

On Lorentzian Two-symmetric Manifolds of Dimension-four

Amirhesam Zaeim*, Mohammad Chaichi, Yadollah Aryanejad

Department of Mathematics, Payame noor University,
P.O. Box 19395-3697, Tehran, Iran.
E-mail: zaeim@pnu.ac.ir
E-mail: chaichi@pnu.ac.ir
E-mail: y.aryanejad@pnu.ac.ir

ABSTRACT. We study curvature properties of four-dimensional Lorentzian manifolds with two-symmetry property. We then consider Einstein-like metrics, Ricci solitons and homogeneity over these spaces.

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1. INTRODUCTION

Symmetries of the mathematical models have a lot of applications in applied sciences. For example, molecular symmetries studied in [20] and [6], obtaining the group of symmetries of the molecules. k -symmetric spaces are a natural generalization of symmetric manifolds. A (pseudo-) Riemannian space (M, g) is called k -symmetric if the following conditions establish:

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0,$$

where $k \geq 1$ and R is the curvature tensor of (M, g) . In the Riemannian setting, contrary to the pseudo-Riemannian case, a k -symmetric space is necessarily locally symmetric, i.e., $\nabla R = 0$ [29]. Examples of pseudo-Riemannian k -symmetric spaces with $k \geq 2$ can be found in [28, 8, 24]. Many interesting

*Corresponding Author

results about Lorentzian two-symmetric spaces were presented in [28], in particular the author proved that any two-symmetric Lorentzian manifold admits a parallel null vector field. A classification of four-dimensional two-symmetric Lorentzian spaces is obtained in [8], based on the Petrov classification of the Weyl tensors, and it is shown that such spaces are some special pp-waves. For wide applications in physics, many authors studied pp-wave manifolds which are spacial kind of pr-wave spaces. In [7], local symmetry, conformal flatness, Einstein-like metrics and existence of non-trivial Ricci solitons studied on the conformally flat pr-wave manifolds. Homogeneous plane wave manifolds, other special kinds of pr-waves, investigated in [9] and one geodesically complete family of the spaces under consideration were found. The generalization of the results of [8] is the subject of [3], where it is proven that a locally indecomposable Lorentzian manifold of dimension $n + 2$ is two-symmetric if and only if there exist local coordinates (v, x^1, \dots, x^n, u) such that

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + (H_{ij}u + F_{ij})x^i x^j (du)^2, \quad (1.1)$$

where H_{ij} is a nonzero diagonal matrix with diagonal elements $\lambda_1 \leq \dots \leq \lambda_n$, and F_{ij} is a symmetric real matrix. According to this general form of Lorentzian two-symmetric manifolds, in the four-dimensional case, there exist local coordinates (x^1, \dots, x^4) such that the metric g of a Lorentzian two-symmetric space is

$$g = 2dx^1 dx^4 + (dx^2)^2 + (dx^3)^2 + (x^4(a(x^2)^2 + b(x^3)^2) + p(x^2)^2 + 2qx^2 x^3 + s(x^3)^2) (dx^4)^2, \quad (1.2)$$

where a, b, p, q, s are real constants and $a^2 + b^2 \neq 0$. Our main goal is to study some geometric properties of four-dimensional Lorentzian two-symmetric spaces.

This paper is organized as the following way. Curvature properties of Lorentzian two-symmetric four-spaces will be studied in the section two and Einstein-like metrics of the spaces under consideration is the subject of section three. Ricci solitons and homogeneous four-dimensional Lorentzian two-symmetric spaces will be studied in the sections four and five respectively.

2. TWO-SYMMETRIC LORENTZIAN FOUR-MANIFOLDS

The first step for study the geometry of (pseudo-)Riemannian manifolds is to determine the Lievi-Civita connection. By using the *Koszul* identity

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),$$

and applying the metric (1.2), one can determine the components of the Levi-Civita connection. We use $\partial_i = \frac{\partial}{\partial x^i}$ as a local basis for the tangent space and have:

Theorem 2.1. *Let (M, g) be an arbitrary two-symmetric Lorentzian four-manifold, where the metric g is described in local coordinates (x^1, x^2, x^3, x^4) by the Equation (1.2). The non-zero components of the Levi-Civita connection are:*

$$\begin{aligned}\nabla_{\partial_2}\partial_4 &= (ax^2x^4 + px^2 + qx^3)\partial_1, \\ \nabla_{\partial_3}\partial_4 &= (bx^3x^4 + sx^3 + qx^2)\partial_1, \\ \nabla_{\partial_4}\partial_4 &= \frac{a(x^2)^2 + b(x^3)^2}{2}\partial_1 \\ &\quad - (ax^2x^4 + px^2 + qx^3)\partial_2 - (bx^3x^4 + qx^2 + sx^3)\partial_3.\end{aligned}\tag{2.1}$$

Applying the relation $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ we immediately determine the curvature tensor. If we set $R(\partial_k, \partial_l)\partial_j = R^i{}_{jkl}\partial_i$, then by contraction on the first and third indices of the curvature tensor, the Ricci tensor ϱ will be deduced. The scalar curvature tensor τ is also obtained by full contraction of coefficients of the curvature tensor.

Theorem 2.2. *A four-dimensional two-symmetric Lorentzian space admits zero scalar curvature. Also, the non-zero components of curvature tensor and Ricci tensor are*

$$\begin{aligned}R(\partial_2, \partial_4) &= (ax^4 + p)\partial_1dx^2 + q\partial_1dx^3 - (ax^4 + p)\partial_2dx^4 - q\partial_3dx^4, \\ R(\partial_3, \partial_4) &= q\partial_1dx^2 + (bx^4 + s)\partial_1dx^3 - q\partial_2dx^4 - (bx^4 + s)\partial_3dx^4, \\ \varrho(\partial_4, \partial_4) &= -(a + b)x^4 - (s + p).\end{aligned}\tag{2.2}$$

A (pseudo-)Riemannian manifold (M, g) is called *Einstein* if $\varrho = cg$, for a real constant c . Being *Ricci flat* means that the Ricci tensor vanishes identically. Also, *conformal flatness* translates into the following system of algebraic equations:

$$\begin{aligned}W_{ijkh} &= R_{ijkh} - \frac{1}{2}(g_{ik}Q_{jh} + g_{jh}Q_{ik} - g_{ih}Q_{jk} - g_{jk}Q_{ih}) \\ &\quad + \frac{\tau}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}) = 0 \quad \text{for all indices } i, j, k, h = 1, \dots, 4,\end{aligned}\tag{2.3}$$

where W denotes the *Weyl tensor* and τ is the scalar curvature. Although two-symmetric spaces clearly aren't flat, we can check Ricci flatness.

Theorem 2.3. *Let (M, g) be a two-symmetric four-dimensional Lorentzian manifold such that the metric g is described by the Equation (1.2) in local coordinates (x^1, x^2, x^3, x^4) . The following statements satisfy:*

- a) (M, g) is Einstein if and only if be Ricci flat if and only if $b = -a$, $s = -p$.
- b) (M, g) is conformally flat if and only if $b = a$, $s = p$, $q = 0$.

Proof. Let (M, g) be an Einstein manifold. Using the Equation (2.2) we set $\varrho = cg$. The following relations must be established.

$$c = (a + b)x^4 + s + p = 0.$$

So the Einstein property is equivalent to satisfying $b = -a$, $s = -p$ and $c = 0$ which is clearly equivalent to Ricci flatness. Using the Equation (2.3), the

non-zero components of the Weyl tensor would be

$$W_{2424} = -W_{3434} = \frac{(b-a)x^4 + s-p}{2}, \quad W_{2434} = -q,$$

it is obvious that the Weyl tensor vanishes if and only if $b = a$, $s = p$ and $q = 0$. \square

3. EINSTEIN-LIKE LORENTZIAN TWO-SYMMETRIC SPACES

Two new classes of Riemannian manifolds which are defined through conditions on the Ricci tensor, introduced by A. Gray in [23]. These types of manifolds which are famous as \mathcal{A} and \mathcal{B} classes, would be extended at once to the pseudo-Riemannian geometry. \mathcal{A} and \mathcal{B} classes are defined in the following way:

Class \mathcal{A} : a pseudo-Riemannian manifold (M, g) belongs to class \mathcal{A} if and only if its Ricci tensor ϱ is *cyclic-parallel*, that is,

$$(\nabla_X \varrho)(Y, Z) + (\nabla_Y \varrho)(Z, X) + (\nabla_Z \varrho)(X, Y) = 0, \quad (3.1)$$

for all vector fields X, Y and Z on M . The Equation (3.1) is equivalent to requiring that ϱ is a *Killing tensor*, that is,

$$(\nabla_X \varrho)(X, X) = 0. \quad (3.2)$$

To note that Equation (3.2), also known as the first odd Ledger condition, is a necessary condition for a (pseudo-)Riemannian manifold to be a D'Atri space. Hence, identifying two-symmetric manifolds of a given dimension satisfying (3.2), is the first step to understand D'Atri spaces.

Class \mathcal{B} : a pseudo-Riemannian manifold (M, g) belongs to class \mathcal{B} if and only if its Ricci tensor be a *Codazzi tensor*, that is,

$$(\nabla_X \varrho)(Y, Z) = (\nabla_Y \varrho)(X, Z). \quad (3.3)$$

A pseudo-Riemannian manifold which belongs to one of the above classes is called an *Einstein-like manifold*. We denote the class of Ricci parallel, Einstein and manifolds with constant scalar curvature by \mathcal{P} , \mathcal{E} and \mathcal{C} respectively. One can easily see that the intersection of two Einstein-like classes consists of Ricci parallel manifolds. This situation can be summarized in the following diagram:

$$\begin{array}{ccc} & \subset \mathcal{A} & \\ \mathcal{E} \subset \mathcal{P} = \mathcal{A} \cap \mathcal{B} & & \subset \mathcal{A} \cup \mathcal{B} \subset \mathcal{C}. \\ & \subset \mathcal{B} & \end{array}$$

Einstein-like metrics are deeply investigated through the different kinds of homogeneous spaces in both Riemannian and pseudo-Riemannian signatures. Three-dimensional Riemannian homogeneous spaces studied in [1]. In [13], authors study three- and four-dimensional Einstein-like Riemannian manifolds which are Ricci-curvature homogeneous. They could completely classify three-dimensional case of the mentioned spaces, while in the four-dimensional case,

they partially classified the special case where the manifold is locally homogeneous. They also presented explicit examples of four-dimensional locally homogeneous Riemannian manifolds whose Ricci tensor is cyclic-parallel and has distinct eigenvalues. These examples invalidated the expectation stated by F. Podestá and A. Spiro in [27]. Three-dimensional ball-homogeneous spaces, semi-symmetric spaces, Sasakian spaces and three-dimensional contact metric manifolds are other Riemannian classes which are subjects of research for the Einstein-like properties [15, 10, 2, 16].

Theorem 3.1. *Every four-dimensional two-symmetric Lorentzian manifold (M, g) belongs to class \mathcal{A} , if and only if $b = -a$.*

Proof. Let $v = v^1\partial_1 + v^2\partial_2 + v^3\partial_3 + v^4\partial_4$ be an arbitrary vector space on (M, g) , where v^1, \dots, v^4 are smooth functions on M . As mentioned before, (M, g) belongs to class \mathcal{A} of Einstein-like manifolds if and only if the Equation (3.2) satisfies. By straight forward calculations it is implied that

$$(\nabla_v \varrho)(v, v) = -(v^4)^3(a + b).$$

Thus, (M, g) belongs to class \mathcal{A} if and only if $b = -a$. □

Theorem 3.2. *Every four-dimensional two-symmetric Lorentzian manifold (M, g) belongs to class \mathcal{B} of the Einstein-like manifolds.*

Proof. Let $v = \sum_{i=1}^4 v^i \partial_i$, $u = \sum_{i=1}^4 u^i \partial_i$ and $w = \sum_{i=1}^4 w^i \partial_i$ be three arbitrary smooth vector fields on (M, g) . Every two-symmetric space (M, g) belongs to class \mathcal{B} of Einstein-like manifolds if and only if the Equation (3.3) satisfies. Direct calculations yield that

$$(\nabla_u \varrho)(v, w) = (\nabla_v \varrho)(u, w) = -u^4 v^4 w^4 (a + b),$$

and so, the Equation (3.3) always establishes and proves the claim. □

4. TWO-SYMMETRIC LORENTZIAN RICCI SOLITONS

We now report some basic information on Ricci solitons, referring to [19] for a survey and further references. A *Ricci soliton* is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field v , such that

$$\mathcal{L}_v g + \varrho = \lambda g, \tag{4.1}$$

where \mathcal{L} denotes the Lie derivative and λ is a real constant. A Ricci soliton is said to be *shrinking*, *steady* or *expanding* depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Ricci solitons are self-similar solutions of the *Ricci flow*.

Originally introduced in the Riemannian case, Ricci solitons have been intensively studied in pseudo-Riemannian settings in recent years. The Ricci soliton equation is also a special case of Einstein field equations.

Theorem 4.1. *Every four-dimensional two-symmetric Lorentzian manifold (M, g) is shrinking, expanding and steady Ricci soliton.*

Proof. Let (M, g) be a four-dimensional two-symmetric Lorentzian manifold, where g is described by the Equation (1.2). Suppose that $v = \sum_{i=1}^4 v^i \partial_i$ is a smooth vector field on (M, g) such that the Equation (4.1) satisfies for a real constant λ . The Lie derivative of g in direction v is

$$\begin{aligned} \mathcal{L}_v g = & 2\partial_1 v^4 (dx^1)^2 + 2(\partial_2 v^4 + \partial_1 v^2) dx^1 dx^2 + 2(\partial_3 v^4 + \partial_1 v^3) dx^1 dx^3 \\ & + 2(\partial_1 x^1 + \partial_4 v^4 + \partial_1 v^4 (a(x^2)^2 x^4 + p(x^2)^2 + 2qx^2 x^3 + b(x^3)^2 x^4 + \\ & s(x^3)^2)) dx^1 dx^4 + 2\partial_2 v^2 (dx^2)^2 + 2(\partial_2 v^3 + \partial_3 v^2) dx^2 dx^3 + \\ & 2(\partial_2 v^4 (a(x^2)^2 x^4 + p(x^2)^2 + 2qx^2 x^3 + b(x^3)^2 x^4 + s(x^3)^2 + 1) + \\ & \partial_2 v^1) dx^2 dx^4 + 2\partial_3 v^3 (dx^3)^2 + 2(\partial_3 v^4 (a(x^2)^2 x^4 + p(x^2)^2 + 2qx^2 x^3 + \\ & b(x^3)^2 x^4 + s(x^3)^2 + 1) + \partial_3 v^1) dx^3 dx^4 + (\partial_4 v^4 (2a(x^2)^2 x^4 + 2p(x^2)^2 + \\ & 4qx^2 x^3 + 2b(x^3)^2 (x^4) + 2s(x^3)^2) + 2\partial_4 v^1 + 2av^2 x^2 x^4 + 2pv^2 x^2 + \\ & 2qv^2 x^3 + 2qv^3 x^2 + 2bv^3 x^3 x^4 + 2sv^3 x^3 + av^4 (x^2)^2 + v^4 b(x^3)^2) (dx^4)^2. \end{aligned}$$

Applying Equations (1.2) and (2.2) in the Ricci soliton Equation (4.1), we have a system of PDEs which admits the following solution

$$\begin{cases} \lambda = 2c, \\ v^1 = \frac{(x^4)^2}{4} (a + b) + \frac{x^4}{2} (s + p) + 2cx^1, \\ v^2 = cx^2, \\ v^3 = cx^3, \\ v^4 = 0, \end{cases}$$

for a real constant c . Since c is arbitrary, (M, g) can be an expanding, shrinking or steady Ricci soliton. \square

A Ricci soliton (M, g) is called *gradient* if the Equation (4.1) holds for a vector field $X = \text{grad} f$, for some potential function f . In this case, (4.1) can be rewritten as

$$2\text{Hes}_f + \varrho = \lambda g,$$

where Hes_f denotes the Hessian of f . Studying locally conformally flat Lorentzian gradient Ricci solitons, as well as quasi-Einstein spaces, in [11] and [12] proved that such spaces are locally isometric to a plane-wave, if the gradient of the potential function is null.

Theorem 4.2. *Every four-dimensional two-symmetric Lorentzian space (M, g) is a gradient Ricci soliton if and only if be a steady Ricci soliton. In this case, the potential function is $f = \frac{a+b}{12} (x^4)^3 + \frac{p+s}{4} (x^4)^2 + c_1 x^4 + c_2$, for arbitrary real constants c_1, c_2 .*

Proof. Let $f = f(x^1, x^2, x^3, x^4)$ be a smooth function on (M, g) and $v = \sum_{i=1}^4 v^i \partial_i$ be a gradient Ricci soliton with the potential function f . So the coefficient v^i must be $v^i = \sum_{j=1}^4 g^{ij} \partial_j(f)$. By applying v to the Equation (4.1)

the following equations must establish

$$\left\{ \begin{array}{l} f_{11} = f_{12} = f_{13} = f_{23} = 0, \\ \lambda = 2f_{14} = 2f_{22} = 2f_{33}, \\ 2f_{24} - 2af_1x^2x^4 - 2pf_1x^2 - 2qf_1x^3 = 0, \\ 2f_{34} - 2bf_1x^3x^4 - 2qf_1x^2 - 2sf_1x^3 = 0, \\ \lambda(x^2)^2(ax^4 + p) + 2\lambda qx^2x^3 + \lambda(x^3)^2(bx^4 + s) + (a + b)x^4 + s + p \\ - 2f_{44} + f_1(a(x^2)^2 + b(x^3)^2) \\ - 2f_2(ax^2x^4 + px^2 + qx^3) - 2f_3(qx^2 + bx^3x^4 + sx^3) = 0, \end{array} \right.$$

where $f_i := \partial_i f$. After solving the above system of PDEs we get that λ must be vanished and f must be the same function of the statement, this matter ends the proof. \square

To note that, from the above Theorem 4.2 we get $\nabla f = \frac{a+b}{4}(x^4)^2 + \frac{p+s}{2}x^4 + c_1$, which is a null vector field. This result is compatible with main Theorem in [11].

5. HOMOGENEOUS TWO-SYMMETRIC FOUR-DIMENTIONAL SPACES

Study of homogeneous spaces is one of the most interesting topics in differential geometry, where a deep connection between geometry and algebra appears. A (pseudo-)Riemannian manifold M is called *homogeneous*, if for any points $p, q \in M$, there is an isometry ϕ of M such that $\phi(p) = q$. In short, $I(M)$, the group of isometries of M , acts transitively on M . If M is homogeneous, then evidently any geometrical properties at one point of M holds at every point. For a detailed introduction to homogeneous spaces see e.g., [5, 26, 31].

Homogeneous Riemannian structures introduced by Ambrose and Singer in [4] and deeply studied in [30]. The notation is generalized to homogeneous pseudo-Riemannian structures by Gadea and Oubiña in [22], in order to obtain a characterization of reductive homogeneous pseudo-Riemannian manifolds. A pretty application of homogeneous structures on three dimensional Lorentzian manifold is shown in [14].

Let (M, g) be a connected pseudo-Riemannian manifold, the following definition introduced by Gadea and Oubiña:

Definition 5.1. [22] A homogeneous pseudo-Riemannian structure on (M, g) is a tensor field T of type $(1, 2)$ on M , such that the connection $\tilde{\nabla} = \nabla - T$ satisfies

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}T = 0. \quad (5.1)$$

The above conditions are equivalent to the following system of equations which are famous as Ambrose-Singer equations.

$$g(T_X Y, Z) + g(Y, T_X Z) = 0, \quad (5.2)$$

$$(\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \quad (5.3)$$

$$(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}, \quad (5.4)$$

for all vector fields X, Y, Z .

Existence of homogeneous pseudo-Riemannian structures shows the homogeneity of the space. This fact is the subject of the following Lemma

Lemma 5.2. [22] *Let (M, g) be a simply connected and complete pseudo-Riemannian manifold, then (M, g) admits a pseudo-Riemannian homogeneous structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

Case1: Reductive cases:

By applying the above lemma, we consider reductive homogeneous four-dimensional two-symmetric Lorentzian spaces. The result is the following theorem,

Theorem 5.3. *Every Lorentzian four-dimensional two-symmetric manifold is not reductive homogeneous.*

Proof. Let (M, g) be a four-dimensional two-symmetric manifold. There exist local coordinates (x^1, \dots, x^4) such that the metric g is defined using the Equations (1.2). According to the Lemma 5.2, (M, g) is (locally) reductive homogeneous if and only if the tensor field T of type $(1, 2)$ exists, such that the Ambrose-Singer equations satisfy. Let $T_{\partial_i} \partial_j = T_{ij}^k \partial_k, 1 \leq i, j, k \leq 4$ be a homogeneous structure on (M, g) where $\partial_1 = \partial_v, \partial_2 = \partial_{x_1}, \partial_3 = \partial_{x_2}, \partial_4 = \partial_u$ and T_{ij}^k are smooth functions on M .

From the Equations (5.2) and (5.3), besides the relations between the components T_{ij}^k , one of the following relations for the constants a, b, p, q, s must be valid:

$$\begin{aligned} \mathbf{1} : & a = p = q = 0, \\ \mathbf{2} : & a \neq 0, q = 0, s = \frac{bp}{a}, \\ \mathbf{3} : & b = a, s = p, q = 0, \\ \mathbf{4} : & b = -a, s = -p, \end{aligned}$$

but each of these solutions makes a contradiction with Equation (5.4) (or equivalently with $\tilde{\nabla} T = 0$). For example, in the case 1, for the components T_{ij}^k have

$$\begin{aligned} T_{1j}^k &= 0, \quad 1 \leq j, k \leq 4, (j, k) \neq (4, 1), \quad T_{14}^1 = -T_{44}^4 = \frac{b}{2(bx^4 + s)}, \\ T_{2j}^k &= 0, \quad 1 \leq j, k \leq 4, \quad k \neq 1, \\ T_{3j}^k &= 0, \quad 1 \leq j, k \leq 4, \quad k \neq 1, \\ T_{4j}^1 &= 0, \quad 1 \leq j \leq 3, \quad T_{44}^1 = \frac{b(x^3)^2}{2}, \\ T_{4j}^k &= -T_{kj}^1, \quad 1 \leq j \leq 4, \quad 2 \leq k \leq 3, \\ T_{4j}^4 &= 0, \quad 1 \leq j \leq 3. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (\tilde{\nabla}_{\partial_4} T)_{44}^4 = & \partial_4 T_{44}^4 + T_{42}^4 (T_{44}^2 + x^2(p + ax^4) + qx^3) \\ & + T_{43}^4 (T_{44}^3 + x^3(s + bx^4) + qx^2) \\ & + T_{24}^4 (x^2(p + ax^4) + qx^3) + T_{34}^4 (x^3(s + bx^4) + qx^2) \\ & + T_{41}^4 (T_{44}^1 - \frac{a(x^2)^2 + b(x^3)^2}{2}) - T_{14}^4 \frac{a(x^2)^2 + b(x^3)^2}{2} + (T_{44}^4)^2. \end{aligned}$$

If we substitute the previous solutions in the above relation we get $(\tilde{\nabla}_{\partial_4} T)_{44}^4 = \frac{3b^2}{4(bx^4+s)^2}$, and so the Equation (5.4) satisfies if $b = 0$ which is a contradiction, since in this case the matrix H in (1.1) vanishes. \square

Case2: Non-reductive cases:

Consider a homogeneous manifold $M = G/H$ (with H connected), the Lie algebra \mathfrak{g} of G , the isotropy subalgebra \mathfrak{h} , and $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ the factor space, which identifies with a subspace of \mathfrak{g} complementary to \mathfrak{h} . The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropy representation

$$\psi : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{m}), \quad \psi(x)(y) = [x, y]_{\mathfrak{m}} \quad \text{for all } x \in \mathfrak{g}, y \in \mathfrak{m}.$$

Given a basis $\{h_1, \dots, h_r, u_1, \dots, u_n\}$ of \mathfrak{g} , where $\{h_j\}$ and $\{u_i\}$ are bases of \mathfrak{h} and \mathfrak{m} , respectively, a bilinear form on \mathfrak{m} is determined by the matrix g of its components with respect to the basis $\{u_i\}$ and is invariant if and only if ${}^t\psi(x) \circ g + g \circ \psi(x) = 0$ for all $x \in \mathfrak{h}$. Invariant pseudo-Riemannian metrics g on the homogeneous space $M = G/H$ are in a one-to-one correspondence with nondegenerate invariant symmetric bilinear forms g on \mathfrak{m} [25].

Then, g uniquely defines its invariant linear Levi-Civita connection, described in terms of the corresponding homomorphism of \mathfrak{h} -modules $\Lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{m})$, such that $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$. Explicitly, one has

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \quad \text{for all } x, y \in \mathfrak{g},$$

where $v : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{m}$ is the \mathfrak{h} -invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}), \quad \text{for all } x, y, z \in \mathfrak{g}.$$

The curvature tensor is then determined by

$$\begin{aligned} R : \mathfrak{m} \times \mathfrak{m} & \rightarrow \mathfrak{gl}(\mathfrak{m}) \\ (x, y) & \mapsto [\Lambda(x), \Lambda(y)] - \Lambda([x, y]). \end{aligned} \tag{5.5}$$

Finally, the Ricci tensor ϱ of g , described in terms of its components with respect to $\{u_i\}$, is given by

$$\varrho(u_i, u_j) = \sum_{r=1}^4 R_{ri}(u_r, u_j), \quad i, j = 1, \dots, 4 \tag{5.6}$$

and the scalar curvature τ is the trace of ϱ .

Non-reductive homogeneous manifolds of dimension 4 were classified in [21], in terms of the corresponding non-reductive Lie algebras. The corresponding

invariant pseudo-Riemannian metrics, together with their connection and curvature, were explicitly described in [17, 18]. These spaces categorized in eight classes, $A_1, \dots, A_5, B_1, B_2, B_3$. The invariant metrics of types A_1, A_2, A_3 can be both of Lorentzian or neutral signature while the cases A_4, A_5 are always Lorentzian and cases B_1, B_2, B_3 admit the neutral signature (2, 2).

Theorem 5.4. *Every Lorentzian four-dimensional non-reductive homogeneous manifold is locally symmetric if and only if $\nabla^2 R = 0$.*

Proof. Let (M, g) be a Lorentzian non-reductive homogeneous four-dimensional manifold, then (M, g) is isometric to a homogeneous space G/H equipped with a Lorentzian invariant metric g , where the corresponding Lie algebras \mathfrak{g} and \mathfrak{h} are described in the cases A_1, \dots, A_5 of [17].

We bring the details of the case A_1 . The other cases can be treated in the similar way. In this case, $\mathfrak{g} = \mathcal{A}_1$ is the decomposable 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$, where $\mathfrak{s}(2)$ is the 2-dimensional solvable algebra. There exists a basis $\{e_1, \dots, e_5\}$ of \mathcal{A}_1 , such that the non-zero products are:

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_4, e_5] = e_4$$

and the isotropy subalgebra is $\mathfrak{h} = \text{Span}\{h_1 = e_3 + e_4\}$. So, we can take

$$\mathfrak{m} = \text{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_5, u_4 = e_3 - e_4\}.$$

With respect to $\{u_i\}$, we have the following isotropy representation H_1 for h_1 and consequently the following description of invariant metric g :

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad g = \begin{pmatrix} a & 0 & -\frac{a}{2} & 0 \\ 0 & b & c & a \\ -\frac{a}{2} & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad (5.7)$$

which is nondegenerate whenever $a(a - 4d) \neq 0$ and is Lorentzian if and only if $a(a - 4d) < 0$. Putting $\Lambda[i] := \Lambda(u_i)$ for all indices $i = 1, \dots, 4$, we find:

$$\Lambda[1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{b}{a} & -\frac{c}{a} & -1 \end{pmatrix}, \quad \Lambda[2] = \begin{pmatrix} 0 & -\frac{8bd}{a(a-4d)} & \frac{c}{a} & 1 \\ -1 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{4bc}{a(a-4d)} & 0 & 0 \\ -\frac{b}{a} & \frac{4bc}{a(a-4d)} & -\frac{b}{2a} & 0 \end{pmatrix}, \quad (5.8)$$

$$\Lambda[3] = \begin{pmatrix} 0 & \frac{c}{a} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{c}{a} & -\frac{b}{2a} & 0 & -\frac{1}{2} \end{pmatrix}, \quad \Lambda[4] = 0.$$

Moreover, applying (5.5) and (5.6), by setting $R_{ij} = R(u_i, u_j)$, some standard calculations give that with respect to $\{u_i\}$, the non-zero curvature components

are determined as following:

$$\begin{aligned}
 R_{12} &= \begin{pmatrix} 0 & \frac{b(20d+a)}{a(-4d+a)} & -\frac{c}{a} & -1 \\ 1 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{12b}{-4d+a} & 0 & 0 \\ \frac{4b}{a} & -\frac{12cb}{a(-4d+a)} & \frac{b}{a} & 0 \end{pmatrix}, & R_{13} &= \begin{pmatrix} 0 & -\frac{c}{a} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{c}{a} & 0 & -\frac{c}{2a} & 0 \end{pmatrix}, \\
 R_{14} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, & R_{23} &= \begin{pmatrix} 0 & -\frac{b(4d+a)}{2a(-4d+a)} & -\frac{c}{2a} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{c}{a} & -\frac{1}{4} & 0 \\ 0 & -\frac{2b}{-4d+a} & 0 & 0 \\ -\frac{b}{a} & \frac{cb(-4d+3a)}{a^2(-4d+a)} & \frac{c^2}{a^2} & \frac{c}{a} \end{pmatrix}, \\
 R_{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{b}{a} & \frac{c}{a} & 1 \end{pmatrix}, & R_{34} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} & 0 \end{pmatrix},
 \end{aligned}$$

and the Ricci tensor ϱ is determined by

$$\varrho = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & \frac{2b(a+12d)}{a(a-4d)} & -\frac{2c}{a} & -2 \\ 1 & -\frac{2c}{a} & -\frac{1}{2} & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

By using description of the curvature tensor and Levi-Civita connection, we set $\nabla R = 0$ and have, the homogeneous spaces G/H is locally symmetric if and only if $b = 0$. Also, $\nabla^2 R = 0$ if and only if $b = 0$, so we conclude that the homogeneous spaces of type A_1 are locally symmetric if and only if $\nabla^2 R = 0$. Similar arguments will be applied for the other Lorentzian cases and this finishes the proof. \square

The following remark is the direct conclusion of the Theorems 5.3 and 5.4.

Remark 5.5. Let (M, g) be a homogeneous Lorentzian four-dimensional manifold, then (M, g) is not a two-symmetric space.

Classification of four-dimensional pseudo-Riemannian homogeneous spaces with non-trivial isotropy has been studied in [25] in order to find the local classification of four-dimensional Einstein-Maxwell homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature. The presented list is a good reference to study pseudo-Riemannian homogeneous four-manifolds. We apply the mentioned classification to find an example of a pseudo-Riemannian two-symmetric homogeneous four-manifold, equipped with an invariant metric of *neutral* signature.

EXAMPLE 5.6. Let $M = G/H$ be a homogeneous four-dimensional manifold of type 1.4¹ : 9 of the Komrakov's list [25]. In this case, the Lie algebras \mathfrak{g} and \mathfrak{h}

are described as following:

$$\mathfrak{g} = \left\{ \left(\begin{array}{cccc} x & 0 & 0 & 0 \\ 0 & 0 & \lambda x & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & -\mu x \end{array} \right) \middle| x \in \mathbb{R}, \lambda, \mu \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \quad \mathfrak{h} = \langle p \rangle,$$

where $\mathfrak{n}_3 = \langle h, p, q \rangle$ with the only non-zero bracket $[p, q] = h$. If

$$\mathfrak{g} = \text{span}\{u_1, u_2, u_3, u_4\}$$

and $\mathfrak{h} = \text{span}\{h_1\}$, the table of Lie brackets is:

| $[,]$ | h_1 | u_1 | u_2 | u_3 | u_4 |
|---------|--------|--------|----------------------------|---------------------------|-----------|
| h_1 | 0 | 0 | u_1 | u_2 | 0 |
| u_1 | 0 | 0 | 0 | u_1 | 0 |
| u_2 | $-u_1$ | 0 | 0 | $\lambda h_1 + u_2 + u_4$ | 0 |
| u_3 | $-u_2$ | $-u_1$ | $-\lambda h_1 - u_2 - u_4$ | 0 | μu_4 |
| u_4 | 0 | 0 | 0 | $-\mu u_4$ | 0 |

The invariant metric will be calculated as following

$$g = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix},$$

for arbitrary real constants $a \neq 0, b, c, d$. This metric admits both Lorentzian and neutral signatures while for $d = -9a$, g is of neutral signature. We also set $\mu = -\frac{5}{2}$ and $\lambda = \frac{3}{4}$. Keeping in mind $\Lambda[i] = \Lambda(u_i)$ for all indices $i = 1, \dots, 4$, the components of the Levi-Civita connection are:

$$\begin{aligned} \Lambda[1] &= 0, & \Lambda[2] &= \begin{pmatrix} 0 & 1 & \frac{c}{2a} & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \\ \Lambda[3] &= \begin{pmatrix} -1 & \frac{c}{2a} & \frac{6ba-c^2}{6a^2} & \frac{5c}{2a} \\ 0 & 0 & \frac{c}{a} & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{c}{6a} & 0 \end{pmatrix}, & \Lambda[4] &= \begin{pmatrix} 0 & -\frac{9}{2} & \frac{5c}{2a} & -\frac{45}{2} \\ 0 & 0 & -\frac{9}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \end{pmatrix}, \end{aligned}$$

also, by using Equation (5.5), if set $R_{ij} = R(u_i, u_j)$, the non-zero components of the curvature tensor are

$$R_{23} = \begin{pmatrix} 0 & 6 & -\frac{2c}{a} & 18 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & -18 & \frac{6c}{a} & -54 \\ 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{pmatrix}.$$

The space is Ricci flat and the only non-zero covariant derivative of the curvature tensor is in the direction of u_3 . We set $(\Lambda[k]R)_{ij} = (\Lambda(u_k)R)(u_i, u_j)$, the non-zero components are

$$(\Lambda[3]R)_{23} = \begin{pmatrix} 0 & 6 & -\frac{2c}{a} & 18 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad (\Lambda[3]R)_{34} = \begin{pmatrix} 0 & -18 & \frac{6c}{a} & -54 \\ 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{pmatrix}.$$

Clearly, $(M = G/H, g)$ is never locally symmetric but by straight forward calculations we get $\nabla^2 R = 0$ which shows that the spaces is two-symmetric.

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