. *Let R be a ring and $\mathfrak{m} \in \max(R)$. Then $\mathfrak{m}$ is a clique of $Z\mathbb{CAY}(R)$.*

Proof. Every two vertices of $\mathfrak{m}$ are adjacent. If x is a vertex of $Z\mathbb{CAY}(R)$ such that x is adjacent to every vertex of $\mathfrak{m}$. Then

$$x + \mathfrak{m} \subseteq \bigcup_{\mathfrak{m} \in \max(R)} \mathfrak{m}.$$

It follows form [9, Theorem 3.64] that

$$Rx + \mathfrak{m} \subseteq \bigcup_{\mathfrak{m} \in \max(R)} \mathfrak{m}.$$

By Prime Avoidance Theorem (see for example [9, Theorem 3.61]) there exists maximal ideal $\mathfrak{m}_0$ of R such that $x + \mathfrak{m} \subseteq \mathfrak{m}_0$. Therefore $x \in \mathfrak{m}_0 = \mathfrak{m}$. Hence $\mathfrak{m}$ is a clique of $Z\mathbb{CAY}(R)$. □

Theorem 6.2. *Let $G = Z\mathbb{CAY}(F_1 \times \cdots \times F_n)$, where F_is are finite fields and $|F_1| \leq |F_2| \leq \cdots \leq |F_n|$. Then $\mathfrak{m} = 0 \times F_2 \times \cdots \times F_n$ is a largest clique of G. In particular $\omega(G) = |R|/|F_1|$.*

Proof. Let $f_i : F_1 \setminus \{0\} \longrightarrow F_i \setminus \{0\}$ be an injective function for all $i \in \{2, 3, \ldots, n\}$ (note that $|F_1| \leq |F_i|$). Let K be an arbitrary clique of G. We define

$$\varphi : K \longrightarrow \mathfrak{m}$$

as follows: If $x \in K \cap \mathfrak{m}$, then $\varphi(x) = x$. If $x = (x_1, x_2, \ldots, x_n) \in K \setminus \mathfrak{m}$, then $\varphi(x_1, x_2, \ldots, x_n) = (0, x_2 + f_2(x_1), \ldots, x_n + f_n(x_1))$. We claim that φ is injective. Suppose that $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in K$ and $\varphi(a) = \varphi(b)$. We consider the following cases:

Case 1: $a, b \in K \cap \mathfrak{m}$. In this case, we have $a = \varphi(a) = \varphi(b) = b$.

Case 2: $a \in K \cap \mathfrak{m}$ and $b \notin K \cap \mathfrak{m}$. Since a, b are adjacent and $0 = a_1 \neq b_1$, there exists $2 \leq i \leq n$ such that $a_i = b_i$. Then $a_i = b_i + f_i(b_1)$, since $\varphi(a) = \varphi(b)$. It follows that $f_i(b_1) = 0$, a contradiction.

Case 3: $a \notin K \cap \mathfrak{m}$ and $b \in K \cap \mathfrak{m}$. Then by an argument symmetric to that of Case 2, we get a contradiction.

Case 4: $a \notin K \cap \mathfrak{m}$ and $b \notin K \cap \mathfrak{m}$. Since a, b are adjacent, there exists $1 \leq i \leq n$ such that $a_i = b_i$. If $i = 1$, then $\varphi(a) = \varphi(b)$ implies that $a = b$. Now, suppose that $a_1 \neq b_1$ and $a_i = b_i$ for some $i \geq 2$. Then $\varphi(a) = \varphi(b)$ implies that $a_i + f_i(a_1) = b_i + f_i(b_1)$. It follows that $f_i(a_1) = f_i(b_1)$. Since f_i is injective we must have $a_1 = b_1$, a contradiction.

So φ is injective and hence $|K| \leq |\mathfrak{m}|$. It follows from Lemma 6.1 that $\mathfrak{m}$ is a largest clique of G. □

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